Modeling rate-independent hysteresis in large deformations of preconditioned soft tissues

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**ABSTRACT**

A phenomenological model is proposed for characterizing rate-independent hysteresis exhibited by preconditioned soft tissues. The preconditioned tissue is modeled as an isotropic composite of a hyperelastic component and a dissipative (inelastic) component. Specifically, the constitutive equations are hyperelastic in the sense that the stress is determined by derivatives of a strain energy function. Inelasticity of the dissipative component is controlled by a yield function with different functional forms for the hardening variable during deformation loading and unloading. The constitutive equations proposed in this paper are simple. In particular, they depend on only seven material constants: three controlling the response of the elastic component and the remainder controlling the response of the dissipative component. More importantly, the material constants can be determined to match rather general loading and unloading behavior. It is observed that the hysteretic response of the model compares well with experimental data for passive uniaxial loading/unloading of Manduca muscle. Moreover, the present model treats partial loading and reloading of preconditioned tissue as elastic–plastic response, which is different from the treatment of pseudo-elastic models used in the literature.

1. Introduction

Most biological soft tissues are inhomogeneous, nearly incompressible materials that exhibit nonlinear inelastic (viscoelastic/viscoplastic) response. Many of these soft tissues are reinforced by fiber families which generally consist of collagen and elastin. The material orientation of these fibers along with the fiber constituents play a dominant role in determining the anisotropic mechanical properties of the tissue. Phenomenological models which include specific fiber orientations have been considered in Holzapfel (2001) and Rubin and Bodner (2002). However, for some applications, it is sufficient to model these tissues as isotropic materials.

In general, the material response of the tissue is rate-dependent and inelastic. More specifically, cyclic loading of tissues at constant strain rate between fixed stress or strain limits typically exhibits time-dependent inelastic hysteresis loops that shift with each cycle towards a steady-state hysteresis loop. As an example, Fig. 1. shows the steady-state hysteresis loop for passive cyclic uniaxial stress loading of a Manduca muscle (Dorfmann et al., 2008).

Fung et al. (1972, 1993) observed that this steady-state hysteresis loop is relatively insensitive to the magnitude of the constant strain rate over more than two orders of magnitude of strain rate. This steady-state hysteresis loop characterizes the state of the material which is referred to here as preconditioned. The transitional process towards this preconditioned state is referred to here as preconditioning. Moreover, it is noted that the hysteresis loops of the preconditioned tissue depend on the magnitudes of strain or stress defining the limits of the cycles.

Several researchers have experimentally observed this insensitivity of the response of the preconditioned soft tissues to strain rate. Specifically, the effect was observed for biaxial stretching of rabbit skin in Lanir and Fung (1974a,b) and for excised skin in Pereira et al. (1991). Zheng et al. (1999) found that the effective Young’s modulus of limb soft tissue was fairly rate insensitive and Vogel (1972) reported that the strain to failure of rat skin was also rate independent.

Often, the hysteresis loop of the preconditioned tissue is ignored and the tissue is modeled as being a hyperelastic material. Since a hyperelastic material exhibits a single loading/unloading curve it is necessary to decide whether the loading curve, the unloading curve or some average of the two curves in the actual preconditioned hysteresis loop will be used to calibrate the strain energy function for the approximate hyperelastic model. For example, Hendriks et al. (2004, 2006) used a Mooney–Rivlin model for human skin and Shergold and Fleck (2005) and Shergold et al. (2006) used the Ogden model for human skin and pig skin, respectively.
The most common model used to characterize the hysteresis loop of a preconditioned tissue was proposed by Fung et al. (1972) and Fung (1993), who suggested that the tissue can be modeled by two hyperelastic materials: one characterizing the loading curve and the other characterizing the unloading curve. This material response is called pseudo-elasticity. In particular, a single strain energy function is used with different material constants for the loading and unloading curves.

Within the context of pseudo-elastic models, it is noted that Tong and Fung (1976) developed a strain energy function for modeling the response to biaxial stretching of rabbit skin observed in the experiments in Lanir and Fung (1974a,b). This model had a number of material parameters which were difficult to determine from experimental data and it was found to be too sensitive to changes in the bounds of the biaxial loading. Yin et al. (1986) modified the pseudo-elastic strain energy function in this model to reduce the number of material constants to a nearly “minimum” set needed to match experimental data. Further modification of this pseudo-elastic energy function can be found in Chaudhrya et al. (1998) and Gambbarotta et al. (2005).

Dorfmann et al. (2007, 2008) and Paetsch et al. (2012) exploited the isotropic pseudo-elastic model developed by Ogden and Roxburgh (1999) to characterize the passive response of muscle tissue. In this model, the strain energy function is taken in the form

\[ W = W(F, \eta) \]

where \( F \) is the deformation gradient tensor. The additional variable \( \eta \) is inactive (remains constant) during loading and is a specified function \( \eta = \eta(F) \) during unloading. The functional form for \( \eta \) is discussed in Ogden and Roxburgh (1999), Dorfmann and Ogden (2003, 2004), Dorfmann et al. (2007) and Paetsch et al. (2012). In particular, the model in Dorfmann and Ogden (2003) is proposed for loading, partial or complete unloading and subsequent reloading and unloading. However, the notion of loading/unloading in this model is unclear and the determination of \( \eta \) for general loading situations is complicated.

