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Irreducibility and Reducibility for the Energy Representation of the Group of Mappings of a Riemannian Manifold into a Compact Semisimple Lie Group*

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The irreducibility of the energy representation of the group of smooth mappings from a Riemannian manifold of dimension $d \geqslant 3$ into a compact semisimple Lie group is proven. The nonequivalence of the representations associated with different Riemann structures is also proven for $d \geqslant 3$. The case $d = 2$ is examined and irreducibility and nonequivalence results are also given. The reducibility in the case $d = 1$ is pointed out (in this case the commutant contains a representation equivalent with the energy representation).

1. INTRODUCTION

In this paper we study the energy representation of the group G^X of smooth mappings with compact support from a Riemannian manifold X into a semisimple compact Lie group G. Such representations have been introduced in [1–4]. The irreducibility and the mutual nonequivalence of

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such representations for different metrics depend strongly on the dimension of X and the first proof of these properties was given by Ismagilov [1] in the case $d \geqslant 5$ and $G = SU2$. The crucial point in the Ismagilov's proof [1, Theorem 3] is a result about the Gaussian measure on the Schwartz space $\mathscr{D}'(U)$, U an open set in $\mathbb{R}^d$, with covariance given by the Laplacian and its translations by linear combinations of Dirac measures. Vershik, Gelfand and Graev [2] were able to improve the method of Ismagilov. First, they show that the algebraic elements in the proof are essentially the same when G is an arbitrary compact semisimple Lie-group (this is remarked without proof in [1]) and more importantly they extend for $d \geqslant 4$ the result about the Gaussian measure, alluded to above, by developing a different method and hence obtaining the irreducibility and the nonequivalence results for $d \geqslant 4$.

In the present paper we tackle the case $d \geqslant 3$ and also give results on the remaining cases, $d = 2$ and $d = 1$. As far as the irreducibility and nonequivalence results are concerned, the algebraic part of our work is the same as that used in [1, 2], however, we differ from those papers by having a stronger method of controlling the Gaussian measure mentioned above. Our main technical tool is in fact Theorem 4.1 below, which is concerned with that Gaussian measure; and this result is in fact an extension of the recent results which served to proved the triviality of the exponential interaction $:e^{\alpha\varphi}:$ in the cases $d \geqslant 3$ for all α or $d = 2$ for $|\alpha|$ big in quantum field theory [8, 9] (see also [10, 11]).

In Section 2 we give the definitions; we describe the problems and state the results. In Section 3 we reduce the proof of the results to estimations on Gaussian measures which are given in Section 4.

2. The Group G^X and Its Energy Representation. Description and Statement of the Results

Let X be a Riemannian manifold of dimension d and let G be a compact semisimple Lie group with Lie algebra g. Let G^X be the space of C^∞-functions from X to G with compact support. G^X is a group with respect to the pointwise multiplication.

The space of smooth 1-forms with values in g will be denoted by $\Omega(g)$. If $\omega \in \Omega(g)$ then $\omega(x)$ is a linear operator from the tangent space $T_x(X)$ at x of X into the Lie algebra g. $\Omega(g)$ is a vector space and it is a real prehilbert space if one defines the scalar product by

$$(\omega_1, \omega_2) = \int_X \mathrm{Sp}\,(\omega_1(x)\,\omega_2(x)^*)\,\rho(x)\,dx, \tag{2.1}$$

where $\rho(x)\,dx$ is a measure on X with strictly positive C^∞-density $\rho(x)$ with respect to the volume measure dx on X, and where $\omega(x)^*$ is the adjoint of $\omega(x)$ with respect to the canonical Euclidean structure on $T_x(X)$ and the Euclidean structure on g given by $(-K)$, K being the Killing form. The symbol Sp denotes the normalized trace in the space of linear operators on g.

Note that, except for $d=2$, we can always reduce ourselves to the case $\rho(x)=1$ by using another Riemann structure on X.

We denote by $H(g)$ the complex Hilbert space generated by $\Omega(g)$. The elements of $H(g)$ can be identified with currents on X with values in the complexification g^c of g, using the canonical identification of g with its dual. These currents are actually (classes for the equality almost everywhere with respect to dx of) locally integrable sections of the bundle $T(X)^* \otimes g^c$.

G^X acts on $H(g)$ by the pointwise adjoint representation V, i.e., for $\psi \in G^X$, $\omega \in H(g)$ we have

$$(V(\psi)\,\omega)(x) = Ad\psi(x)\,\omega(x),$$

which we also write as

$$(V(\psi)\,\omega)(x) = \psi(x)\,\omega(x)\,\psi(x)^{-1}. \tag{2.2}$$

Due to the invariance of the Killing form with respect to the adjoint representation, this formula defines a unitary representation V of G^X acting on the classes of sections.

