A new generalized Laguerre spectral approximation and its applications

Guo Ben-yu\textsuperscript{a,b,*,1}, Zhang Xiao-yong\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Shanghai Normal University, Guilin Road, 100, Shanghai 200234, China
\textsuperscript{b}Division of Computational Science of E-institute of Shanghai Universities, Shanghai, China
\textsuperscript{c}Department of Mathematics, Shanghai Maritime University, Pudong Road, 1550, Shanghai 200135, China

Received 15 November 2003; received in revised form 23 December 2004

Abstract

A new family of generalized Laguerre polynomials is introduced. Various orthogonal projections are investigated. Some approximation results are established. As an example of their important applications, the mixed spherical harmonic-generalized Laguerre approximation is developed. A mixed spectral scheme is proposed for a three-dimensional model problem. Its convergence is proved. Numerical results demonstrate the high accuracy of this new spectral method.

© 2005 Elsevier B.V. All rights reserved.

MSC: 65N35; 41A10

Keywords: New generalized Laguerre approximation; Mixed spherical harmonic-generalized Laguerre spectral method

1. Introduction

Many problems in science, engineering and finance are set on unbounded domains. In the past decade, considerable progress has been made in numerical simulations of such problems; for instance, the spectral
methods based on some specific orthogonal systems on unbounded domains. In particular, the Hermit and Laguerre spectral methods were used widely, see, e.g. [3,4,6–9,11–14]. In the standard Laguerre spectral method for half-line, we used the weight function $e^{-\rho}$. But this weight is not the most appropriate oftentimes. Thus, it is necessary to consider certain orthogonal systems with other weight functions. For example, the Black–Sholes equation plays an important role in financial mathematics, which is of the form

$$
\partial_t v(\rho, t) + (r - q)\rho \partial_\rho v(\rho, t) + \frac{1}{2} \rho^2 \partial^2_\rho v(\rho, t) - rv(\rho, t) = F(\rho, t),
$$

(1.1)

where $\partial^k_z v(\rho, t)$ denotes the derivative of order $k$ of $v(\rho, t)$ with respect to $z = \rho$ or $t$. Due to the appearance of the coefficients $\rho$ and $\rho^2$ in the terms involving $\partial_\rho v(\rho, t)$ and $\partial^2_\rho v(\rho, t)$, respectively, it is natural to use the generalized Laguerre approximation with the weight function like $\rho^2 e^{-\rho}$. Another interesting problem is how to solve spherically symmetric problems arising in quantum mechanics, which are reduced often to the following equation:

$$
\partial_t^2 v(\rho, t) = \frac{1}{\rho^2} \partial_\rho(\rho^2 \partial_\rho v(\rho, t)) + f(v(\rho, t), \rho, t).
$$

(1.2)

In this case, we also have to use the generalized Laguerre approximation. Recently, the generalized Laguerre approximation was also applied to two-dimensional exterior problems successfully, see [10]. Furthermore, the asymptotic behaviors at infinity of exact solutions usually depend on some parameters in the underlying differential equations. Thus, it seems reasonable to adopt the orthogonal systems with the more general weight function $\omega_{x, \beta}(\rho) = \rho^x e^{-\beta \rho}$, $x > -1$, $\beta > 0$. Indeed, by better choices of $\beta$, the numerical solutions could fit the asymptotic behaviors of exact solutions more properly.

In this paper, we introduce a new family of generalized Laguerre polynomials, which is mutually orthogonal on the half-line, with respect to the weight function $\omega_{x, \beta}(\rho)$. We investigate various orthogonal projections and derive some approximation results which serve as the mathematical foundation of related spectral methods for unbounded domains. As an example of their important applications, we propose the mixed spherical harmonic-generalized Laguerre approximation in three-dimensional space and establish a basic approximation result in nonuniformly weighted Sobolev space, which is frequently used for numerical solutions of a large class of differential equations. Then we consider a model problem and provide a mixed spherical harmonic-generalized Laguerre spectral scheme for its alternative form. Owing to the orthogonality of spherical harmonic functions, we can evaluate the corresponding coefficients of expansion of the numerical solution separately. In other words, we only need to solve a set of ordinary differential equations like (1.2). On the other hand, by using some results on this new mixed spherical harmonic-generalized Laguerre approximation, we obtain the optimal error estimate for the proposed spectral scheme. Moreover, by choosing suitable basis functions, we provide an efficient numerical algorithm in which the corresponding discrete linear system for unknown coefficients of the numerical solution is symmetric and sparse, and so can be resolved very efficiently. The numerical results demonstrate the high accuracy of this method. It is also shown that a suitable parameter $\beta$ could increase the accuracy of the numerical solution.

The paper is organized as follows. In Section 2, we introduce the new family of generalized Laguerre polynomials, and investigate several orthogonal projections. In Section 3, we study the new mixed spherical harmonic-generalized Laguerre approximation and derive some useful results. In Section 4, we
propose the mixed spectral scheme for a model problem, and prove its convergence. In Section 5, we present some numerical results. The final section is for some concluding remarks.

2. Generalized Laguerre approximation

In this section, we study the new generalized Laguerre approximation. We first introduce some notations. Let \( A = \{ \rho \mid 0 < \rho < \infty \} \) and \( \chi(\rho) \) be a certain weight function in the usual sense. We define

\[
L^2_{\chi}(A) = \{ v \mid v \text{ is measurable on } A \text{ and } \|v\|_{\chi,A} < \infty \}
\]

with the following inner product and norm:

\[
(u, v)_{\chi,A} = \int_A u(\rho)v(\rho)\chi(\rho) \, d\rho, \quad \|v\|_{\chi,A} = (v, v)_{\chi,A}^{1/2}.
\]

For any integer \( m \geq 0 \), we define the space

\[
H^m_\chi(A) = \{ v \mid \partial^k_\rho v \in L^2_{\chi}(A), \ 0 \leq k \leq m \},
\]
equipped with the following inner product, semi-norm and norm:

\[
(u, v)_{m,\chi,A} = \sum_{0 \leq k \leq m} \left( \partial^k_\rho u, \partial^k_\rho v \right)_{\chi,A},
\]
\[
|v|_{m,\chi,A} = \|\partial^m_\rho v\|_{\chi,A}, \quad \|v\|_{m,\chi,A} = (v, v)_{m,\chi,A}^{1/2}.
\]

For any \( r > 0 \), the space \( H^r_\chi(A) \) and its norm \( \|v\|_{r,\chi,A} \) are defined by space interpolation as in \([1]\). In particular, \( 0H^1_\chi(A) = \{ v \in H^1_\chi(A) \mid v(0) = 0 \} \).

Let \( \omega_{x,\beta}(\rho) = \rho^x e^{-\beta \rho}, \ x > -1, \text{ and } \beta > 0 \). The corresponding generalized Laguerre polynomials of degree \( l \) are defined by

\[
\ell^{(x,\beta)}_l(\rho) = \frac{1}{l!} \rho^{-x} e^{\beta \rho} \partial_\rho^l (\rho^{l+x} e^{-\beta \rho}), \quad l = 0, 1, 2, \ldots .
\]

They are the eigenfunctions of the Sturm–Liouville problem

\[
\partial_\rho (\omega_{x+1,\beta}(\rho) \partial_\rho v(\rho)) + \lambda_l^{(\beta)} \omega_{x,\beta}(\rho) v(\rho) = 0, \quad 0 < \rho < \infty \tag{2.1}
\]

with the corresponding eigenvalues \( \lambda_l^{(\beta)} = \beta l \). They fulfill the following recurrence relations:

\[
\ell^{(x,\beta)}_l(\rho) = \frac{1}{\beta} \left( \partial_\rho \ell^{(x,\beta)}_l(\rho) - \partial_\rho L^{(x,\beta)}_l(\rho) \right), \tag{2.2}
\]

\[
(l + 1) \ell^{(x,\beta)}_{l+1}(\rho) + (\beta \rho - 2l - x - 1) \ell^{(x,\beta)}_l(\rho) + (l + x) \ell^{(x,\beta)}_{l-1}(\rho) = 0, \tag{2.3}
\]

\[
\partial_\rho \ell^{(x,\beta)}_l(\rho) = -\beta \ell^{(x+1,\beta)}_{l-1}(\rho), \tag{2.4}
\]

\[
-\rho \partial_\rho \ell^{(x,\beta)}_l(\rho) = (l + x) \ell^{(x,\beta)}_{l-1}(\rho) - l \ell^{(x,\beta)}_l(\rho). \tag{2.5}
\]
The set of \( \mathcal{L}^{(x, \beta)}_l (\rho) \) is the complete \( L^2_{\omega_{x, \beta}}(A) \)-orthogonal system, namely,

\[
(\mathcal{L}^{(x, \beta)}_l, \mathcal{L}^{(x, \beta)}_m)_{\omega_{x, \beta}, A} = \begin{cases} \gamma_l^{(x, \beta)}, & l = m, \\ 0, & l \neq m, \end{cases}
\]

where

\[
\gamma_l^{(x, \beta)} = \frac{\Gamma(l + x + 1)}{\beta^{2l+1} l!}.
\]

Thus, for any \( v \in L^2_{\omega_{x, \beta}}(A) \),

\[
v(\rho) = \sum_{l=0}^{\infty} \gamma_l^{(x, \beta)} \mathcal{L}^{(x, \beta)}_l (\rho)
\]

with the coefficients

\[
\gamma_l^{(x, \beta)} = \frac{1}{\gamma_l^{(x, \beta)}} (v, \mathcal{L}^{(x, \beta)}_l)_{\omega_{x, \beta}, A}, \quad l \geq 0.
\]

Now, let \( N \) be any positive integer and \( \mathcal{P}_N(A) \) be the set of all algebraic polynomials of degree at most \( N \). Furthermore, \( \mathcal{P}_N(A) = \{ v \in \mathcal{P}_N(A) \mid v(0) = 0 \} \). The orthogonal projection \( P_{N, x, \beta} : L^2_{\omega_{x, \beta}}(A) \to \mathcal{P}_N(A) \) is defined by

\[
(P_{N, x, \beta} v - v, \phi)_{\omega_{x, \beta}, A} = 0, \quad \forall \phi \in \mathcal{P}_N(A).
\]

In order to describe the approximation results, we introduce the weighted space \( A^r_{x, \beta}(A) \). For any integer \( r \geq 0 \),

\[
A^r_{x, \beta}(A) = \{ v \mid v \text{ is measurable on } A \text{ and } \| v \|_{A^r_{x, \beta}, A} < \infty \},
\]

equipped with the following semi-norm and norm:

\[
\| v \|_{A^r_{x, \beta}, A} = \| \tilde{c}^r_{\rho} v \|_{\omega_{x+r, \bar{\beta}}, A}, \quad \| v \|_{A^r_{x, \beta}, A} = \left( \sum_{k=0}^{r} |v|_{A^k_{x, \beta}, A}^2 \right)^{1/2}.
\]

For any \( r > 0 \), we define the space \( A^r_{x, \beta}(A) \) and its norm by space interpolation as in [1].