A single loading/unloading curve associated with the Mullins effect (Mullins, 1969; Diani et al., 2009) looks identical to that for a preconditioned tissue. In fact, the strain energy function used to model the Mullins effect has the same form \( W = W(F, \eta) \). However, for the Mullins effect \( \eta \) is used to characterize damage that only occurs when loading is applied beyond the previous maximum point of loading. Therefore, for the Mullins effect unloading and reloading occur on the same curve with no hysteresis until the material is further damaged. In contrast, unloading and reloading of a preconditioned tissue occur on different curves with hysteresis always being present.

Viscoelastic (Sverdlik and Lanir, 2002) and elastic-viscoplastic (Rubin and Bodner, 2002; Mazza et al., 2005) constitutive equations have been developed which can model the time dependent response of tissues and the process of preconditioning. However, there is still a need for a simple model that characterizes dissipation of the hysteresis loop of a preconditioned tissue. Consequently, the objective of this work is to develop simple isotropic constitutive equations for large deformations of preconditioned biological soft tissues which exhibit rate-independent hysteresis curves and which are valid for general loading histories. In contrast with the standard pseudo-elastic formulation, here the preconditioned tissue is modeled as a composite of a hyperelastic component and a dissipative component. Specifically, the dissipative component is modeled as a rate-independent elastic–plastic material using a yield function, which depends on the elastic distortional deformation of the inelastic component and on a hardening variable. Furthermore, in contrast with the standard uniaxial stress response of metals for cyclic loading, the axial stress in a preconditioned tissue does not change sign during unloading until the axial strain changes sign. To account for this fact, the hardening variable is taken to be a function of the total distortional deformation which vanishes in the unstressed reference state of the tissue. Also, different functional forms for the hardening variable are proposed for deformation loading and unloading, which allow for easy modeling of the hysteresis loop exhibited by preconditioned tissues.

2. Basic equations of the preconditioned tissue

This section briefly reviews constitutive equations for a nonlinear isotropic elastic material and provides background for the developments in the following sections. To this end, it is recalled that a material point \( X \) in the fixed reference configuration moves to the point \( x \) in the present configuration at time \( t \), with the deformation gradient \( F \) and the dilatation \( J \) defined by

\[ F = \frac{\partial x}{\partial X}, \quad J = \det(F) > 0 \]  

Also, the velocity \( \mathbf{v} \) of a material point, the velocity gradient \( L \) and the rate of deformation tensor \( D \) are defined by

\[ \mathbf{v} = \dot{x}, \quad L = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \quad D = \frac{1}{2} (L + L^T) \]  

where the superposed dot denotes material time differentiation holding \( \mathbf{X} \) fixed.

It can be shown that \( F \) and \( J \) satisfy the evolution equations

\[ \dot{F} = LF, \quad \dot{J} = J(D \cdot I) \]  

where \( I \) is the second order unity tensor and \( A \cdot B = \text{tr}(AB^T) \) denotes the inner product between two second order tensors \( (A, B) \).

The preconditioned tissue is considered to be a composite of an elastic component and a dissipative component. In particular, the specific (per unit mass) strain energy \( \Sigma \) of the tissue is modeled as a sum of the specific strain energy \( \Sigma_e \) of the elastic component and the specific strain energy \( \Sigma_d \) of the dissipative component

\[ \Sigma = \Sigma_e + \Sigma_d \]  

Moreover, the total Cauchy stress \( \mathbf{T} \) in this model separates additively into two parts

\[ \mathbf{T} = \mathbf{T}_e + \mathbf{T}_d \]  

where \( \mathbf{T}_e \) is the stress in the elastic component and \( \mathbf{T}_d \) is the stress in the dissipative component. Within the context of the purely mechanical theory, the rate of material dissipation is given by

\[ D = \dot{T} - \rho \dot{\Sigma} \geq 0 \]  

where the conservation of mass relates the mass density \( \rho_0 \) in the reference configuration to the mass density \( \rho \) in the present configuration.
\[\rho \ddot{\mathbf{J}} = \rho_0 \]  
(7)
Since the elastic component is non-dissipative, it follows that
\[\mathbf{T}_e \cdot \mathbf{D} = \rho \dot{\mathbf{\Sigma}}_e\]  
(8)
so material dissipation is due solely to the dissipative component
\[\mathcal{D} = \mathbf{T}_d \cdot \mathbf{D} - \rho \dot{\mathbf{\Sigma}}_d \geq 0\]  
(9)

