Sperner Families over a Subset

KO-WEI LIH

Institute of Mathematics, Academia Sinica, Taipei, Taiwan, Republic of China

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Let $|X| = n > 0$, $|Y| = k > 0$, and $Y \subseteq X$. A family $A$ of subsets of $X$ is a Sperner family of $X$ over $Y$ if $A_1 \not\subseteq A_2$ for every pair of distinct members of $A$ and every member of $A$ has a nonempty intersection with $Y$. The maximum cardinality $f(n, k)$ of such a family is determined in this paper. $f(n, k) = \binom{n}{\lfloor n/2 \rfloor} - \binom{n-k}{\lfloor n/2 \rfloor}$.

1. INTRODUCTION

Let $X$ be a finite set of cardinality $|X| = n > 0$. A family $A$ of subsets of $X$ is called a Sperner family of $X$ if $A_1 \not\subseteq A_2$ for every pair of distinct members of $A$. A fundamental result of Sperner [5] states that the cardinality of a Sperner family of $X$ is bounded by $\binom{n}{\lfloor n/2 \rfloor}$. This bound can be attained by the family of all subsets of cardinality $\lfloor n/2 \rfloor$. During the past two decades, there has been extensive research into problems generalizing Sperner's result in various directions. Such tremendous efforts cumulate to the so-called extremal theory of finite sets. See [2-4]. However, the following relativized version of Sperner's problem does not seem to have attracted enough attention. Let $Y$ be a fixed subset of $X$ of cardinality $|Y| = k > 0$. A family $A$ is called a Sperner family of $X$ over $Y$ if $A$ is a Sperner family of $X$ and each member of $A$ has a nonempty intersection with $Y$. The relativized Sperner problem is to determine the maximum cardinality $f(n, k)$ of Sperner families of $X$ over $Y$. In this paper we will show that $f(n, k) = \binom{n}{\lfloor n/2 \rfloor} - \binom{n-k}{\lfloor n/2 \rfloor}$. Our discourse will be conducted in the context of graded posets. In this way the whole problem is put into a wider perspective which enables us to formulate a conjecture in the concluding section.

2. MAIN THEOREMS

A finite graded poset $P$ is a finite partially ordered set with a rank function $r$. That is, $r$ is an integer-valued function defined on $P$ such that $r(x) = 0$
for every minimal element \( x \), and \( r(x) = r(y) + 1 \) whenever \( y < x \) and \( y < z < x \) for no \( z \). We call \( r(x) \) the rank of \( x \). A set \( A \) of elements of \( P \) is an antichain in \( P \) if two arbitrary distinct elements of \( A \) are not related by the partial order of \( P \). Let \( P_m \) denote the set of elements in \( P \) having rank \( m \). The number \( |P_m| \) is called the \( m \)th Whitney number of \( P \) by Crapo and Rota [1]. \( P \) is said to be Sperner if \( \max_m |P_m| = \max|A| \mid A \) is an antichain in \( P \). The common value is henceforth called the Sperner number of \( P \). An order-filter \( F \) in \( P \) is a subset of \( P \) such that, for any \( a \in F \) and \( b \in P \), \( a \leq b \) implies \( b \in F \). The principal order-filter \( \langle a \rangle \) generated by \( a \) is defined to be the set \( \{ b \mid a \leq b \} \). An order-filter \( F \) is generated by \( a_1, a_2, \ldots, a_k, k > 0 \), if \( F = \langle a_1 \rangle \cup \langle a_2 \rangle \cup \cdots \cup \langle a_k \rangle \). We write \( F = \langle a_1, a_2, \ldots, a_k \rangle \). If we furthermore suppose that \( a_1, a_2, \ldots, a_k \) are of a fixed rank, then the rank function \( r \) of \( P \) will induce a canonical rank function \( r' \) on \( F \). That is, \( r'(x) = r(x) - r(a_1) \) for all \( x \in F \). \( F \) thus becomes a graded poset.