In the sequel, we denote by \( c \) a generic positive constant which does not depend on \( x, \beta, N \) and any function.

**Theorem 2.1.** For any integers \( 0 \leq s \leq r \),

\[
\| \tilde{c}^s_{\rho} (P_{N, x, \beta} v - v) \|_{\omega_{x+s, \bar{\beta}}, A} \leq c(\beta N)^{(s-r)/2} \| \tilde{c}^r_{\rho} v \|_{\omega_{x+r, \bar{\beta}}, A}.
\]

Moreover, for any \( v \in A^r_{x, \beta}(A) \), real number \( s \) and \( 0 \leq s \leq r \),

\[
\| P_{N, x, \beta} v - v \|_{A^s_{x, \beta}, A} \leq c(\beta N)^{(s-r)/2} |v|_{A^r_{x, \beta}, A}.
\]
Therefore, a combination of (2.9)–(2.11) yields that
\[ P_{N,\alpha,\beta} v(\rho) - v(\rho) = - \sum_{l=N+1}^{\infty} \hat{v}_l^{(x,\beta)} \ell_l^{(x,\beta)}(\rho). \]

By virtue of (2.4), we have that
\[ \hat{e}_x^s(P_{N,\alpha,\beta} v - v) = - \sum_{l=N+1}^{\infty} (-\beta)^s \hat{v}_l^{(x,\beta)} \ell_l^{(x,s,\beta)}(\rho), \]
whence
\[ \| \hat{e}_x^s(P_{N,\alpha,\beta} v - v) \|_{C_{x+r,\beta}}^2 = \sum_{l=N+1}^{\infty} \beta^{2s} \ell_l^{(x,s,\beta)}(\hat{v}_l^{(x,\beta)})^2. \] (2.9)

On the other hand,
\[ \| \hat{e}_x^r v \|_{C_{x+r,\beta}}^2 = \sum_{l=r}^{\infty} \beta^{2r} \ell_l^{(x+r,\beta)}(\hat{v}_l^{(x,\beta)})^2. \] (2.10)

By using (2.6) again, we verify that for \( l \geq r - 1, s \),
\[ \frac{\ell_l^{(x+s,\beta)}}{\ell_l^{(x+\beta,\beta)}} = \beta^{r-s} \frac{(l-r)!}{(l-s)!} = \beta^{r-s} \frac{1}{(l-s)(l-s-1) \cdots (l-r+1)} \leq c \beta^{r-s} N^{s-r}. \] (2.11)

Therefore, a combination of (2.9)–(2.11) yields that
\[ \| \hat{e}_x^s(P_{N,\alpha,\beta} v - v) \|_{C_{x+r,\beta}}^2 \leq c(\beta N)^{s-r} \| \hat{e}_x^r v \|_{C_{x+r,\beta}}^2, \]
which implies result (2.7). Consequently, we obtain (2.8) for integers \( 0 \leq s \leq r \). Finally, we complete the proof by space interpolation. □

In numerical analysis of the generalized Laguerre spectral method for unbounded domains, we need to consider other orthogonal approximations related to underlying differential equations. For this purpose, we take \( \alpha, \gamma > -1, \beta, \delta > 0 \), and define the space
\[ H^1_{\alpha,\beta,\alpha \gamma,\delta}(A) = \{ v \mid v \text{ is measurable on } A \text{ and } \| v \|_{1,\alpha,\beta,\alpha \gamma,\delta} < \infty \}, \]
equipped with the norm
\[ \| v \|_{1,\alpha,\beta,\alpha \gamma,\delta} = (\| \hat{e}_x v \|_{1,\alpha,\beta}^2 + \| v \|_{\alpha \gamma,\delta}^2)^{1/2}. \] (2.12)

In particular,
\[ \mathcal{H}^1_{\alpha,\beta,\alpha \gamma,\delta}(A) = \{ v \in H^1_{\alpha,\beta,\alpha \gamma,\delta}(A) \mid v(0) = 0 \}. \]
The orthogonal projection \( P^1_{N,\alpha,\beta,\alpha \gamma,\delta} : H^1_{\alpha,\beta,\alpha \gamma,\delta}(A) \to \mathcal{P}_N(A) \) is defined by
\[ (\hat{e}_x(P^1_{N,\alpha,\beta,\alpha \gamma,\delta} v - v), \hat{e}_x \phi)_{\alpha,\beta,\delta} + (P^1_{N,\alpha,\beta,\alpha \gamma,\delta} v - v, \phi)_{\alpha \gamma,\delta} = 0, \quad \forall \phi \in \mathcal{P}_N(A). \] (2.13)
We also define the orthogonal projection $0P_{N,\beta}(A) : 0H^1_{\alpha,\beta}(A) \rightarrow 0H^1_N(A)$ by
\[
(\partial \rho (0P_{N,\beta}v - v), \partial \rho \phi)_{\alpha,\beta,A} = 0, \quad \forall \phi \in 0H^1_N(A). \tag{2.14}
\]

In order to derive the approximation results, we need several embedding inequalities. For simplicity of statements, let $\rho_0 > 0$ for $\gamma \leq 1$, and $\rho_0 > (2/\delta)\sqrt{\gamma(\gamma - 1)}$ for $\gamma > 1$. We shall also use the following notations:
\[
c_{x,\beta,\gamma,\delta} = \begin{cases} 
\frac{\gamma - x}{e(\delta - \beta)}, & \text{for } x \leq \gamma, \quad \beta < \delta, \\
\rho_0^{-\gamma} e^{(\beta - \delta)\rho_0}, & \text{for } \gamma < x, \quad \beta \leq \delta,
\end{cases}
\]
\[
d_{x,\beta,\gamma,\delta} = \begin{cases} 
1, & \text{for } -1 < \gamma \leq 0, \\
1 + \frac{2\gamma}{\delta \rho_0}, & \text{for } 0 < \gamma < 1, \\
\frac{2(\delta^2 \rho_0^2 - 4\gamma(\gamma - 1))}{2(\delta^2 \rho_0^2 - 4\gamma(\gamma - 1)) + (\gamma + 1)^2}, & \text{for } \gamma > 1
\end{cases}
\]
and
\[
q_{x,\beta,\gamma,\delta} = 4 \max \left( \frac{1}{\delta^2 c_{x,\beta,\gamma,\delta} d_{x,\beta,\gamma,\delta}}, \frac{e^{\delta \rho_0} \rho_0^{\gamma + 2 - \gamma}}{(\gamma + 1)^2} \right).
\]

**Lemma 2.1.** Let $-1 < x \leq \gamma + 2, \beta = \delta$ or $\gamma > -1, -1 < x \leq \gamma + 2, \beta < \delta$. Also assume that $v \in H^1_{\alpha,\beta,\gamma,\delta}(A)$ and there exists $\rho_0$ such that $v(\rho_0) = 0$ where $\rho_0 > 0$ for $\gamma \leq 1$, and $\rho_0 > 2\sqrt{\gamma(\gamma - 1)}/\delta$ for $\gamma > 1$. Then
\[
\|v\|_{\omega_{x,\delta},A}^2 \leq q_{x,\beta,\gamma,\delta} \|\partial \rho v\|_{\omega_{x,\delta},A}^2.
\]

**Proof.** Let $A_1 = (\rho_0, \infty), A_2 = (0, \rho_0)$ and
\[
\|v\|_{\omega_{x,\delta},A_j} = \left( \int_{A_j} \omega_{x,\delta}(\rho) v^2(\rho) \, d\rho \right)^{1/2}, \quad j = 1, 2.
\]
For any $\rho \in A_1$, we have that
\[
\omega_{x,\delta}(\rho) v^2(\rho) = \int_{\rho_0}^{\rho} \omega_{x,\delta}(\xi) v^2(\xi) \, d\xi = 2 \int_{\rho_0}^{\rho} \omega_{x,\delta}(\xi) v(\xi) \, d\xi + \gamma \int_{\rho_0}^{\rho} \omega_{x,\delta}(\xi) v^2(\xi) \, d\xi - \delta \int_{\rho_0}^{\rho} \omega_{x,\delta}(\xi) v^2(\xi) \, d\xi, \tag{2.15}
\]
Due to $v \in H^1_{\alpha,\beta,\gamma,\delta}(A)$, we have $\omega_{x,\delta}(\rho) v^2(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. Letting $\rho \rightarrow \infty$ in (2.15) and using the Cauchy–Schwarz inequality, we obtain that for any $\delta > 0$,
\[
\delta \|v\|_{\omega_{x,\delta},A_1}^2 \leq \frac{\delta}{2} \|v\|_{\omega_{x,\delta},A_1}^2 + \frac{2}{\delta} \|\partial \rho v\|_{\omega_{x,\delta},A_1}^2 + \gamma \|v\|_{\omega_{x,\delta},A_1}^2.
\]
Thus
\[
\|v\|_{\Omega_{r,\delta}, A_1}^2 \leq \frac{4}{\delta^2} \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2 + \frac{2\gamma}{\delta} \|v\|_{\Omega_{r-1,\delta}, A_1}^2.
\] (2.16)