3. Constitutive equations for the elastic component

Constitutive equations for elastically isotropic materials are well known and are recorded here briefly for completeness. Specifically, it is convenient to use the work of Flory (1961) to define a unimodular tensor \(\mathbf{B}\) and its two invariants \(\{\beta_1, \beta_2\}\) by
\[\mathbf{B} = \mathbf{F}^T, \quad \mathbf{B} = J^{1/3} \mathbf{B}, \quad \det(\mathbf{B}) = 1, \quad \beta_1 = \mathbf{B} \cdot \mathbf{I}, \quad \beta_2 = \mathbf{B} \cdot \mathbf{B}'\]  
(10)
where \(\{\mathbf{B}, \beta_1, \beta_2\}\) satisfy the evolution equations
\[\dot{\beta}_1 = 4 \mathbf{g}'^e \cdot \mathbf{D}, \quad \dot{\beta}_2 = 4 \left[ \mathbf{B}'^2 - \frac{1}{3} (\mathbf{B}' \cdot \mathbf{I}) \mathbf{I} \right] \cdot \mathbf{D}\]  
(11)
with \(\mathbf{g}'^e\) being a deviatoric distortional strain tensor. Moreover, the strain energy \(\mathbf{\Sigma}_e\) per unit mass is taken in the form
\[\mathbf{\Sigma}_e = \mathbf{\Sigma}(J, \beta_1, \beta_2)\]  
(12)
With the help of (8), (11) and (12) it follows that the constitutive equation for the elastic part \(\mathbf{T}_e\) of the Cauchy stress is given by
\[\mathbf{T}_e = -p \mathbf{I} + \mathbf{T}_e', \quad p = -\rho_0 \frac{\partial \mathbf{\Sigma}_e}{\partial J}\]  
(13)
\[\mathbf{T}_e' = 4J^{-1} \rho_0 \frac{\partial \mathbf{\Sigma}_e}{\partial \beta_1} \mathbf{g}'^e + 4J^{-1} \rho_0 \frac{\partial \mathbf{\Sigma}_e}{\partial \beta_2} \left[ \mathbf{B}'^2 - \frac{1}{3} (\mathbf{B}' \cdot \mathbf{I}) \mathbf{I} \right] \]  
where \(p\) is the pressure. Moreover, the functional form of \(\mathbf{\Sigma}_e\) is restricted so that this stress vanishes in the reference configuration
\[\mathbf{T}_e = 0 \quad \text{for} \quad J = 1, \quad \mathbf{B} = \mathbf{I}\]  
(14)
As a special case, \(\mathbf{\Sigma}_e\) is specified in the form
\[\rho_0 \mathbf{\Sigma}_e = Kf(J) + \frac{\mu_e}{2} \left[ (J^3 - 1) + \frac{b_e}{n_e + 1} (J^3 - 1)^{n_e+1} \right], \quad \mu_e > 0, \quad b_e > 0, \quad n_e > 0\]  
(15)
where \(K\) is a constant having the units of stress that controls the bulk modulus, \(f(J)\) is a function to be specified which satisfies the restrictions
\[f(1) = 0, \quad \frac{df}{dJ}(1) = 0, \quad \frac{d^2f}{dJ^2}(1) = 1\]  
(16)
\(\mu_e\) is a positive constant having the units of stress, \(n_e\) is a positive constant and \(b_e\) is a non-negative constant which vanishes for the simplest Neo-Hookean model. Then, the elastic Cauchy stress \(\mathbf{T}_e\) associated with (15) is given by
\[\mathbf{T}_e = -p \mathbf{I} + \mathbf{T}_e', \quad p = -K \frac{df}{dJ}, \quad \mathbf{T}_e' = 2J^{-1} \mu_e \left[ 1 + b_e (J^3 - 1)^{n_e} \right] \mathbf{g}'^e\]  
(17)

4. Constitutive equations for the dissipative component

Following the work in Eckart (1948), Leonov (1976) and Rubin and Attia (1996) the response of the dissipative component in the model is characterized by an elastic unimodular distortional deformation tensor \(\mathbf{B}'\), which satisfies the evolution equation
\[\mathbf{B}' = \Gamma \mathbf{B}'^e + \mathbf{B}'^e \mathbf{L}^T - \frac{2}{3} (\mathbf{D} \cdot \mathbf{I}) \mathbf{B}' - \Gamma \mathbf{A}_p\]  
(18)
where \(\Gamma\) controls the magnitude of the rate of inelasticity and \(\mathbf{A}_p\) controls its direction
\[\mathbf{A}_p = \mathbf{B}' - \left( \frac{3}{\beta_1^e} - 1 \right) \mathbf{I}\]  
(19)
Since \(\mathbf{B}'\) is a unimodular tensor, it has two non-trivial scalar invariants
\[\alpha_1 = \mathbf{B}' \cdot \mathbf{I}, \quad \alpha_2 = \mathbf{B}' \cdot \mathbf{B}'\]  
(20)
which satisfy the evolution equations
\[\dot{\alpha}_1 = 4 \mathbf{g}'^e \cdot \mathbf{D} - \Gamma \mathbf{A}_p \cdot \mathbf{I}, \quad \mathbf{g}'^e = \frac{1}{2} \left[ \mathbf{B}'_0 - \frac{3}{2} (\mathbf{B}' \cdot \mathbf{I}) \mathbf{I} \right]\]  
(21)
\[\dot{\alpha}_2 = 4 \left[ \mathbf{B}'_0' - \frac{3}{2} (\mathbf{B}' \cdot \mathbf{I}) \mathbf{I} \right] \mathbf{D} - 2\Gamma \mathbf{A}_p \cdot \mathbf{B}'_0\]  
(22)
where \(\mathbf{g}'^e\) being a deviatoric distortional strain tensor associated with the dissipative component.