The right logarithmic derivative defined by

$$\beta(\psi)(x) = d\psi(x)\,\psi(x)^{-1} \tag{2.3}$$

is an element of $\Omega(g)$ and β is a one-cocycle (the so-called Maurer–Cartan-cocycle) with respect to the representation V. This means that for all ψ, $\varphi \in G^X$

$$\beta(\psi\varphi) = V(\psi)\,\beta(\varphi) + \beta(\psi). \tag{2.4}$$

Let now $e^{H(g)}$ be the Fock space built over the one particle space $H(g)$, i.e.,

$$e^{H(g)} \equiv \sum_{n=0}^{\infty}{}^{\oplus} H(g)^{\otimes n}_{sym},$$

where $H(g)^{\otimes n}_{\mathrm{sym}}$ is the symmetric Hilbert n-fold tensor product of $H(g)$ with itself and the sum is a direct Hilbert one.

Then one defines a representation U of G^X in $e^{H(g)}$, the so called *energy representation of* G^X (associated with the weight ρ) ([1–4]), by giving the action of $U(\psi)$, $\psi \in G^X$, on the total set of coherent states $\{e^{\omega} | \omega \in H(g)\}$, with $e^{\omega} \equiv \sum_{n=0}^{\infty} (1/n!)(\omega \otimes \cdots \otimes \omega)$ (see, e.g., [5, 14]) in the following way:

$$U(\psi)\, e^{\omega} = e^{-\|\beta(\psi)\|^2/2}\, e^{-(V(\psi)\omega, \beta(\psi))}\, e^{V(\psi)\omega + \beta(\psi)}, \tag{2.5}$$

where $\| \;\|$ is the norm in $H(g)$.

In other words U is the unitary representation of G^X in the representation space $e^{H(g)}$ which one obtains by exponentiating the action of the Euclidean group of transformations of $H(g)$ given by $\omega \to V(\psi)\,\omega + \beta(\psi)$, $\psi \in G^X$, $\omega \in \Omega(g)$.

An equivalent description of this representation is the following. Let μ be the Gaussian measure canonically associated with the dual $\Omega'(g)$ of $\Omega(g)$. Then the canonical isomorphism of $e^{H(g)}$ with $L^2(d\mu)$ identifies U with the unitary representation in $L^2(d\mu)$ given by $f(\omega') \to e^{i\langle \beta(\psi), \omega' \rangle} f(V^{-1}(\psi)\, \omega')$, $\omega' \in \Omega'(g)$, $\psi \in G^X$. For more details on this representation see the original references [1–3].

The irreducibility of the energy representation has been proven for $\dim X \geqslant 5$ (in the case $G = SU2$) by Ismagilov [1] and for $\dim X \geqslant 4$ by Vershik, Gelfand and Graev [2]. In the present paper we improve this to $d \geqslant 3$ and we also discuss the interesting situations that arise for $d = 1, 2$. Let us formulate shortly the main results of the present paper:

THEOREM 2.1. *For $d \geqslant 3$ the energy representation of G^X associated to any weight ρ is irreducible. For $d = 2$ the energy representation is irreducible if the roots λ of the Lie algebra satisfy $|\lambda| > \sqrt{32\pi\rho(x)}$ for all $x \in X$ (where $|\lambda|$ is the length of the root vector λ in g).*

There is a natural conjecture concerning the case $d = 2$ and $\lambda < \sqrt{4\pi\rho(x)}$ for any root of g and some point $x \in X$, namely, that the energy representation associated with the weight ρ should be reducible. This is based on results obtained [13] (see also [8, 9]) for the exponential interaction in quantum field theory. The remaining cases for $d = 2$ are also in close relation with as yet unsolved problems of quantum field theory with exponential interaction; on the basis of [10] one should actually obtain irreducibility in the whole region $|\lambda| > \sqrt{8\pi\rho(x)}$. Concerning reducibility in the region $|\lambda| < \sqrt{4\pi\rho(x)}$ we have presently only the following weaker result:

THEOREM 2.2. *Let $d = 2$ and let Ω be the open set of $x \in X$ such that $|\lambda| < \sqrt{4\pi\rho(x)}$ for all roots λ of g. Let $\exp(t^{\Omega})$ be the set of elements of G^X which are of the form*

$$(\exp A)(x) = \exp(A(x))$$

for some smooth function A with compact support in Ω and values in an abelian Lie algebra t of g. Then the restriction of U to $\exp(t^{\Omega})$ *generates an abelian algebra of operators with a spectrum which is not simple.*

We have also the following result concerning the nonequivalence of the energy representations given by different weights on the Riemannian manifold X:

THEOREM 2.3. *Let U_0, U be the two energy representations of G^X associated with two different weights ρ_0 and ρ_1. Then U_0 and U_1 are inequivalent if either $d=2$ and $|\lambda| > \sqrt{32\pi\rho_\alpha(x)}$ for $\alpha = 0, 1$, and all $x \in X$ and all roots λ of g, or $d \geqslant 3$.*

