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Sperner Families over a Subset

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Let $|X| = n > 0$, $|Y| = k > 0$, and $Y \subseteq X$. A family A of subsets of X is a Sperner family of X over Y if $A_1 \not\subseteq A_2$ for every pair of distinct members of A and every member of A has a nonempty intersection with Y. The maximum cardinality $f(n, k)$ of such a family is determined in this paper. $f(n, k) = \binom{n}{\lceil n/2 \rceil} - \binom{n-k}{\lceil n/2 \rceil}$.

1. INTRODUCTION

Let X be a finite set of cardinality $|X| = n > 0$. A family A of subsets of X is called a Sperner family of X if $A_1 \not\subseteq A_2$ for every pair of distinct members of A. A fundamental result of Sperner [5] states that the cardinality of a Sperner family of X is bounded by $\binom{n}{\lceil n/2 \rceil}$. This bound can be attained by the family of all subsets of cardinality $\lceil n/2 \rceil$. During the past two decades, there has been extensive research into problems generalizing Sperner's result in various directions. Such tremendous efforts cumulate to the so-called extremal theory of finite sets. See [2–4]. However, the following relativized version of Sperner's problem does not seem to have attracted enough attention. Let Y be a fixed subset of X of cardinality $|Y| = k > 0$. A family A is called a Sperner family of X over Y if A is a Sperner family of X and each member of A has a nonempty intersection with Y. The relativized Sperner problem is to determine the maximum cardinality $f(n, k)$ of Sperner families of X over Y. In this paper we will show that $f(n, k) = \binom{n}{\lceil n/2 \rceil} - \binom{n-k}{\lceil n/2 \rceil}$. Our discourse will be conducted in the context of graded posets. In this way the whole problem is put into a wider perspective which enables us to formulate a conjecture in the concluding section.

2. MAIN THEOREMS

A finite graded poset P is a finite partially ordered set with a rank function r. That is, r is an integer-valued function defined on P such that $r(x) = 0$

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for every minimal element x, and $r(x) = r(y) + 1$ whenever $y < x$ and $y < z < x$ for no z. We call $r(x)$ the rank of x. A set A of elements of P is an antichain in P if two arbitrary distinct elements of A are not related by the partial order of P. Let P_m denote the set of elements in P having rank m. The number $|P_m|$ is called the mth Whitney number of P by Crapo and Rota [1]. P is said to be Sperner if $\max_m |P_m| = \max\{|A| \mid A \text{ is an antichain in } P\}$. The common value is henceforth called the Sperner number of P. An order-filter F in P is a subset of P such that, for any $a \in F$ and $b \in P$, $a \leqslant b$ implies $b \in F$. The principal order-filter $\langle a \rangle$ generated by a is defined to be the set $\{b \mid a \leqslant b\}$. An order-filter F is generated by $a_1, a_2, \ldots, a_k$, $k > 0$, if $F = \langle a_1 \rangle \cup \langle a_2 \rangle \cup \cdots \cup \langle a_k \rangle$. We write $F = \langle a_1, a_2, \ldots, a_k \rangle$. If we furthermore suppose that $a_1, a_2, \ldots, a_k$ are of a fixed rank, then the rank function r of P will induce a canonical rank function r' on F. That is, $r'(x) = r(x) - r(a_1)$ for all $x \in F$. F thus becomes a graded poset.

The graded poset that mainly concerns us here is B^n, the Boolean algebra of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. B^n has the rank function $r(x) = |x|$. The classical theorem of Sperner in fact says that B^n is Sperner and its Sperner number is $\binom{n}{\lfloor n/2 \rfloor}$. We are going to establish the following stronger results.

THEOREM 1. *Let $a_1, a_2, \ldots, a_k$, $0 < k \leqslant n$, be distinct elements of rank 1 in B^n. Then $F = \langle a_1, a_2, \ldots, a_k \rangle$ is Sperner.*

Proof. We first elucidate two basic facts about F.

Fact 1. For $S \subseteq F_m$, $0 \leqslant m < n - 1$, let $S^* = \{y \in F_{m+1} \mid (\exists x \in S)(x \leqslant y)\}$. Then $(n - m - 1)|S| \leqslant (m + 2)|S^*|$.

Fact 2. For $S \subseteq F_m$, $0 < m \leqslant n - 1$, let $S_* = \{y \in F_{m-1} \mid (\exists x \in S)(y \leqslant x)\}$. Then $m|S| \leqslant (n - m)|S_*|$.

F_m and F_{m+1} altogether form a bipartite graph such that $u \in F_m$ and $v \in F_{m+1}$ are adjacent if and only if $u \leqslant v$. Similarly, F_m and F_{m-1} form a bipartite graph. Now, for $x \in F_m$, there are exactly $n - m - 1$ elements in F_{m+1} adjacent to x since $y \in B^n_{m+2}$ and $x \leqslant y$ imply $y \in F_{m+1}$. Looking downward, we have two possibilities. In the first case, there are at least a_i and a_j, $i \neq j$, such that $a_i \leqslant x$ and $a_j \leqslant x$. Thus, if $y \in B^n_m$ and $y \leqslant x$, then it must be the case $y \in F_{m-1}$. So x is adjacent to $m + 1$ elements in F_{m-1}. In the second case, there is exactly one a_i such that $a_i \leqslant x$. Then, except the element $x \backslash a_i$, x is adjacent to m elements in F_{m-1}. Facts 1 and 2 now can be visualized simply by counting edges between S and S^*, between S and S_*, respectively.