If $\gamma \leq 0$, then (2.16) implies
\[
\|v\|_{\Omega_{r,\delta}, A_1}^2 \leq \frac{4}{\delta^2} \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2.
\] (2.17)

Moreover, a direct calculation shows that for $b \geq 0$,
\[
\sup_{\rho_0 \leq \rho < \infty} e^{-a \rho} \rho^b = \begin{cases} \left( \frac{b}{ae} \right)^b & \text{if } b \geq 0, \\ \rho_0^b e^{-a \rho_0} & \text{if } b < 0. \end{cases}
\]

Hence, for $\gamma \leq \alpha$ and $\beta \leq \delta$,
\[
\|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2 \leq \rho_0^{\gamma-2} e^{(\beta-\delta)\rho_0} \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2.
\] (2.18)

while for $\alpha \leq \gamma$ and $\beta < \delta$,
\[
\|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2 \leq \left( \frac{\gamma-\alpha}{e(\delta-\beta)} \right)^{\gamma-2} \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2.
\] (2.19)

Therefore, for $\gamma \leq 0$, (2.17) implies that
\[
\|v\|_{\Omega_{r,\delta}, A_1}^2 \leq \frac{4c_{\alpha,\beta,\gamma,\delta}}{\delta^2} \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2.
\] (2.20)

Next, an integration by parts yields
\[
\frac{2\gamma}{\delta} \|v\|_{\Omega_{r,\delta}, A_1}^2 = \frac{4\gamma}{\delta^2} \int_{A_1} \rho^{\gamma-1} e^{-\delta \rho} v(\rho) \partial_\rho v(\rho) \, d\rho + \frac{2\gamma(\gamma-1)}{\delta^2} \|v\|_{\Omega_{r-1,\delta}, A_1}^2.
\] (2.21)

Moreover, by the Cauchy–Schwarz inequality,
\[
\frac{4\gamma}{\delta^2} \int_{A_1} \rho^{\gamma-1} e^{-\delta \rho} v(\rho) \partial_\rho v(\rho) \, d\rho \leq \frac{4\gamma}{\delta^2} \|\partial_\rho v\|_{\Omega_{r-1,\delta}, A_1}^2 + \frac{\gamma}{\delta} \|v\|_{\Omega_{r-1,\delta}, A_1}^2.
\]

Inserting the above into (2.21), we deduce that for $0 < \gamma \leq 1$,
\[
\frac{2\gamma}{\delta} \|v\|_{\Omega_{r-1,\delta}, A_1}^2 \leq \frac{8\gamma}{\delta^3} \|\partial_\rho v\|_{\Omega_{r-1,\delta}, A_1}^2.
\]

Substituting (2.18), (2.19) and the above inequality into (2.16), we obtain that $0 < \gamma \leq 1$,
\[
\|v\|_{\Omega_{r,\delta}, A_1}^2 \leq \frac{4}{\delta^2} \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2 + \frac{8\gamma}{\delta^3} \|\partial_\rho v\|_{\Omega_{r-1,\delta}, A_1}^2 \leq \frac{4}{\delta^2} \left( 1 + \frac{2\gamma}{\delta \rho_0} \right) \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2
\]
\[
\leq \frac{4c_{\alpha,\beta,\gamma,\delta}}{\delta^2} \left( 1 + \frac{2\gamma}{\delta \rho_0} \right) \|\partial_\rho v\|_{\Omega_{r,\delta}, A_1}^2.
\] (2.22)
We now consider the case $\gamma > 1$. We have
\[
\frac{4\gamma}{\delta^2} \int_{A_1} \rho^{\gamma-1} e^{-\delta \rho v(\rho)} \partial_{\rho} v(\rho) \, d\rho \leq \frac{2\gamma}{\delta^2 (\gamma - 1)} ||\partial_{\rho} v||_{C^{0,\beta}, A_1}^2 + \frac{2\gamma (\gamma - 1)}{\delta^2} ||v||_{C^{0,\beta}, A_1}^2.
\]
This inequality together with (2.21) leads to that
\[
\frac{2\gamma}{\delta^2} ||v||_{C^{0,\beta}, A_1}^2 \leq \frac{2\gamma}{\delta^2 (\gamma - 1)} ||\partial_{\rho} v||_{C^{0,\beta}, A_1}^2 + \frac{4\gamma (\gamma - 1)}{\delta^2} ||v||_{C^{0,\beta}, A_1}^2.
\]
Thus, we find from the above and (2.16) that for $\gamma > 1$,
\[
||v||_{C^{0,\beta}, A_1}^2 \leq \frac{6\gamma - 4}{\delta^2 (\gamma - 1)} ||\partial_{\rho} v||_{C^{0,\beta}, A_1}^2 + \frac{4\gamma (\gamma - 1)}{\delta^2} ||v||_{C^{0,\beta}, A_1}^2
\leq \frac{6\gamma - 4}{\delta^2 (\gamma - 1)} ||\partial_{\rho} v||_{C^{0,\beta}, A_1}^2 + \frac{4\gamma (\gamma - 1)}{\delta^2} ||v||_{C^{0,\beta}, A_1}^2.
\]
If $\rho_0 > 2\sqrt{\gamma (\gamma - 1)}/\delta$, then the above result with (2.18) and (2.19) leads to that for $\gamma > 1$,
\[
||v||_{C^{0,\beta}, A_1}^2 \leq \frac{2\rho_0^2 (3\gamma - 2)}{(\delta^2 \rho_0^2 - 4\gamma (\gamma - 1)) (\gamma - 1)} ||\partial_{\rho} v||_{C^{0,\beta}, A_1}^2
\leq \frac{2\rho_0^2 (3\gamma - 2) c_{x, \beta, \gamma, \delta}}{(\delta^2 \rho_0^2 - 4\gamma (\gamma - 1)) (\gamma - 1)} ||\partial_{\rho} v||_{C^{0,\beta, A_1}}^2. \tag{2.23}
\]
On the other hand, for any $\rho \in A_2$,
\[
\rho^{\gamma+2} v^2(\rho) = - \int_{\rho}^{\rho_0} \partial_\xi (\xi^{\gamma+1} v^2(\xi)) \, d\xi
= - 2 \int_{\rho}^{\rho_0} \xi^{\gamma+1} v(\xi) \partial_\xi v(\xi) \, d\xi - (\gamma + 1) \int_{\rho}^{\rho_0} \xi^\gamma v^2(\xi) \, d\xi.
\]
Letting $\rho \to 0$ and using the Cauchy–Schwarz inequality, we find that for $\gamma > -1$,
\[
(\gamma + 1) \int_{\rho}^{\rho_0} \rho^{\gamma+2} (\partial_{\rho} v(\rho))^2 \, d\rho \leq \frac{2}{\gamma + 1} \int_{\rho}^{\rho_0} \rho^{\gamma+2} (\partial_{\rho} v(\rho))^2 \, d\rho + \frac{\gamma + 1}{2} \int_{\rho}^{\rho_0} \rho^{\gamma+2} v^2(\rho) \, d\rho.
\]
Therefore
\[
\int_{\rho}^{\rho_0} \rho^{\gamma+2} v^2(\rho) \, d\rho \leq \frac{4}{(\gamma + 1)^2} \int_{\rho}^{\rho_0} \rho^{\gamma+2} (\partial_{\rho} v(\rho))^2 \, d\rho \leq \frac{4\rho_0^{\gamma+2}}{(\gamma + 1)^2} ||\partial_{\rho} v||_{C^{0,\beta, A_1}}^2.
\]
Accordingly, for $\gamma > -1$,
\[
||v||_{C^{0,\beta, A_1}}^2 \leq \frac{4\rho_0^{\gamma-2+2} c_{x, \beta, \gamma, \delta}}{(\gamma + 1)^2} ||\partial_{\rho} v||_{C^{0,\beta, A_1}}^2. \tag{2.24}
\]
Then the desired result comes from a combination of (2.20) and (2.22)–(2.24).
Lemma 2.2. We have that

(i) for any \( v \in H^1_{\omega_{2,\nu}}(A) \) and \( \alpha < 1 \),
\[
\|v\|^2_{\omega_{2,\nu},A} \leq c_{\alpha,\beta} \|\partial_{\rho} v\|^2_{\omega_{2,\nu},A},
\]
where \( c_{\alpha,\beta} = 4/\beta^2 \) for \( \alpha \leq 0 \), and \( c_{\alpha,\beta} = (4 - 2\alpha)/\beta^2(1 - \alpha) \) for \( 0 < \alpha < 1 \);

(ii) for any \( v \in H^1_{\omega_{1,\nu}}(A) \cap L^2_{\omega_{1,\nu,1}}(A) \),
\[
\|v\|^2_{\omega_{1,\nu},A} \leq \frac{2}{\beta^2} (\sqrt{2} + 1)(\|\partial_{\rho} v\|^2_{\omega_{1,\nu},A} + \|v\|^2_{\omega_{1,\nu,1},A});
\]

(iii) for any \( v \in H^1_{\omega_{1,\nu}}(A) \cap L^2_{\omega_{1,\nu,1}}(A) \) and \( \alpha > 1 \),
\[
\|v\|^2_{\omega_{1,\nu},A} \leq \frac{2(3\alpha - 2)}{\beta^2(\alpha - 1)} \|\partial_{\rho} v\|^2_{\omega_{1,\nu},A} + \frac{4\alpha(\alpha - 1)}{\beta^2} \|v\|^2_{\omega_{1,\nu,1},A}.
\]

Proof. Following the same line as in the derivation of (2.16), we deduce that for \( v(0) = 0 \) or \( \alpha > 0 \),
\[
\|v\|^2_{\omega_{1,\nu},A} \leq \frac{4}{\beta^2} \|\partial_{\rho} v\|^2_{\omega_{1,\nu},A} + \frac{2\alpha}{\beta} \|v\|^2_{\omega_{1,\nu,1},A}. \tag{2.25}
\]
Clearly, result (i) with \( \alpha \leq 0 \) follows from (2.25) immediately.