As a special case, \(\mathbf{\Sigma}_e\) is specified by
\[\rho_0 \mathbf{\Sigma}_d = \frac{\mu_d}{2(n_d + 1)} (\alpha_1 - 3)^{n_d+1}, \quad \mu_d > 0, \quad n_d > 0\]  
(23)
where \(\mu_d\) is a positive constant having the units of stress and \(n_d\) is a non-negative constant which vanishes for the simplest Neo-Hookean model. Furthermore, the dissipative part \(\mathbf{T}_d\) of the Cauchy stress, associated with (22), is given by the deviatoric tensor \(\mathbf{T}_d\)
\[\mathbf{T}_d = \mathbf{T}_d = 2J^{-1} \mu_d (\alpha_1 - 3)^{n_d+1} \mathbf{g}'^e\]  
(24)
With the help of (21–23) it follows that the rate of dissipation (9) reduces to
\[\mathcal{D} = \frac{1}{2} J^{-1} \mu_d (\alpha_1 - 3)^{n_d+1} \Gamma \mathbf{A}_p \cdot \mathbf{I} \geq 0\]  
(25)
Using the fact that \(\mathbf{B}'\) is unimodular, it can be shown that \(\{\mathbf{A}_p \cdot \mathbf{I}\}\) is non-negative, so (24) requires \(\Gamma\) to also be non-negative
\[\Gamma \geq 0\]  
(26)
For later convenience, the scalar measures \(\gamma, \gamma_e\) of total and elastic distortional deformation are defined, respectively, by
\[\gamma = \sqrt{\frac{2}{3} \mathbf{g}'^e \cdot \mathbf{g}'^e}, \quad \gamma_e = \sqrt{\frac{2}{3} \mathbf{g}_0^e \cdot \mathbf{g}_0^e}\]  
(27)
which satisfy the evolution equations
\[\dot{\gamma} = \gamma - \Gamma \gamma_e, \quad \dot{\gamma}_e = \frac{3}{2\gamma_e} (\mathbf{g}_0^e \mathbf{B}'_0) \cdot \mathbf{D} - \frac{2}{3} (\mathbf{D} \cdot \mathbf{I}) \gamma_e\]  
(28)
Moreover, combining the responses (17) and (23), it follows that the deviatoric part \(\mathbf{T}'\) of the total Cauchy stress is given by
\[\mathbf{T}' = 2J^{-1} \mu_e \left[ 1 + b_e (J^3 - 1)^{n_e} \right] \mathbf{g}'^e + 2J^{-1} \mu_d (\alpha_1 - 3)^{n_d+1} \mathbf{g}'^e\]  
where for small deformations from the stress-free reference configuration \(J = 1, \beta_1 = 1\) and \(\mathbf{A}_p = 0\) yields the approximations
\[\mathbf{T}' \approx 2 \mu_e \mathbf{g}'^e \quad \text{for} \quad n_d > 0, \quad \mathbf{T}' \approx 2 \mu_d \mathbf{g}'^e + 2 \mu_d \mathbf{g}_0^e \quad \text{for} \quad n_d = 0\]  
(29)
This shows that the constants \(\{\mu_e, \mu_d\}\) control the resistance of the elastic and dissipative components, respectively, to distortion from
the stress-free reference configuration for the simplest Neo-Hookean model of the dissipative component.

5. A simple yield function for the dissipative component

The inelastic response of the dissipative component is characterized by a yield function of the form

\[ g = \gamma_e - \kappa \leq 0 \] (30)

with elastic response for \( g < 0 \). Here, the yield strength \( \kappa \) and an additional history dependent variable \( \gamma_e \) are defined by

\[ \kappa = \kappa_I (\gamma) \quad \text{for} \quad \gamma \geq 0, \quad \kappa = \kappa_{UL} (\gamma, \gamma_l) \quad \text{for} \quad \gamma < 0. \]

\[ \gamma_l = \gamma \quad \text{for} \quad \gamma \geq 0, \quad g = 0, \quad g > 0, \quad \gamma_l = 0 \quad \text{otherwise} \] (31)

where \( \{\kappa_I (\gamma), \kappa_{UL} (\gamma, \gamma_l)\} \) are the hardening functions during loading and unloading, respectively, that need to be specified and \( g \) is an auxiliary quantity which is specified later by (32). Moreover, in this model the variable \( \gamma_l \) merely records the value of \( \gamma \) during deformation loading (\( \gamma > 0 \)). Alternatively, if during reloading the data does not indicate that the response returns to the original loading curve, then it might be better to propose a more complicated evolution equation for \( \gamma_l \).