Returning to the proof of Theorem 1, we assume that, among all maximum-sized antichains in F, A has the minimum value of $d(A) = \max\{r'(x) \mid x \in A\} - \min\{r'(x) \mid x \in A\}$. Now suppose $d(A) > 0$.

Case I. $\min\{r'(x) \mid x \in A\} = m < \lceil n/2 \rceil - 1$. Since $m \leqslant \lceil n/2 \rceil - 2 \leqslant (n-3)/2$, we have that $n - m - 1 \geqslant m + 2$. It follows from fact 1 that $|S| \leqslant |S^*|$ for any $S \subseteq F_m$. Now let $S = A \cap F_m$ and replace S in A by S^* to obtain A'. Since A is an antichain, none of those elements in S^* are included in elements of A. This shows that A' is also a maximum-sized antichain in F. However $d(A') = d(A) - 1$, a contradiction.

Case II. $\max\{r'(x) \mid x \in A\} = m > \lceil n/2 \rceil - 1$. In this case $m \geqslant n - m$, from which $|S| \leqslant |S_*|$ follows by fact 2 for any $S \subseteq F_m$. Reasoning as in case I, we may replace $A \cap F_m$ by $(A \cap F_m)_*$. We immediately see that there exists a maximum-sized antichain A' in F with $d(A') = d(A) - 1$, a contradiction.

If A does not satisfy either case I or II, then all elements in A are of the constant rank $\lceil n/2 \rceil - 1$ which contradicts the assumption $d(A) > 0$. After all these contradictions, we come to the conclusion $d(A) = 0$, i.e., $A \subseteq F_m$ for some m. Obviously, each F_m is an antichain. Hence we can choose A to be that F_m with maximum Whitney number.

Remark. If we adjoin ϕ to F, then the extended family turns out to be a Sperner lattice and the ranks agree with the cardinalities of the sets.

THEOREM 2. *Let F be the same as in Theorem* 1. *Then the Sperner number of F is* $\binom{n}{\lceil n/2 \rceil} - \binom{n-k}{\lceil n/2 \rceil}$.

Proof. It is easy to see $|F_m| = \binom{n}{m+1} - \binom{n-k}{m+1}$. Theorem 2 follows from Theorem 1 if we can show that $d_m = \binom{n}{m} - \binom{n-k}{m}$ attains its maximum over the segment $0 < m \leqslant n$ when $m = \lceil n/2 \rceil$. Of course, we are here under the usual convention $\binom{n}{m} = 0$ if $m > n$. Consider $d_{m+1} - d_m = [\binom{n}{m+1} - \binom{n}{m}] - [\binom{n-k}{m+1} - \binom{n-k}{m}]$. When $1 < m + 1 \leqslant \lceil (n-k)/2 \rceil$, both differences inside brackets are nonnegative and the first difference is greater than the second difference. This latter fact can be seen by an easy induction on n. Hence $d_{m+1} - d_m \geqslant 0$. When $\lceil (n-k)/2 \rceil < m + 1 \leqslant \lceil n/2 \rceil$, the first difference is still nonnegative but the second difference is nonpositive. Again $d_{m+1} - d_m \geqslant 0$. In other words, d_m is increasing as m steps up from 1 to $\lceil n/2 \rceil$. When $\lceil n/2 \rceil \leqslant m \leqslant n$, we need another form for d_m, i.e., $d_m = \sum_{j=1}^{k} \binom{n-j}{m-1}$. This can be shown by induction on n since

$$\binom{n+1}{m} - \binom{n+1-k}{m}$$
$$= \left[\binom{n}{m} - \binom{n-k}{m}\right] + \left[\binom{n}{m-1} - \binom{n-k}{m-1}\right]$$

$$= \sum_{j=1}^{k} \binom{n-j}{m-1} + \sum_{j=1}^{k} \binom{n-j}{m-2}$$

$$= \sum_{j=1}^{k} \binom{n+1-j}{m-1}.$$

Now $d_m - d_{m+1} = \sum_{j=1}^{k} [\binom{n-j}{m-1} - \binom{n-j}{m}]$ and $\lfloor (n-j)/2 \rfloor \leqslant \lfloor n/2 \rfloor - 1 \leqslant m-1$ for $j = 1, 2, \ldots, k$. So we have $\binom{n-j}{m-1} - \binom{n-j}{m} \geqslant 0$ for $j = 1, 2, \ldots, k$; hence $d_m - d_{m+1} \geqslant 0$. In other words, d_m is decreasing as m steps up from $\lceil n/2 \rceil$ to n. This completes the proof.

Incidentally, the above proof also establishes the following.

THEOREM 3. *Let F be the same as in Theorem* 1. *Then the sequence of Whitney numbers of F is a unimodal sequence.*

3. CONCLUDING REMARK

It seems that not enough invesugation has been penetrated into the substructures of Sperner graded posets. In view of Theorem 1, we offer the following plausible

Conjecture. If F is an order-filter in B^n generated by elements of a fixed rank, then F is Sperner.

Note that the conjecture is false if B^n is replaced by an arbitrary Sperner poset. The following counterexample is supplied by the referee. Consider the poset P with these sets ordered by inclusion: $\{1\}$, $\{2\}$, $\{1, 2\}$, $\{2, 3\}$, $\{4\}$, $\{5\}$, $\{4, 5\}$. Let F consist of all sets in P except $\{1\}$. Then P is Sperner and F is not.

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