We next prove result (i) with \( 0 < \alpha < 1 \). Like (2.21), an integration by parts yields that
\[
2\alpha \|v\|^2_{\omega_{1,\nu},A} = 4\alpha \int_A \rho^{\alpha - 1} e^{-\beta \rho} v(\rho) \partial_{\rho} v(\rho) \, d\rho + 2\alpha(\alpha - 1) \|v\|^2_{\omega_{1,\nu,1},A}. \tag{2.26}
\]
By the Cauchy–Schwartz inequality,
\[
4\alpha \int_A \rho^{\alpha - 1} e^{-\beta \rho} v(\rho) \partial_{\rho} v(\rho) \, d\rho \leq \frac{2\alpha}{1 - \alpha} \|\partial_{\rho} v\|^2_{\omega_{1,\nu},A} + 2\alpha(1 - \alpha) \|v\|^2_{\omega_{1,\nu,1},A}.
\]
Thus (2.26) implies
\[
2\alpha \|v\|^2_{\omega_{1,\nu,1},A} \leq \frac{2\alpha}{\beta(1 - \alpha)} \|\partial_{\rho} v\|^2_{\omega_{1,\nu},A}.
\]
Substituting the above into (2.25), we obtain
\[
\|v\|^2_{\omega_{1,\nu},A} \leq \frac{4 - 2\alpha}{\beta^2(1 - \alpha)} \|\partial_{\rho} v\|^2_{\omega_{1,\nu},A}.
\]

We now prove result (ii). An integration by parts leads to
\[
2\|v\|^2_{\omega_{1,\nu},A} = 4\beta \int_A e^{-\beta \rho} v(\rho) \partial_{\rho} v(\rho) \, d\rho \leq \frac{2}{\beta} (\sqrt{2} - 1) \|\partial_{\rho} v\|^2_{\omega_{1,\nu},A} + \frac{2}{\beta} (\sqrt{2} + 1) \|v\|^2_{\omega_{1,\nu,1},A}.
\]
The above with (2.25) implies result (ii).
Finally we prove result (iii). In this case, (2.26) is also valid. Moreover, by the Cauchy–Schwartz inequality,

$$4\alpha \int_a^b \rho^{a-1} e^{-\beta \rho} v(\rho) \partial_\rho v(\rho) \, d\rho \leq \frac{2\alpha}{\alpha - 1} \| \partial_\rho v \|_{\mathcal{O}_{\alpha, \beta}^2}^2 + 2\alpha (\alpha - 1) \| v \|_{\mathcal{O}_{\alpha-2, \beta}^1}^2.$$  

Therefore, we have from (2.26) that

$$2\alpha \beta \| v \|_{\mathcal{O}_{\alpha-1, \beta}^1}^2 \leq \frac{2\alpha}{\alpha - 1} \| \partial_\rho v \|_{\mathcal{O}_{\alpha, \beta}^2}^2 + 4\alpha (\alpha - 1) \| v \|_{\mathcal{O}_{\alpha-2, \beta}^1}^2.$$  

Substituting the above into (2.25), we obtain result (iii).  

The above two lemmas are the generalizations of the corresponding results in [10]. We now are in a position to derive some approximation results.

**Theorem 2.2.** Let $-1 < \gamma < \alpha < \alpha + 2$, $\beta = \delta$ or $\gamma > -1$, $-1 < \alpha < \gamma + 2$, $\beta < \delta$. If $v \in H^{1}_{\mathcal{O}_{\alpha, \beta}^1}(A)$, $\partial_\rho v \in A^{r-1}_{\alpha, \beta}(A)$ and integer $r \geq 1$, then

$$\| P_{N, x, \beta}^1 v - v \|_{1, \mathcal{O}_{\alpha, \beta}^1} \leq c(q_{\alpha, \beta, \gamma} + 1)^{1/2} (\beta N)^{(1-r)/2} \| \partial_\rho v \|_{A^{r-1}_{\alpha, \beta}}.$$  

**Proof.** By definition (2.13) and projection theorem,

$$\| P_{N, x, \beta}^1 v - v \|_{1, \mathcal{O}_{\alpha, \beta}^1} \leq \| \phi - v \|_{1, \mathcal{O}_{\alpha, \beta}^1}, \quad \forall \phi \in \mathcal{P}_N(A).$$  

We take

$$\phi(\rho) = \int_0^\rho P_{N-1, x, \beta} \partial_\rho v(\rho) \, d\rho + \lambda,$$

where $\lambda$ is chosen in such a way that $\phi(\rho_0) = v(\rho_0)$, and $\rho_0$ is the same as in Lemma 2.1. According to Lemma 2.1 and Theorem 2.1 with $s = 0$, we deduce that for any integer $r \geq 1$,

$$\| \phi - v \|_{1, \mathcal{O}_{\alpha, \beta}^1} \leq (q_{\alpha, \beta, \gamma} + 1)^{1/2} \| \partial_\rho (\phi - v) \|_{\mathcal{O}_{\alpha, \beta}^1} \leq (q_{\alpha, \beta, \gamma} + 1)^{1/2} \| P_{N-1, x, \beta} \partial_\rho v - \partial_\rho v \|_{\mathcal{O}_{\alpha, \beta}^1} \leq c(q_{\alpha, \beta, \gamma} + 1)^{1/2} (\beta N)^{(1-r)/2} \| \partial_\rho v \|_{A^{r-1}_{\alpha, \beta}}.$$  

This completes the proof.  

**Theorem 2.3.** If $v \in L^2_{\mathcal{O}_{\alpha, \beta}^2}(A)$, $\partial_\rho v \in A^{r-1}_{\alpha, \beta}(A)$ and $v(0) = 0$, then for integer $r \geq 1$,

$$\| \partial_\rho (P_{N, x, \beta}^1 v - v) \|_{\mathcal{O}_{\alpha, \beta}^2} \leq c(\beta N)^{(1-r)/2} \| \partial_\rho v \|_{A^{r-1}_{\alpha, \beta}}.$$  

If, in addition, $|x| < 1$, then

$$\| P_{N, x, \beta}^1 v - v \|_{1, \mathcal{O}_{\alpha, \beta}^1} \leq c\| \partial_\rho v \|_{A^{r-1}_{\alpha, \beta}}.$$  

If, in addition, $|r| < 1$, then

$$\| P_{N, x, \beta}^1 v - v \|_{1, \mathcal{O}_{\alpha, \beta}^1} \leq c\| \partial_\rho v \|_{A^{r-1}_{\alpha, \beta}}.$$
**Proof.** By definition (2.14), for any \( \phi \in 0\mathcal{P}_N(A) \),
\[
\| \hat{\partial}_\rho (q^1_P_{N,z,\beta} v - v)\|_{\omega_{z,\beta},A} \leq \| \hat{\partial}_\rho (\phi - v)\|_{\omega_{z,\beta},A}.
\]
Take
\[
\phi(\rho) = \int_0^\rho P_{N-1,z,\beta} \partial_{\xi} v(\xi) \, d\xi \in 0\mathcal{P}_N(A).
\]
Then by Theorem 2.1 with \( s = 0 \),
\[
\| \hat{\partial}_\rho (P_{N,z,\beta}^1 v - v)\|_{\omega_{z,\beta},A} \leq \| P_{N-1,z} \partial_{\rho} v - \partial_{\rho} v\|_{\omega_{z,\beta},A} \leq c(\beta N)^{(1-r)/2} \| \hat{\partial}_\rho v\|_{A_{z,\beta}^{-1},A}.
\]
Furthermore, we use the above result and result (i) of Lemma 2.2 to derive that for \( |x| < 1 \),
\[
\|0 P_{N,z,\beta}^1 v - v\|_{\omega_{z,\beta},A} \leq c_{z,\beta}^{1/2} \| \hat{\partial}_\rho (q^1_P_{N,z,\beta} v - v)\|_{\omega_{z,\beta},A} \leq c c_{z,\beta}^{1/2} (\beta N)^{(1-r)/2} \| \hat{\partial}_\rho v\|_{A_{z,\beta}^{-1},A}.
\]
This completes the proof. \( \square \)

3. Mixed spherical harmonic-generalized Laguerre approximation

As an application of the results in the previous section, we now investigate the mixed spherical harmonic-generalized Laguerre approximation in three-dimensional space. We need several preparations.

We first consider an auxiliary projection related to the generalized Laguerre approximation with \( \alpha = 2 \) and \( \gamma = 0 \). For the sake of simplicity, let \( \omega_\beta(\rho) = \omega_{0,\beta}(\rho) = e^{-\beta \rho} \), and \( \eta_\beta(\rho) = \omega_{2,\beta}(\rho) = \rho^2 e^{-\beta \rho} \). We also denote \( A^r_{0,\beta}(A) \), \( \| v \|_{A^r_{0,\beta},A} \) and \( |v|_{A^r_{0,\beta},A} \) by \( A^r_{\beta}(A) \), \( \| v \|_{A^r_{\beta},A} \) and \( |v|_{A^r_{\beta},A} \), respectively.