The graded poset that mainly concerns us here is \( B^n \), the Boolean algebra of all subsets of \( \{1, 2, \ldots, n\} \) ordered by inclusion. \( B^n \) has the rank function \( r(x) = |x| \). The classical theorem of Sperner in fact says that \( B^n \) is Sperner and its Sperner number is \( \binom{n}{\lfloor n/2 \rfloor} \). We are going to establish the following stronger results.

Theorem 1. Let \( a_1, a_2, \ldots, a_k, 0 < k \leq n, \) be distinct elements of rank 1 in \( B^n \). Then \( F = \langle a_1, a_2, \ldots, a_k \rangle \) is Sperner.

Proof. We first elucidate two basic facts about \( F \).

**Fact 1.** For \( S \subseteq F_m \), \( 0 \leq m < n-1 \), let \( S^* = \{ y \in F_{m+1} \mid (\exists x \in S)(x \leq y) \} \). Then \( (n-m-1) |S| \leq (m+2) |S^*| \).

**Fact 2.** For \( S \subseteq F_m \), \( 0 < m \leq n-1 \), let \( S_* = \{ y \in F_{m-1} \mid (\exists x \in S)(y \leq x) \} \). Then \( m |S| \leq (n-m) |S_*| \).

\( F_m \) and \( F_{m+1} \) altogether form a bipartite graph such that \( u \in F_m \) and \( v \in F_{m+1} \) are adjacent if and only if \( u \leq v \). Similarly, \( F_m \) and \( F_{m-1} \) form a bipartite graph. Now, for \( x \in F_m \), there are exactly \( n-m-1 \) elements in \( F_{m+1} \) adjacent to \( x \) since \( y \in B^n_{m+2} \) and \( x \leq y \) imply \( y \in F_{m+1} \). Looking downward, we have two possibilities. In the first case, there are at least \( a_i \) and \( a_j \), \( i \neq j \), such that \( a_i \leq x \) and \( a_j \leq x \). Thus, if \( y \in B^n_m \) and \( y \leq x \), then it must be the case \( y \in F_{m-1} \). So \( x \) is adjacent to \( m+1 \) elements in \( F_{m-1} \). In the second case, there is exactly one \( a_i \) such that \( a_i \leq x \). Then, except the element \( x \langle a_i \rangle \), \( x \) is adjacent to \( m \) elements in \( F_{m-1} \). Facts 1 and 2 now can be visualized simply by counting edges between \( S \) and \( S^* \), between \( S \) and \( S_* \), respectively.

Returning to the proof of Theorem 1, we assume that, among all maximum-sized antichains in \( F \), \( A \) has the minimum value of \( d(A) = \max \{ r'(x) \mid x \in A \} - \min \{ r'(x) \mid x \in A \} \). Now suppose \( d(A) > 0 \).
Case I. \( \min \{|r'(x)| \mid x \in A\} = m < \lfloor n/2 \rfloor - 1 \). Since \( m \leq \lfloor n/2 \rfloor - 2 \leq (n-3)/2 \), we have that \( n - m - 1 \geq m + 2 \). It follows from fact 1 that \( |S| \leq |S^*| \) for any \( S \subseteq F_m \). Now let \( S = A \cap F_m \) and replace \( S \) in \( A \) by \( S^* \) to obtain \( A' \). Since \( A \) is an antichain, none of those elements in \( S^* \) are included in elements of \( A \). This shows that \( A' \) is also a maximum-sized antichain in \( F \). However \( d(A') - d(A) = 1 \), a contradiction.

Case II. \( \max \{|r'(x)| \mid x \in A\} = m > \lfloor n/2 \rfloor - 1 \). In this case \( m > n - m \), from which \( |S| \leq |S^*_n| \) follows by fact 2 for any \( S \subseteq F_m \). Reasoning as in case I, we may replace \( A \cap F_m \) by \( (A \cap F_m)^* \). We immediately see that there exists a maximum-sized antichain \( A' \) in \( F \) with \( d(A') = d(A) - 1 \), a contradiction.