In order to determine the function \( \Gamma \) for this rate-independent model it is convenient to define the auxiliary quantity \( \tilde{g} \), such that

\[ \tilde{g} = \frac{\gamma_e}{\gamma_e} \quad \frac{\partial \kappa}{\partial \gamma} \quad \tilde{g} = \tilde{g} - \Gamma \gamma_e \] (32)

where \( \gamma_e \) is given by (27). Then, \( \Gamma \) is determined by the consistency condition

\[ \Gamma = 0 \quad \text{for} \quad \gamma > 0 \quad \text{and} \quad \tilde{g} > 0, \]

\[ \Gamma = 0 \quad \text{for} \quad \text{all other cases} \] (33)

which ensures that \( \Gamma \) satisfies the restriction (25). Furthermore, in this model it is important to distinguish between deformation loading conditions defined by

\[ \text{Deformation} : \begin{cases} \text{loading} & \gamma > 0 \\ \text{neutral loading} & \gamma = 0 \\ \text{unloading} & \gamma < 0 \end{cases} \]

and inelastic loading conditions defined by

\[ \text{Inelastic} \quad (g = 0) : \begin{cases} \text{loading} & \tilde{g} > 0 \\ \text{neutral loading} & \tilde{g} = 0 \\ \text{unloading} & \tilde{g} < 0 \end{cases} \]

Also, it is noted that the material response is elastic when \( g < 0 \) or \( g = 0 \) with neutral loading or unloading.

As special cases, the functional forms for \( \kappa_I \) and \( \kappa_{UL} \) are specified by

\[ \kappa_I (\gamma) = \frac{\kappa_I (\gamma)}{\kappa_I + |\gamma|}, \quad \frac{\partial \kappa_I}{\partial \gamma} = \frac{\kappa_I^2}{(\kappa_I + |\gamma|)} \]

\[ \kappa_{UL} (\gamma, \gamma_l) = \frac{\kappa_{UL}}{\kappa_{UL} + |\gamma_l|} (\gamma_l + m(\gamma - \gamma_l)), \quad \frac{\partial \kappa_{UL}}{\partial \gamma} = \frac{m \kappa_{UL}}{\kappa_{UL} + |\gamma_l|}, \quad m > 1 \] (36)

where \( \kappa_I \) is the saturated value of \( \kappa_I \) and \( m \) controls the value of \( \gamma \) when \( \kappa_{UL} \) vanishes

\[ \kappa_{UL} = 0 \quad \text{for} \quad \gamma \leq \frac{(m - 1) |\gamma_l|}{m} \]

(37)

Notice from (26) and (28) that the effect of the dissipative component vanishes when \( \gamma_e \) vanishes (with \( g^2 = 0 \)). Thus, the functional form (36) for \( \kappa_{UL} \) together with the yield function \( g \) in (30) cause the effect of the dissipative component to vanish during unloading when \( \kappa_{UL} \) vanishes. This is consistent with the response of many preconditioned tissues.

6. Examples

For the examples considered in this section, the material is taken to be incompressible \( (J = 1) \) so that \( f(J) \) in (15) vanishes. Then, with the help of (15) and (22) the strain energy function \( \Sigma \) in (4) is specified by

\[ \rho_v \Sigma = \frac{\mu_c}{2} \left[ (\beta_1 - 3)^{n_s + 1} + \frac{\mu_g}{2(n_g + 1)} (x_1 - 3)^{n_g + 1} \right] \]

(38)

Furthermore, using (5) and (28) the Cauchy stress is given by

\[ T = -p I + T' \quad T' = 2\mu_c [1 + b_c(\beta_1 - 3)^{n_c} g^c + 2\mu_c(x_1 - 3)^{n_g} g^c] \]

(39)

where the pressure \( p \) is an arbitrary function of space and time that is determined by the equations of motion and boundary conditions. This constitutive equation for stress and the functional forms (36) for \( \{\kappa_I, \kappa_{UL}\} \) are characterized by seven material constants

\[ \{\mu_c, b_c, n_c, \mu_g, n_g, \kappa_I, \kappa_{UL}\} \]

(40)

with \( \{n_c, b_c, n_g\} \) determining the stress response of the elastic component, \( \{\mu_g, n_g\} \) determining the stress response of the dissipative component and \( \{\kappa_I, \kappa_{UL}\} \) determining the hardening functions.

It is noted that the robust, strongly objective numerical integration algorithm developed by Rubin and Papes (2011) and Hollenstein et al. (2013) can be used to obtain the value \( B_* (t_2) \) of \( B_* \) at the end of a time step defined by \( t \in [t_1, t_2] \), assuming that the value \( B_* (t_1) \) of \( B_* \) at the beginning of the time step is known. Also, it is assumed that the tissue is stress-free in its reference configuration so that the initial condition for \( B_* \) is specified by \( B_* (0) = I \), although in some situations the tissue is actually internally loaded by residual stresses as discussed in (e.g. Chuong and Fung, 1986).