The orthogonal projection \( P^1_{N,\beta} : H^1_{\omega_{0,\beta},o_{\beta}}(A) \cap L^2_{\omega_{0,\beta}}(A) \to \mathcal{P}_N(A) \) is defined by
\[
(\hat{\partial}_\rho (P^1_{N,\beta} v - v), \partial_{\rho} \phi)_{\eta_{\beta},A} + (P^1_{N,\beta} v - v, \phi)_{\eta_{\beta},A} + (P^1_{N,\beta} v - v, \phi)_{\omega_{\beta},A} = 0, \quad \forall \phi \in \mathcal{P}_N(A).
\]

**Lemma 3.1.** For any \( v \in H^1_{\omega_{0,\beta},o_{\beta}}(A) \cap A^r_{\beta}(A) \) with integer \( r \geq 2 \),
\[
\| P^1_{N,\beta} v - v \|_{1,\eta_{\beta},A} + \| P^1_{N,\beta} v - v \|_{\omega_{\beta},A} \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r/2} N^{1-r/2} |v|_{A^r_{\beta},A}.
\]

**Proof.** By the projection theorem,
\[
\| P^1_{N,\beta} v - v \|_{1,\eta_{\beta},A} + \| P^1_{N,\beta} v - v \|_{\omega_{\beta},A} \leq \| v - v \|_{1,\eta_{\beta},A} + \| v - v \|_{\omega_{\beta},A}, \quad \forall \phi \in \mathcal{P}_N(A).
\]

Let \( P^1_{N,z,\beta,\gamma,\delta} \) be the orthogonal projection defined by (2.13). We take
\[
\phi(\rho) = \int_0^\rho P^1_{N-1,2,\beta,0,\gamma} \partial_{\xi} v(\xi) \, d\xi + v(0).
\]
Clearly \( \phi \in \mathcal{P}_N(A) \). Thus, it suffices to estimate \( \| \hat{\partial}_\rho (\phi - v)\|_{\eta_{\beta},A} \), \( \| \phi - v \|_{\eta_{\beta},A} \) and \( \| \phi - v \|_{\omega_{\beta},A} \). By Theorem 2.2 with \( \alpha = 2 \), \( \gamma = 0 \) and \( \beta = \delta \),
\[
\| \hat{\partial}_\rho (\phi - v)\|_{\omega_{\beta},A}^2 = \| P^1_{N-1,2,\beta,0,\rho} \partial_{\rho} v - \partial_{\rho} v\|_{\omega_{\beta},A}^2 \leq c(q_{2,\beta,0,\beta} + 1)(\beta N)^{2-r} \| \hat{\partial}_\rho v\|_{A_{2,\beta}^{-2},A}^2.
\]
Since \( \gamma = 0 \), we can take \( \rho_0 = 1/\beta \) and so \( q_{2,\theta,0,\beta} \) is bounded above by a positive constant. Thus

\[
\| \partial_{\rho} (\phi - v) \|_{\omega_{2,\beta, A}}^2 \leq c(\beta N)^{2-r} |v|_{A^\beta_{p,r} A}^2.
\]  

(3.1)

Due to \( \phi(0) = v(0) \), we use (3.1) and Lemma 2.2 with \( \varepsilon = 0 \) to derive that

\[
\| \phi - v \|_{\omega_{2,\beta, A}}^2 \leq \frac{c}{\beta^2} \| \partial_{\rho} (\phi - v) \|_{\omega_{2,\beta, A}}^2 \leq c\beta^{-r} N^{2-r} |v|_{A^\beta_{p,r} A}^2.
\]  

(3.2)

By result (iii) of Lemma 2.2 with \( \varepsilon = 2, \) and Theorem 2.2 with \( \varepsilon = 2, \gamma = 0 \), we have that

\[
\| \partial_{\rho} (\phi - v) \|_{\omega_{2,\beta, A}}^2 \leq \frac{c}{\beta^2} (\| \partial_{\rho}^2 (\phi - v) \|_{\omega_{2,\beta, A}}^2 + \| \partial_{\rho} (\phi - v) \|_{\omega_{2,\beta, A}}^2) = \frac{c}{\beta^2} \| \partial_{\rho}^2 (\phi - v) \|_{1, \omega_{2,\beta, A}}^2
\]

\[
= \frac{c}{\beta^2} \| P_{N-1,\beta,0,\rho} \partial_{\rho} v - \partial_{\rho}^2 v \|_{1, \omega_{2,\beta, A}}^2 \leq c\beta^{-r} N^{2-r} |v|_{A^\beta_{p,2} A}^2
\]

(3.3)

By using result (iii) of Lemma 2.2 with \( \varepsilon = 2, \) and Lemma 2.2 with (3.2) and (3.3), we obtain that

\[
\| \phi - v \|_{\omega_{2,\beta, A}}^2 \leq \frac{c}{\beta^2} (\| \partial_{\rho} (\phi - v) \|_{\omega_{2,\beta, A}}^2 + \| \phi - v \|_{\omega_{2,\beta, A}}^2) \leq c\beta^{-r-2} N^{2-r} |v|_{A^\beta_{p,2} A}^2.
\]  

(3.4)

Finally, a combination of (3.2)–(3.4) leads to the desired result. \( \square \)

We next recall some results on the spherical harmonic approximation. Let \( \lambda \) and \( \theta \) be the longitude and the latitude, respectively. Denote by \( S \) the unit spherical surface,

\[
S = \left\{ (\lambda, \theta) \mid 0 \leq \lambda < 2\pi, -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} \right\}.
\]

The differentiations with respect to \( \lambda \) and \( \theta \) are denoted by \( \partial_{\lambda} \) and \( \partial_{\theta} \). We define the space \( L^2(S) \) in the usual way, with the inner product

\[
(u, v)_S = \int_S u(\lambda, \theta) v(\lambda, \theta) \, dS = \int_{\pi/2}^{-\pi/2} \int_0^{2\pi} u(\lambda, \theta) v(\lambda, \theta) \cos \theta \, d\lambda \, d\theta,
\]

and the norm \( \| v \|_S = (v, v)_S^{1/2} \). Furthermore, let

\[
H^1(S) = \left\{ v \mid v, \frac{1}{\cos \theta} \partial_{\lambda} v, \partial_{\theta} v \in L^2(S) \right\},
\]

equipped with the semi-norm and the norm as

\[
|v|_{1,S} = \left( \frac{1}{\cos \theta} \left| \partial_{\lambda} v \right|_{S}^2 + \left| \partial_{\theta} v \right|_{S}^2 \right)^{1/2}, \quad \| v \|_{1,S} = (\| v \|_S^2 + |v|_{1,S}^2)^{1/2}.
\]

For any positive integer \( r \), we define the space \( H^r(S) \) with the norm \( \| v \|_{r,S} \) by induction. In particular,

\[
H^r_p(S) = \{ v \mid v \in H^r(S) \text{ and } \partial_{\lambda}^k v(\lambda + 2\pi, \theta) = \partial_{\lambda}^k v(\lambda, \theta), \ 0 \leq k \leq r - 1 \}.
\]

For any \( r > 0 \), the spaces \( H^r(S) \) and \( H^r_p(S) \) are defined by the space interpolation as in [1].
Let $L_l(x)$ be the Legendre polynomial of degree $l$. The normalized associated Legendre function is given by

$$L_{l,m}(x) = \sqrt{\frac{2m + 1}{2(m + l)!}} \left(1 - x^2\right)^{l/2} c_l^m L_m(x) \quad \text{for } l \geq 0, \ m \geq |l|,$$

$\quad L_{l,m}(x) = L_{-l,m}(x)$ for $l < 0, \ m \geq |l|$. The spherical harmonic function

$$Y_{l,m}(\lambda, \theta) = \frac{1}{\sqrt{2\pi}} e^{il\lambda} L_{l,m}(\sin \theta), \quad m \geq |l|.$$

The set of $Y_{l,m}(\lambda, \theta)$ is the complete $L^2(S)$-orthogonal system on $S$, i.e.,

$$\int_S Y_{l,m}(\lambda, \theta) \overline{Y_{l',m'}}(\lambda, \theta) \, dS = \delta_{l,l'} \delta_{m,m'}.$$  \hspace{1cm} (3.5)

Thus for any $v \in L^2(S)$,

$$v(\lambda, \theta) = \sum_{l=-\infty}^{\infty} \sum_{m \geq |l|} \hat{v}_{l,m} Y_{l,m}(\lambda, \theta),$$

where

$$\hat{v}_{l,m} = \int_S v(\lambda, \theta) \overline{Y_{l,m}(\lambda, \theta)} \, dS.$$

Let $M$ be any positive integer, and define the finite-dimensional space $\tilde{V}_M(S)$ as

$$\tilde{V}_M(S) = \text{span}\{Y_{l,m}(\lambda, \theta) \mid |l| \leq M, \ |l| \leq m \leq M\}.$$ Denote by $V_M(S)$ the subset of $\tilde{V}_M(S)$ containing all real-valued functions. The $L^2(S)$-orthogonal projection $P_{M,S} : L^2(S) \rightarrow V_M(S)$ is defined by

$$(P_{M,S}v - v)_S = 0, \quad \phi \in V_M(S),$$

or equivalently,

$$P_{M,S}v(\lambda, \theta) = \sum_{l=-M}^{M} \sum_{m=|l|}^{M} \hat{v}_{l,m} Y_{l,m}(\lambda, \theta).$$