If \( A \) does not satisfy either case I or II, then all elements in \( A \) are of the constant rank \( \lfloor n/2 \rfloor - 1 \) which contradicts the assumption \( d(A) > 0 \). After all these contradictions, we come to the conclusion \( d(A) = 0 \), i.e., \( A \subseteq F_m \) for some \( m \). Obviously, each \( F_m \) is an antichain. Hence we can choose \( A \) to be that \( F_m \) with maximum Whitney number.

Remark. If we adjoin \( \phi \) to \( F \), then the extended family turns out to be a Sperner lattice and the ranks agree with the cardinalities of the sets.

\section*{Theorem 2.} Let \( F \) be the same as in Theorem 1. Then the Sperner number of \( F \) is \( \binom{n}{\lfloor n/2 \rfloor} (\binom{n-k}{\lfloor n/2 \rfloor}) \).

\textbf{Proof.} It is easy to see \( |F_m| = \binom{n}{m+1} - \binom{n-k}{m+1} \). Theorem 2 follows from Theorem 1 if we can show that \( d_m = \binom{n}{m} - \binom{n-k}{m} \) attains its maximum over the segment \( 0 < m \leq n \) when \( m = \lfloor n/2 \rfloor \). Of course, we are here under the usual convention \( \binom{n}{m} = 0 \) if \( m > n \). Consider \( d_{m+1} - d_m = \left[\binom{n}{m+1} - \binom{n}{m}\right] - \left[\binom{n-k}{m+1} - \binom{n-k}{m}\right] \). When \( 1 < m + 1 \leq (n-k)/2 \), both differences inside brackets are nonnegative and the first difference is greater than the second difference. This latter fact can be seen by an easy induction on \( n \). Hence \( d_{m+1} - d_m > 0 \). When \( (n-k)/2 < m + 1 \leq \lfloor n/2 \rfloor \), the first difference is still nonnegative but the second difference is nonpositive. Again \( d_{m+1} - d_m > 0 \). In other words, \( d_m \) is increasing as \( m \) steps up from 1 to \( \lfloor n/2 \rfloor \). When \( \lfloor n/2 \rfloor \leq m \leq n \), we need another form for \( d_m \), i.e., \( d_m = \sum_{j=1}^k \binom{n-j}{m-1} \). This can be shown by induction on \( n \) since

\[
\binom{n+1}{m} - \binom{n+1-k}{m} = \left[\binom{n}{m} - \binom{n-k}{m}\right] + \left[\binom{n}{m-1} - \binom{n-k}{m-1}\right]
\]
Now \( d_m - d_{m+1} = \sum_{j=1}^{k} \left( \binom{n-j}{m-1} - \binom{n-j}{m} \right) \) and \( |(n-j)/2| \leq |n/2| - 1 \leq m - 1 \) for \( j = 1, 2, \ldots, k \). So we have \( \binom{n-j}{m-1} - \binom{n-j}{m} \geq 0 \) for \( j = 1, 2, \ldots, k \); hence \( d_m - d_{m+1} \geq 0 \). In other words, \( d_m \) is decreasing as \( m \) steps up from \( |n/2| \) to \( n \). This completes the proof.

Incidentally, the above proof also establishes the following.

**Theorem 3.** Let \( F \) be the same as in Theorem 1. Then the sequence of Whitney numbers of \( F \) is a unimodal sequence.

### 3. Concluding Remark

It seems that not enough investigation has been penetrated into the substructures of Sperner graded posets. In view of Theorem 1, we offer the following plausible conjecture.

**Conjecture.** If \( F \) is an order-filter in \( B^n \) generated by elements of a fixed rank, then \( F \) is Sperner.

Note that the conjecture is false if \( B^n \) is replaced by an arbitrary Sperner poset. The following counterexample is supplied by the referee. Consider the poset \( P \) with these sets ordered by inclusion: \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{4\}, \{5\}, \{4, 5\}. Let \( F \) consist of all sets in \( P \) except \{1\}. Then \( P \) is Sperner and \( F \) is not.

### References