6.1. Uniaxial stress

In this subsection, the response of the proposed model is examined for uniaxial stress. Specifically, with reference to a fixed rectangular Cartesian coordinate system with base vectors \( e_i \) \((i = 1, 2, 3)\), the deformation gradient \( F \) and the elastic distortional deformation \( B_* \) can be expressed in the forms

\[ F = \lambda (e_1 \otimes e_1) + \frac{1}{\sqrt{\lambda}} (e_2 \otimes e_2 + e_3 \otimes e_3) \]

\[ B_* = a^2 (e_1 \otimes e_1) + \frac{1}{a_e} (e_2 \otimes e_2 + e_3 \otimes e_3) \]

(41)

where \( \lambda \) is the specified axial stretch in the \( e_1 \) direction, \( a_e \) is a non-negative scalar to be determined by the evolution equation for \( B_* \) and \( a \otimes b \) denotes the tensor product between two vectors \( (a, b) \). Also, with the help of (10), (20) and (26) it follows that

\[ \beta_1 = \lambda^2 + \frac{2}{\lambda}, \quad \gamma = \frac{|\lambda|^2 - 1}{2\lambda}, \quad \lambda_1 = a^2 + \frac{2}{a_e}, \quad \gamma_e = \frac{|a^2 - 1|}{2a_e} \]

(42)

Moreover, for uniaxial stress the lateral component of \( T \) vanishes, so with the help of (39) the pressure \( p \) can be determined by the condition

\[ T \cdot (e_2 \otimes e_2) = -p - \frac{1}{3} \mu_c [1 + b_c(\beta_1 - 3)^{n_c}] (\lambda^2 - \frac{1}{\lambda}) \]

\[ -\frac{1}{3} \mu_g (x_1 - 3)^{n_g} (a^2 - \frac{1}{a_e}) = 0 \]

(43)
controls the magnitude of the nonlinearity of the vanishes, so there is always a of the elastic component. It can be seen on the response. Specif-

\[ \Pi = JFGT \cdot \left( e_t \otimes e_t \right) = \Pi_e + \Pi_d, \]

\[ \Pi_e = \mu_e \left[ 1 + b_e \left( \lambda^2 + \frac{2}{\lambda^2} - 3 \right) \nu \left( \lambda - \frac{1}{\lambda} \right) \right], \]

\[ \Pi_d = \mu_d \left( a_e^2 + \frac{2}{a_e} - 3 \right) \nu \left( a_e^2 - \frac{1}{a_e} \right) \frac{1}{\lambda} \]

(44)

where \( \left( \Pi_e, \Pi_d \right) \) are the axial engineering stresses of the elastic and dissipative components, respectively.

Fig. 2 shows a typical steady-state hysteretic response of a pre-conditioned tissue for cyclic uniaxial stress loading. The tissue is loaded (AB) and then unloaded (BCDEFA). For the proposed model, the portion (CDFEA) of the unloading curve is purely elastic and is used to determine the values of \( \{ \mu_d, b_e, n_e \} \) characterizing the elastic component. Moreover, the values of \( \{ \mu_e, n_e \} \) characterizing the elastic response of the dissipative component and \( \{ \kappa_s \} \) characterizing the hardening function \( \kappa_s \) during loading are determined from the initial portion (BC) of the unloading curve. In this regard, the determination of the transition point C from the experimental data is somewhat subjective. Also, the value of \( \{ m \} \) is determined so that the hardening function \( \kappa_h \) during unloading vanishes at the point C. As an example, the optimum values of the material constants for this model, which were obtained using this procedure with the experimental data for the passive response of a Manduca muscle (Dorfmann et al., 2008), are recorded in Table 1 and generate the excellent agreement with the experiment shown in Fig. 1. The simulation of this experimental data using the optimum material parameters in Table 1 is denoted by (Opt.) in the following figures.

Since the model is fully nonlinear, the values of the material constants obtained by the above procedure are not uniquely defined. In order to help determine values of the material constants for specific experimental data, it is convenient to understand the influence of each of the material constants on the uniaxial stress response. This is explored using the Cases 1–10 defined in Table 2. Specifically, for Cases 1–8 only one material constant is changed relative to the optimum values in Table 1. On the other hand, for Cases 9 and 10 the values of \( \{ \mu_e, n_e, \kappa_s \} \) are coupled by the conditions that the model reproduces the experimental values at the points B and C in Fig. 2, with point C being determined by the elastic component only. Consequently, for these Cases all three material constants \( \{ \mu_e, n_e, \kappa_s \} \) change when the value of \( \{ \mu_d \} \) is specified.

6.1.1. Cases 1 and 2: \( m = \{ 1.0, 50.0 \} \)

It is noted that the derivative of \( \kappa_u \) with respect to \( \gamma \) is discontinuous during unloading when \( \kappa_u \) vanishes, so there is always a discontinuity in the slope of the stress-stretch curve of the model at this transition point. A mild slope change can be seen in Fig. 2 near the point C. Moreover, \( (37) \) indicates that the location of this transition point is controlled by the value of \( \{ m \} \).

Fig. 3 shows the optimum response (Opt.) predicted using the constants in Table 1 together with the predictions for Cases 1 and 2, which examine the influence of \( \{ m \} \) on the response. Specifically, it can be seen from Fig. 3(a) that for small values of \( \{ m \} \) the value of \( \Pi_u \) associated with the dissipative component can change sign during unloading causing an unphysical shape of the stress-stretch curve. From (32) and (36) it can be observed that for very large values of \( \{ m \} \) inelastic loading will occur at the onset of total deformation unloading from the point B in Fig. 2. This causes a steep slope of the unloading curve and a pronounced discontinuity of slope at the transition point where \( \kappa_u \) and \( \Pi_u \) vanish (see Fig. 3(b)).