It was proved in [5] that for any $v \in H^r_p(S)$ and $0 \leq \mu \leq r$,

$$\|P_{M,S}v - v\|_{\mu,S} \leq c M^{r-\mu} \|v\|_{r,S}. \hspace{1cm} (3.6)$$

Now, we are going to establish the main result on the mixed spherical harmonic-generalized Laguerre approximation. For this purpose, let $\Omega = A \times S$ and $\gamma(\rho)$ be a weight function in the usual sense. The spaces
$L^2_\lambda(\Omega)$ and $H^r_\lambda(\Omega)$ are defined in the usual way. Their norms are denoted by $\|v\|_{L^\infty,\Omega}$ and $\|v\|_{r,\lambda,\Omega}$, respectively. The inner product of the space $L^2_\lambda(\Omega)$ is denoted by $(u,v)_{\lambda,\Omega}$. Moreover, we set $\xi_\beta(\rho) = \omega_1(\rho) = \rho e^{-\beta \rho}$, and define the space

$$H^1_{\lambda,\eta_\beta,\xi_\beta}(\Omega) = \{v \mid v(\rho, \lambda + 2\pi, \theta) = v(\rho, \lambda, \theta) \text{ and } \|v\|_{1,\eta_\beta,\xi_\beta,\Omega} < \infty\},$$

where

$$\|v\|_{1,\eta_\beta,\xi_\beta,\Omega} = \left(\|\partial_\rho v\|^2_{\eta_\beta,\Omega} + \frac{1}{\cos \theta} \|\partial_\lambda v\|^2_{\eta_\beta,\Omega} + \|\partial_\theta v\|^2_{\eta_\beta,\Omega} + \|v\|^2_{\xi_\beta,\Omega} + \|v\|^2_{\xi_\beta,\Omega}\right)^{1/2}.$$

Now, let $a, b > 0$ and $V_{N,M}(\Omega) = \mathcal{P}_N(A) \otimes V_M(S)$. The orthogonal projection $P^1_{N,M,\beta,a,b}: H^1_{\lambda,\eta_\beta,\xi_\beta}(\Omega) \to V_{N,M}(\Omega)$ is defined by

$$
(\partial_\rho (P^1_{N,M,\beta,a,b} v - v), \partial_\rho v)_{\eta_\beta,\Omega} + \left(\frac{1}{\cos \theta} \partial_\lambda (P^1_{N,M,\beta,a,b} v - v), \frac{1}{\cos \theta} \partial_\lambda v\right)_{\eta_\beta,\Omega}
+ (\partial_\theta (P^1_{N,M,\beta,a,b} v - v), \partial_\theta v)_{\eta_\beta,\Omega} + a(P^1_{N,M,\beta,a,b} v - v, v)_{\eta_\beta,\Omega}
+ b(P^1_{N,M,\beta,a,b} v - v, \phi)_{\xi_\beta,\Omega} = 0, \quad \forall \phi \in V_{N,M}(\Omega). \tag{3.7}
$$

For simplicity of statements, we introduce the nonisotropic space

$$\mathcal{A}^s(\Omega) = A^s(\Omega, H^1_p(S)) \cap L^2_{\eta_\beta}(\Omega, H^s_p(S)) \cap H^1_{\eta_\beta}(\Omega, H^{s-1}_p(S)),$$

equipped with the norm

$$\|v\|_{\mathcal{A}^s(\Omega)} = (\|v\|^2_{A^s(\Omega,H^1_p(S))} + \|v\|^2_{L^2_{\eta_\beta}(\Omega,H^s_p(S))} + \|v\|^2_{H^1_{\eta_\beta}(\Omega,H^{s-1}_p(S))})^{1/2}.$$

**Theorem 3.1.** For any $v \in H^1_{p,\eta_\beta,\xi_\beta}(\Omega) \cap \mathcal{A}^s(\Omega)$, integer $r \geq 2$ and real number $s \geq 1$,

$$\|P^1_{N,M,\beta,a,b} v - v\|_{1,\eta_\beta,\xi_\beta,\Omega} \leq c \left(1 + \frac{1}{\beta}\right) \beta^{-r/2} N^{1-r/2} + M^{1-s}) \|v\|_{\mathcal{A}^s(\Omega)}.$$

**Proof.** By definition (3.7),

$$\|P^1_{N,M,\beta,a,b} v - v\|_{1,\eta_\beta,\xi_\beta,\Omega} \leq \|v - \phi\|_{1,\eta_\beta,\xi_\beta,\Omega}, \quad \forall \phi \in V_{N,M}(\Omega). \tag{3.8}$$

We take

$$\phi(\rho, \lambda, \theta) = P^1_{N,M,S} v(\rho, \lambda, \theta) \in V_{N,M}(\Omega).$$
Thus it remains to estimate $\| \hat{\varphi}_p (\Pi_{N,\beta}^1 P_{M,S} v - v) \|_{\eta_{\beta,\Omega}, \varphi} \| (1/\cos \theta) \hat{\varphi}_s (\Pi_{N,\beta}^1 P_{M,S} v - v) \|_{\varphi,\Omega}, \| \hat{\varphi}_s (\Pi_{N,\beta}^1 P_{M,S} v - v) \|_{\varphi,\Omega}$, $\| \Pi_{N,\beta}^1 P_{M,S} v - v \|_{\varphi,\Omega}$ and $\| \Pi_{N,\beta}^1 P_{M,S} v - v \|_{\varphi,\Omega}$. Firstly, by virtue of Lemma 3.1 and (3.6),

$$\| \hat{\varphi}_p (\Pi_{N,\beta}^1 P_{M,S} v - v) \|_{\eta_{\beta,\Omega}, \varphi} \leq 2 \| \hat{\varphi}_p (\Pi_{N,\beta}^1 P_{M,S} v - P_{M,S} v) \|_{\eta_{\beta, \varphi}}^2 + 2 \| P_{M,S} \hat{\varphi}_p v - \hat{\varphi}_p v \|_{\eta_{\beta, \varphi}}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r} N^{2-r} \| P_{M,S} v \|_{A_{\beta}^r(\Lambda, L^2(S))}^2 + c M^{2-2s} \| \hat{\varphi}_p v \|_{L_{\varphi}^2(\Lambda, H^{s-1}(S))}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r/2} N^{1-r/2} + M^{1-s} \right)^2 \| v \|_{\varphi,\Omega}^2. \tag{3.9}$$

Next, using Lemma 3.1 and (3.6) again yields

$$\left\| \frac{1}{\cos \theta} \hat{\varphi}_s (\Pi_{N,\beta}^1 P_{M,S} v - v) \right\|_{\varphi,\Omega}^2 \leq 2 \left\| \frac{1}{\cos \theta} \hat{\varphi}_s (\Pi_{N,\beta}^1 P_{M,S} v - P_{M,S} v) \right\|_{\varphi,\Omega}^2 + 2 \left\| \frac{1}{\cos \theta} \hat{\varphi}_s (P_{M,S} v - v) \right\|_{\varphi,\Omega}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r} N^{2-r} \left\| \frac{1}{\cos \theta} \hat{\varphi}_s P_{M,S} v \right\|_{A_{\beta}^r(\Lambda, L^2(S))}^2 + c \| P_{M,S} v - v \|_{L_{\varphi}^2(\Lambda, H^1(S))}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r/2} N^{1-r/2} + M^{1-s} \right)^2 \| v \|_{\varphi,\Omega}^2. \tag{3.10}$$

Similarly

$$\| \hat{\varphi}_s (\Pi_{N,\beta}^1 P_{M,S} v - v) \|_{\varphi,\Omega}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r/2} N^{1-r/2} + M^{1-s} \right)^2 \| v \|_{\varphi,\Omega}^2. \tag{3.11}$$

Furthermore, we use Lemma 3.1 and (3.6) to deduce

$$\| \Pi_{N,\beta}^1 P_{M,S} v - v \|_{\eta_{\beta,\Omega}}^2 \leq 2 \| \Pi_{N,\beta}^1 P_{M,S} v - P_{M,S} v \|_{\eta_{\beta,\Omega}}^2 \| P_{M,S} v - v \|_{\varphi,\Omega}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r} N^{2-r} \| P_{M,S} v \|_{A_{\beta}^r(\Lambda, L^2(S))}^2 + c \| P_{M,S} v - v \|_{L_{\varphi}^2(\Lambda, L^2(S))}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r} N^{2-r} \| v \|_{A_{\beta}^r(\Lambda, L^2(S))}^2 + c M^{2-2s} \| v \|_{L_{\varphi}^2(\Lambda, H^{s-1}(S))}^2 \leq c \left( 1 + \frac{1}{\beta} \right) \beta^{-r/2} N^{1-r/2} + M^{1-s} \right)^2 \| v \|_{\varphi,\Omega}^2. \tag{3.12}$$
In the same manner, we verify that
\[
\|\Pi_{N,\beta}^1 P_{M,SV} - v\|^2_{s,\beta,\Omega} \leq 2\|\Pi_{N,\beta}^1 P_{M,SV} - P_{M,SV}\|^2_{s,\beta,\Omega} + 2\|P_{M,SV} - v\|^2_{s,\beta,\Omega}
\]
\[
\leq (1 + \frac{1}{\beta}) 2 \beta^{-r} N^{2-r} \|P_{M,SV}\|^2_{A_{\rho}(X, L^2(S))} + c \|P_{M,SV} - v\|^2_{L^2_{\phi_{\beta}}(X, L^2(S))}
\]
\[
\leq (1 + \frac{1}{\beta}) 2 \beta^{-r} N^{2-r} \|v\|^2_{A_{\rho}(X, L^2(S))} + c \|P_{M,SV} - v\|^2_{L^2_{\phi_{\beta}}(X, H^{s-1}(S))}
\]
\[
\leq (1 + \frac{1}{\beta}) \beta^{-r/2} N^{1-r/2} + M^{1-s})^2 \|v\|^2_{s,\beta,\Omega}.
\]