6.1.2. Cases 3 and 4: \( \mu_d = \{ 8.0, 50.0 \} \) [kPa]

Fig. 4 shows the optimum response (Opt.) together with the predictions for Cases 3 and 4 which examine the influence of \( \{ \mu_d \} \) on the response \( \Pi_u \) of the elastic component. It can be seen that \( \{ \mu_d \} \) controls the initial slope of the unloading portion of the stress-stretch curve, which is determined by the elastic component.

6.1.3. Cases 5 and 6: \( b_e = \{ 10.0, 100.0 \} \)

Fig. 5 shows the optimum response (Opt.) together with the predictions for Cases 5 and 6 which examine the influence of \( \{ b_e \} \) on the response \( \Pi_u \) of the elastic component. It can be seen that \( \{ b_e \} \) controls the magnitude of the nonlinearity of the unloading portion of the stress-stretch curve which is determined by the elastic component. Also, notice that the initial slope of this unloading portion of the stress-stretch curve is unaffected by changes in \( \{ b_e \} \).

6.1.4. Cases 7 and 8: \( n_e = \{ 0.15, 2.0 \} \)

Fig. 6 shows the optimum response (Opt.) together with the predictions for Cases 7 and 8 which examine the influence of \( \{ n_e \} \) on the response \( \Pi_u \) of the elastic component. It can be seen that \( n_e \) controls the shape of the nonlinearity of the unloading portion of the stress-stretch curve which is determined by the elastic component.

6.1.5. Cases 9 and 10: \( \mu_e = \{ 0.2, 92.0 \} \) [MPa]

Fig. 7 shows the optimum response (Opt.) together with the predictions for Cases 9 and 10, which examine the influence of \( \{ \mu_e \} \) on the response. As noted earlier, the parameters \( \{ \mu_e, n_e, \kappa_s \} \) are coupled by the conditions that the model reproduces the points B and C on the experimental curve (see Fig. 2). Therefore, although the value of \( \{ \mu_e \} \) has been specified in Cases 9 and 10, the values of \( \{ n_e, \kappa_s \} \) have been modified accordingly. Also, it was found that the value of \( \{ \mu_e \} \) for a solution with positive values of \( \{ n_e, \kappa_s \} \) has a minimum value near that for Case 9 and has a maximum value near that for Case 10. The results in Fig. 7 show that Case 9 predicts much more hysteresis than predicted by Case 10.

### Table 1

<table>
<thead>
<tr>
<th>( \mu_e ) [kPa]</th>
<th>( b_e )</th>
<th>( n_e )</th>
<th>( \mu_d ) [MPa]</th>
<th>( n_d )</th>
<th>( \kappa_s )</th>
<th>( m )</th>
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</table>
6.2. Uniaxial stress: partial unloading with reloading and cyclic loading

For the remaining examples in this paper, attention is limited to the material response determined by the material constants in Table 1. These examples explore the response of the model to more general loadings for which experimental data is not available.

6.2.1. Partial unloading with reloading

Fig. 8 shows the response predicted for loading from a stress-free state at point A ($\lambda = 1$) to point B ($\lambda = 1.2$); partial unloading to point C ($\lambda = 1.1$); reloading (CDBE) to point E ($\lambda = 1.25$); followed by unloading to point F ($\lambda = 1.2$). It is noted that the non-linearity of the elastic loading curve (CD) is due to both the elastic and inelastic components. Also, notice that the portion (DB) of the loading curve (CDBE) coincides with the loading curve (ADB). In

<table>
<thead>
<tr>
<th>Case</th>
<th>$\mu_c$ [kPa]</th>
<th>$b_c$</th>
<th>$n_e$</th>
<th>$\mu_d$ [MPa]</th>
<th>$n_d$</th>
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Fig. 3. Case 1 (a) and Case 2 (b): influence of the parameter $m$ on the unloading response of the stress $\Pi_d$ in the dissipative component.

Fig. 4. Cases 3 and 4: influence of the parameter $\mu_c$ on the response $\Pi_e$ of the elastic component.

Fig. 5. Cases 5 and 6: influence of the parameter $b_c$ on the response $\Pi_e$ of the elastic component.

Fig. 6. Cases 7 and 8: influence of the parameter $n_e$ on the response $\Pi_e$ of the elastic component.

6.2. Uniaxial stress: partial unloading with reloading and cyclic loading

For the remaining examples in this paper, attention is limited to the material response determined by the material constants in Table 1. These examples explore the response of the model to more general loadings for which experimental data is not available.

6.2.1. Partial unloading with reloading

Fig. 8 shows the response predicted for loading from a stress-free state at point A ($\lambda = 1$) to point B ($\lambda = 1.2$); partial unloading to point C ($\lambda = 1.1$); reloading (CDBE) to point E ($\lambda = 1.25$); followed by unloading to point F ($\lambda = 1.2$). It is noted that the non-linearity of the elastic loading curve (CD) is due to both the elastic and inelastic components. Also, notice that the portion (DB) of the loading curve (CDBE) coincides with the loading curve (ADB). In
the present model, deformation loading with inelastic loading causes $\gamma_s = \kappa_s(\gamma)$ so that reloading of a partially loaded state follows the same primary loading curve [like (ADBE) in Fig. 8] after an initial elastic response.