The above with (3.12) implies that
\[
\|\Pi_{N,\beta}^1 P_{M,SV} - v\|^2_{s,\beta,\Omega} \leq c \left( (1 + \frac{1}{\beta}) \beta^{-r/2} N^{1-r/2} + M^{1-s} \right)^2 \|v\|^2_{s,\beta,\Omega}.
\] (3.13)

The combination of (3.9)–(3.13) leads to the desired result. \(\square\)

4. Mixed spectral method for a model problem

In this section, we develop the mixed spherical harmonic-generalized Laguerre spectral method for three-dimensional problems. The Laplacian
\[
\Delta v(\rho, \lambda, \theta) = \frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho v(\rho, \lambda, \theta)) + \frac{1}{\rho^2 \cos \theta} \partial_\theta (\cos \theta \partial_{\theta} v(\rho, \lambda, \theta)) + \frac{1}{\rho^2 \cos^2 \theta} \partial_{\lambda}^2 v(\rho, \lambda, \theta).
\]

We consider the following model problem:
\[
- \Delta W(\rho, \lambda, \theta) + \mu W(\rho, \lambda, \theta) = F, \quad \text{in} \ \Omega,
\]
\[
\lim_{\rho \to \infty} W(\rho, \lambda, \theta) = 0,
\]
\[
W(\rho, \lambda + 2\pi, \theta) = W(\rho, \lambda, \theta).
\] (4.1)

Moreover, we know from [2] that \(\partial_{\lambda} W(\rho, \lambda, \theta) = 0\) for \(|\theta| = \frac{1}{4}\pi\).

Problem (4.1) is well-posed in certain standard Sobolev space. But it is not well-posed in the weighted space which is needed in numerical methods for unbounded domains. On the other hand, the linear system induced by the spectral scheme based directly on (4.1) is not symmetric and sparse. Therefore we make the variable transformation
\[
W(\rho, \lambda, \theta) = e^{-(1/2)\beta\rho} U(\rho, \lambda, \theta), \quad F(\rho, \lambda, \theta) = \frac{1}{\rho^2} e^{-(1/2)\beta\rho} f(\rho, \lambda, \theta).
\]

Then (4.1) is changed to
\[
- \rho^2 \partial_\rho^2 U(\rho, \lambda, \theta) - (2\rho - \beta\rho^2) \partial_\rho U(\rho, \lambda, \theta) - \frac{1}{\cos \theta} \partial_\theta (\cos \theta \partial_{\theta} U(\rho, \lambda, \theta))
\]
\[
- \frac{1}{\cos^2 \theta} \partial_{\lambda}^2 U(\rho, \lambda, \theta) + \left( \mu\rho^2 + \beta\rho - \frac{\beta^2}{4\rho^2} \right) U(\rho, \lambda, \theta) = f(\rho, \lambda, \theta).
\] (4.2)
For simplicity, let $\nabla_S v(\rho, \lambda, \theta) = ((1/\cos \theta) \partial_\lambda v(\rho, \lambda, \theta), \partial_\theta v(\rho, \lambda, \theta))$, $dS = \cos \theta d\lambda d\theta$, and introduce the bilinear form

$$A(u, v) = \int_\Omega \rho^2 e^{-\beta \rho} \partial_\rho u(\rho, \lambda, \theta) \partial_\rho v(\rho, \lambda, \theta) dS d\rho + \int_\Omega e^{-\beta \rho} (\partial_\rho v(\rho, \lambda, \theta))^2 dS d\rho \quad (4.5)$$

Clearly

$$|A(u, v)| \leq \max \left( \mu + \frac{\beta^2}{4}, \beta \right) \|u\|_{1, \eta_\beta, \xi_\beta, \Omega} \|v\|_{1, \eta_\beta, \xi_\beta, \Omega}. \quad (4.3)$$

If $\mu > \beta^2/4$, then for suitably small $c_0 > 0$,

$$A(v, v) = \int_\Omega \rho^2 e^{-\beta \rho} (\partial_\rho v(\rho, \lambda, \theta))^2 dS d\rho + \int_\Omega e^{-\beta \rho} (\nabla_S u(\rho, \lambda, \theta))^2 dS d\rho \quad (4.4)$$

Multiplying (4.2) by $\omega_\beta(\rho) v(\rho, \lambda, \theta)$ and integrating the result, we derive a weak formulation of (4.2). It is to find $U \in H^{1, \eta_\beta, \xi_\beta}(\Omega)$ such that

$$A(u, v) = (f, v)_{\omega_\beta, \Omega}, \quad \forall v \in H^{1, \eta_\beta, \xi_\beta}(\Omega). \quad (4.5)$$

If $\mu > \beta^2/4$ and $f \in (H^{1, \eta_\beta, \xi_\beta}(\Omega))'$, then by (4.3), (4.4) and the Lax–Milgram Theorem, (4.5) has a unique solution.

The mixed spectral scheme for (4.5) is to seek $u_{N,M} \in V_{N,M}(\Omega)$ such that

$$A(u_{N,M}, \phi) = (f, \phi)_{\omega_\beta, \Omega}, \quad \forall \phi \in V_{N,M}(\Omega). \quad (4.6)$$

We now turn to the convergence of scheme (4.6). Let $U_{N,M} = P^1_{N,M,\beta,a,b} U$ as in (3.7), with $a = \mu - \beta^2/4$ and $b = \beta$. By the definition of $P^1_{N,M,\beta,a,b} U$, we obtain from (4.5) that

$$A(U_{N,M}, \phi) = (f, \phi)_{\omega_\beta, \Omega}, \quad \forall \phi \in V_{N,M}(\Omega). \quad (4.7)$$

Let $\tilde{u}_{N,M} = u_{N,M} - U_{N,M}$. Subtracting (4.7) from (4.6) yields that

$$A(\tilde{u}_{N,M}, \phi) = 0, \quad \forall \phi \in V_{N,M}(\Omega). \quad (4.8)$$

Taking $\phi(\rho, \lambda, \theta) = \tilde{u}_{N,M}(\rho, \lambda, \theta)$ in (4.8), we have $A(\tilde{u}_{N,M}, \tilde{u}_{N,M}) = 0$, and so $u_{N,M} = U_{N,M}$ exactly. Finally, the above fact with Theorem 3.1 leads to that for integer $r \geq 2$ and real number $s \geq 1$,

$$\|U - u_{N,M}\|_{1, \eta_\beta, \xi_\beta, \Omega} \leq c \left( \left( 1 + \frac{1}{\beta} \right) \beta^{-r/2} N^{1-r/2} + N^{1-s} \right) \|U\|_{l^\beta, s, \Omega}. \quad (4.9)$$
Remark 4.1. We see from (4.9) that the accuracy of numerical solution depends on the parameter $\beta$. In other words, scheme (4.6) with suitably larger $\beta$ provides a more accurate numerical result, as long as $\beta < 2\sqrt{\rho}$.

Remark 4.2. Let $w_{N,M} = e^{-(1/2)\beta \rho} u_{N,M}$. By (4.9), we have that
\[
\| W - w_{N,M} \|_{\rho, \Omega} = C \left( \left( 1 + \frac{1}{\beta} \right) \beta^{-r/2} N^{1-r/2} + N^{1-s} \right) .
\]

Remark 4.3. We may use other methods to solve the simple problem (4.1). But the method proposed in this paper can be applied easily to many other problems, such as non-linear problems and exterior problems.

5. Numerical results

In this section, we first discuss the implementation for scheme (4.6). For simplicity, let $\mathcal{L}(\rho) = \mathcal{L}(0, \rho)$ and set
\[
\psi_k(\rho) = \mathcal{L}(\rho)_k, \quad 0 \leq k \leq N ,
\]
\[
Z^1_{l,m}(\lambda, \theta) = \frac{1}{\sqrt{2\pi}} \sin(l\lambda) L_{l,m}(\sin \theta), \quad -M \leq l \leq M, |l| \leq m \leq M ,
\]
\[
Z^2_{l,m}(\lambda, \theta) = \frac{1}{\sqrt{2\pi}} \cos(l\lambda) L_{l,m}(\sin \theta), \quad -M \leq l \leq M, |l| \leq m \leq M ,
\]
\[
\phi^1_{k,l,m}(\rho, \lambda, \theta) = \psi_k(\rho) Z^1_{l,m}(\lambda, \theta), \quad 0 \leq k \leq N, -M \leq l \leq M, |l| \leq m \leq M ,
\]
\[
\phi^2_{k,l,m}(\rho, \lambda, \theta) = \psi_k(\rho) Z^2_{l,m}(\lambda, \theta), \quad 0 \leq k \leq N, -M \leq l \leq M, |l| \leq m \leq M ,
\]

Let
\[
A_S v(\lambda, \theta) = \frac{1}{\cos \theta} \partial_\theta (\cos \theta \partial_\theta v(\lambda, \theta)) + \frac{1}{\cos^2 \theta} \partial^2_\theta v(\lambda, \theta).
\]

We have (see [5])
\[
A_S Y_{l,m}(\lambda, \theta) = m(m + 1) Y_{l,m}(\lambda, \theta) .
\] (5.1)

Now, we expand the numerical solution as
\[
u_{N,M}(\rho, \lambda, \theta) = \sum_{k=1}^{N} \sum_{l=-M}^{M} \sum_{m=-|l|}^{M} (u^1_{k,l,m} \phi^1_{k,l,m}(\rho, \lambda, \theta) + u^2_{k,l,m} \phi^2_{k,l,m}(\rho, \lambda, \theta)).
\]

Similarly,
\[
f(\rho, \lambda, \theta) = \sum_{l=-M}^{M} \sum_{m=-|l|}^{M} (f^1_{l,m}(\rho) Z^1_{l,m}(\lambda, \theta) + f^2_{l,m}(\rho) Z^2_{l,m}(\lambda, \theta)).
\]
Take \( \phi(\rho, \lambda, \theta) = \phi_{n,l,m}^{q}(\rho, \lambda, \theta), \ q = 1, 2, \) in (4.6). By (2.4), (5.1) and the orthogonality of the spherical harmonic system, we derive that

\[
\sum_{k=1}^{N} \int_{A} \rho^{2} \mathrm{e}^{-\beta \rho \partial_{\rho} \psi_{k}(\rho)} \partial_{\rho} \psi_{n}(\rho) \, d\rho + m(m + 1) \int_{A} \mathrm{e}^{-\beta \rho \psi_{k}(\rho)} \psi_{n}(\rho) \, d\rho \\
+ \int_{A} \left( \mu \rho^{2} + \beta \rho - \frac{\beta^{2}}{4} \rho^{2} \right) \mathrm{e}^{-\beta \rho \psi_{k}(\rho)} \psi_{n}(\rho) \, d\rho \right) u_{k,l,m}^{q} = g_{n,l,m}^{q}, \tag{5.2}
\]

where

\[
g_{n,l,m}^{q} = \int_{A} \mathrm{e}^{-\beta \rho \psi_{1}(\rho)} \psi_{n}(\rho) \, d\rho.
\]

Next, let the matrices \( A = (a_{n,k}), B = (b_{n,k}), C = (c_{n,k}) \) and \( D = (d_{n,k}) \), with the following entries:

\[
a_{n,k} = \int_{A} \rho^{2} \mathrm{e}^{-\beta \rho \partial_{\rho} \psi_{k}(\rho)} \partial_{\rho} \psi_{n}(\rho) \, d\rho,
\]

\[
b_{n,k} = \int_{A} \rho^{2} \mathrm{e}^{-\beta \rho \psi_{n}(\rho)} \psi_{k}(\rho) \, d\rho,
\]

\[
c_{n,k} = \int_{A} \rho \mathrm{e}^{-\beta \rho \psi_{n}(\rho)} \psi_{k}(\rho) \, d\rho,
\]

\[
d_{n,k} = \int_{A} \mathrm{e}^{-\beta \rho \psi_{n}(\rho)} \psi_{k}(\rho) \, d\rho.
\]

The values of \( a_{n,k}, b_{n,k}, c_{n,k} \) and \( d_{n,k} \) are given in the appendix. Furthermore, let

\[
X_{l,m}^{q} = (u_{1,l,m}^{q}, u_{2,l,m}^{q}, \ldots, u_{N,l,m}^{q}), \quad q = 1, 2,
\]

\[
G_{l,m}^{q} = (g_{1,l,m}^{q}, g_{2,l,m}^{q}, \ldots, g_{N,l,m}^{q}), \quad q = 1, 2.
\]

Finally, we have from (5.1) that

\[
\left[ A + \left( \mu - \frac{\beta^{2}}{4} \right) B + \beta C + m(m + 1) D \right] X_{l,m}^{q} = G_{l,m}^{q}, \quad q = 1, 2.
\]

We now present some numerical results. We take \( \mu = 3 \) in (4.1) and the test function

\[
U(\rho, \lambda, \theta) = \frac{\sin \rho}{\rho + 1} \sin(2\lambda) \cos^{2} \theta.
\]

To describe the numerical errors of (4.6), we set

\[
E_{N,M} = \| U - u_{N,M} \|_{L^{2}_{\omega}(\Omega)} = \| W - w_{N,M} \|_{L^{2}(\Omega)}.
\]
In Fig. 1, we plot $\log_{10} E_{N,M}$ vs. $N = M$, with $\beta = 1, 2, 3$. They indicate the spectral accuracy of numerical solutions. In particular, the numerical results with $\beta = 2, 3$ are better than the numerical results with $\beta = 1$ as in the usual Laguerre spectral method. It coincides with the theoretical analysis well.

6. Some concluding remarks

In this paper, we introduced a new generalized Laguerre approximation and established some error estimates in the nonuniformly weighted Sobolev spaces for various orthogonal projections. These results improve and generalize previously published results ($\alpha = 0$ or $\beta = 1$), and are valid for the generalized Laguerre approximation with $\alpha > -1$ and $\beta > 0$. They form the mathematical foundation of related spectral methods for unbounded domains. The suitable choice of $\alpha$ is based on the considered practical problems, and the better choice of $\beta$ depends on the asymptotic behavior of the exact solutions, which usually in turn depends on certain coefficients in underlying differential equations. Using this reason, we can solve a large class of differential equations numerically and fit the asymptotic behaviors of exact solutions more properly.

As an example of applications, we developed the mixed spherical harmonic-generalized Laguerre approximation, proposed a mixed spectral scheme for a model problem in three-dimensional space, and provided an efficient numerical algorithm. By the orthogonality of the spherical harmonic system, we could evaluate all coefficients of spherical harmonic expansion of numerical solution separately. This feature saves a lot of work. Especially, it is very suitable for parallel computation. Moreover, by using the basic results on the mixed spherical harmonic-generalized Laguerre approximation, we obtained very sharp error estimates for the proposed scheme.

Although we only considered a simple model problem in this paper, the method developed in this paper is also useful for other problems, such as the Black–Sholes equation, spherically symmetric solutions, nonlinear problems and exterior problems. We shall report the related results in the future papers.
Appendix

Lemma A.1. For any $1 \leq k, n \leq N$, we have that

$$
\int_A e^{-\beta \rho} \psi_n(\rho) \psi_k(\rho) \, d\rho = \frac{1}{\beta} \delta_{n,k}, \quad (A.1)
$$

$$
\int_A \rho e^{-\beta \rho} \psi_n(\rho) \psi_k(\rho) \, d\rho = \frac{1}{\beta^2} \{(k + n + 1) \delta_{n,k} - n \delta_{n-1,k} - k \delta_{n,k-1} \}, \quad (A.2)
$$

$$
\int_A \rho^2 e^{-\beta \rho} \psi_n(\rho) \psi_k(\rho) \, d\rho = \frac{1}{\beta^3} \{(k + n + 2)(k + n + 1) + 2kn\delta_{n,k} - 2n(k + n + 1)\delta_{n-1,k} - 2k(k + n + 1)\delta_{n,k-1} + n(n - 1)\delta_{n-2,k} + k(k - 1)\delta_{n,k-2} \}, \quad (A.3)
$$

$$
\int_A \rho^2 e^{-\beta \rho} \hat{\partial}_\rho \psi_n(\rho) \hat{\partial}_\rho \psi_k(\rho) \, d\rho = \frac{1}{\beta} (2nk\delta_{n,k} - nk\delta_{n,k-1} - nk\delta_{n-1,k}). \quad (A.4)
$$

Proof. Let $\alpha = 0$ in (2.6). Due to $\psi_k(\rho) = \mathcal{L}_k^{(\beta)}(\rho)$, we obtain (A.1). By (2.2) and an integration by parts, we deduce that

$$
\int_A \rho^2 e^{-\beta \rho} \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) \, d\rho \\
= \frac{1}{\beta} \int_A e^{-\beta \rho} \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) + \rho \hat{\partial}_\rho \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) + \rho \mathcal{L}_n^{(\beta)}(\rho) \hat{\partial}_\rho \mathcal{L}_k^{(\beta)}(\rho) \, d\rho \\
= \frac{1}{\beta} \left[ (k + n + 1) \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) - n \mathcal{L}_{n-1}^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) - k \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_{k-1}^{(\beta)}(\rho) \right] \, d\rho \\
= \frac{1}{\beta^2} \{(k + n + 1) \delta_{n,k} - n \delta_{n-1,k} - k \delta_{n,k-1} \}.
$$

This implies (A.2). Next, by (A.2) and an integration by parts, we obtain that

$$
\int_A \rho^2 e^{-\beta \rho} \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) \, d\rho \\
= \frac{1}{\beta} \int_A e^{-\beta \rho} \left[ 2\rho \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) + \rho^2 \hat{\partial}_\rho \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) + \rho^2 \mathcal{L}_n^{(\beta)}(\rho) \hat{\partial}_\rho \mathcal{L}_k^{(\beta)}(\rho) \right] \, d\rho \\
= \frac{k + n + 2}{\beta} \int_A \rho e^{-\beta \rho} \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) \, d\rho - \frac{n}{\beta} \int_A e^{-\beta \rho} \mathcal{L}_{n-1}^{(\beta)}(\rho) \mathcal{L}_k^{(\beta)}(\rho) \, d\rho \\
- \frac{k}{\beta^2} \int_A e^{-\beta \rho} \rho \mathcal{L}_n^{(\beta)}(\rho) \mathcal{L}_{k-1}^{(\beta)}(\rho) \, d\rho \\
= \frac{1}{\beta^3} \left[ (k + n + 2)(k + n + 1) + 2kn \delta_{n,k} - 2n(k + n + 1) \delta_{n,k-1} - 2k(k + n + 1) \delta_{n,k-1} + n(n - 1) \delta_{n-2,k} + k(k - 1) \delta_{n-2,k} \right].
$$

So result (A.3) follows. Taking $\alpha = 0$ in (2.5) yields that

$$
\rho \hat{\partial}_\rho \mathcal{L}_k^{(\beta)}(\rho) = k(\mathcal{L}_k^{(\beta)}(\rho) - \mathcal{L}_{k-1}^{(\beta)}(\rho)).
$$
Thus by (2.6),
\[
\int_{A} \rho^2 e^{-\beta \rho} \psi_n(\rho) \psi_k(\rho) \, d\rho = \frac{1}{\beta} \left( 2nk\delta_{n,k} - nk\delta_{n,k-1} - nk\delta_{n-1,k} \right).
\]
We complete the proof. □

References