The response shown in Fig. 8 should be contrasted with the response of standard models of rubber materials that experience the Mullins effect (Ogden and Roxburgh, 1999; Dorfmann and Ogden, 2003). Within the context of a model of the Mullins effect the curve (ADBE) in Fig. 8 is called the primary loading curve. During loading the material damages which tends to weaken the unloading response, causing unloading to follow a curve like (BC) in Fig. 8. However, the unloading curve in a model for the Mullins effect remains elastic until further damage occurs. This means that reloading from the point C in Fig. 8 would retrace the curve (CB) until the point B where damage again evolves and the response follows the primary loading curve. In contrast, it can be seen in Fig. 8 that for the model proposed here for preconditioned materials, loading from the point C follows the elastic curve (CD) until point D when the loading continues on the primary loading curve (ADBE).

6.2.2. Cyclic loading

Fig. 9 shows cyclic loading from point A ($\lambda = 1.0$) to point B ($\lambda = 1.2$), followed by unloading (BCD) to point D ($\lambda = 0.8$) and reloading (CAB) to point B ($\lambda = 1.2$). Notice that the portion of the reloading curve for $\lambda > 1$ is the same as that of the loading curve from A to B.

6.3. Simple shear

Examining the response to large deformation simple shear has become a common test of nonlinear constitutive equations. For simple shear, the deformation gradient $F$ and the dilatation $J$ can be expressed in the forms

$$F = I + \gamma_{12}(e_1 \otimes e_2), \quad J = 1$$

where the engineering shear strain $\gamma_{12}$ has a piecewise constant rate during loading, unloading and reloading. Moreover, the shearing component $\Pi_{12}$ of the first Piola-Kirchhoff stress is given by

$$\Pi_{12} = JF^{-T} \cdot (e_1 \otimes e_2)$$

Fig. 10 shows the response to loading from point A ($\gamma_{12} = 0$) to point B ($\gamma_{12} = 0.1$), followed by unloading (BCD) to point D ($\gamma_{12} = -0.1$) with reloading (DCB) to point B.

7. Conclusions

In the present paper a new simple constitutive model has been developed for the rate-independent response of preconditioned
soft tissues undergoing large deformations. Specifically, the soft tissue has been modeled as an isotropic composite of a nonlinear hyperelastic component and another nonlinear rate-independent dissipative component. Moreover, the dissipative component has been modeled as an elastic–plastic material using a yield function with a hardening variable. Specifically, a distinction has been made between deformation loading/unloading which depends on the rate of total distortional deformation $\gamma$ (34) and inelastic loading/unloading (35). Moreover, different functional forms for the hardening variable $\kappa$ have been proposed for deformation loading $[\kappa_3$ in (36)] and for deformation unloading $[\kappa_{UL}$ in (36)]. In particular, the value of $\kappa_{UL}$ vanishes when $\gamma$ satisfies the condition (37), which causes the magnitude of the axial stress $\Pi_\alpha$ in the dissipative component to vanish, keeping the lower portion (CDEFA) of the unloading curve in Fig. 2 elastic. This response is typically observed for preconditioned soft tissues.

Section 6.1 examines the character of the proposed constitutive model for uniaxial stress. In particular, the material parameters listed in Table 1 have been determined to obtain excellent agreement with the experimental loading/unloading curves for the passive response of a preconditioned Mandrau muscle tissue given in Dorfmann et al. (2008) (see Fig. 1). Furthermore, the influence of each of the material parameters has been studied in Sections 6.1.1–6.1.5 to help simplify the process of determining optimum parameters for matching other experimental data.

Section 6.2 considers examples of uniaxial stress with deformation loading, partial unloading and reloading. In particular, the response to partial unloading and reloading is contrasted with the response of pseudo-elastic equations used to model the Mullins effect. Also, the current model simplifies the notions of loading/unloading relative to those for the pseudo-elastic model of the preconditioned tissue discussed in Dorfmann et al. (2008). Section 6.3 shows that the response of the model for large deformation simple shear is physically reasonable.

In summary, the constitutive equations proposed in this paper are simple. In particular, they depend on only seven material constants (40): three controlling the response of the elastic component (which are determined by the lower portion of the unloading curve) and the remainder controlling the response of the dissipative component (which are determined by the initial portion of the unloading curve). Moreover, the material constants can be determined to match rather general loading and unloading behavior. Also, it is possible to modify the proposed functional forms for hardening during deformation loading $\kappa_3$ and hardening during deformation unloading $\kappa_{UL}$ to better match more complicated material response of preconditioned soft tissues. Furthermore, the proposed strain energy for the dissipative component can be added to the strain energy of anisotropic phenomenological models which include specific fiber orientations (e.g. Holzapfel, 2001; Rubin and Bodner, 2002) without difficulty.

Acknowledgments

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References