3. ALGEBRAIC PART OF THE PROOFS

Let us start by choosing a maximal torus T in G and let t be the associated Cartan subalgebra in g. Let $t^{\perp}$ be the orthogonal complement of t in g. The following objects are defined as before, replacing G by T and g by t resp. $t^{\perp}$: T^X, $\Omega(t)$, $\Omega(t^{\perp})$, $H(t)$, $H(t^{\perp})$. Remark that $e^{H(t)}$ and $e^{H(t^{\perp})}$ are canonically imbedded in $e^{H(g)}$, as $e^{H(t)} \otimes e^0$ resp. $e^0 \otimes e^{H(t^{\perp})}$, and one has

$$e^{H(g)} = e^{H(t)} \otimes e^{H(t^{\perp})} \tag{3.1}$$

(see, e.g., [5]).

Let t^X be the set of smooth functions with compact support in X and with values in t. The element $\exp A$ of $T^X \subset G^X$ defined by

$$(\exp A)(x) = \exp A(x) \tag{3.2}$$

satisfies

$$\beta(\exp A) = dA \tag{3.3}$$

and one can easily check that

$$U(\exp A) = W(dA) \otimes e^{V(\exp A)}, \tag{3.4}$$

where $W(dA)$ is the Weyl operator associated with $dA \in \Omega(t) \subset H(t) \subset H(g)$ defined by

$$W(dA)\, e^{\omega} = e^{-\|dA\|^2/2}\, e^{-(\omega, dA)}\, e^{\omega + dA} \tag{3.5}$$

for all $\omega \in H(t)$, and we use the notation e^M for the operator on $e^{H(g)}$ defined by

$$e^M e^{\omega} = e^{M\omega} \tag{3.6}$$

for $\omega \in H(g)$ and M an operator on $H(g)$.

In formula (3.4) the part $e^{V(\exp A)}$ is easily diagonalized. First of all, it leaves invariant the subspaces $H(t)^{\otimes n}_{\mathrm{sym}}$ and in each of these spaces, using a decomposition of $(t^{\perp})^c$ with root vectors in g^c, one sees that $e^{V(\exp A)}$ acts multiplicatively. Then there is a direct integral decomposition of the restriction of U to the set $\exp(t^X)$, on a constant Borel field of Hilbert spaces all isomorphic to $e^{H(t)}$, given by

$$U(\exp A) = \int_{\Phi}^{\oplus} W(dA)\, e^{i\chi(A)}\, d\nu(\chi), \tag{3.7}$$

where Φ is the set of elements of $(t^X)'$ which are of the form

$$\chi(A) = \sum_{j=1}^{n} \alpha_j(A(x_j)) \tag{3.8}$$

for some $x_j \in X$, $\alpha_j \in \mathscr{R}$, denoting by $\mathscr{R}$ the space of roots of g. In (3.7) one also uses the measure space structure on Φ which is given by identifying Φ with the disjoint union of $(X \times \mathscr{R})^n_{\mathrm{sym}}$ for $n = 0, 1, 2, \ldots$, (we use the convention $(X \times \mathscr{R})^0_{\mathrm{sym}} = \{0\}$). Φ has a natural Borel structure. ν is the measure on Φ whose restriction ν^n to $(X \times \mathscr{R})^n_{\mathrm{sym}}$ is $(\rho(x)\, dx \otimes N)^{\otimes n}$, where N is the counting measure (see, e.g., [14]). ν is thus the canonical Poisson measure on Φ. As it is well known, the spectral measure associated with $W(dA)$ is the Gaussian measure μ on $(t^X)'$ with Fourier transform

$$\tilde{\mu}(A) = \exp(-\tfrac{1}{2} \|dA\|^2). \tag{3.9}$$

We can summarize the preceding discussion by the following.

PROPOSITION 3.1. *The spectral measure associated with the restriction of U to* $\exp t^X$ *is the convolution $\mu * \nu$ of μ and ν, where μ is the Gaussian measure on $(t^X)'$ with Fourier transform* (3.9) *and ν is the canonical Poisson measure on Φ.*

The crucial point in the proof of Theorem 2.1 is the following lemma, which will be proven in Section 4. Let us say that two probability measures μ and μ' are disjoint if there exist two measurable sets which are disjoint and have measure 1 for μ and μ', respectively.

LEMMA 3.2. *Let μ be the Gaussian measure on $(t^X)'$ with Fourier transform* (3.9). *Then $\mu * \nu_1$ and $\mu * \nu_2$ are disjoint, for any mutually disjoint probability measures ν_1 and ν_2 on Φ.*

Let us now formulate two important corollaries of this lemma.

COROLLARY 3.3. *The commutant of* $U(\exp t^X)$ *is contained in the set of decomposable operators of the integral decomposition* (3.7).

COROLLARY 3.4. *The operator* $W(dA)$ *for all* $A \in g^X$, *is in the von Neumann algebra generated by* $U(G^X)$.

For the proofs of these Corollaries see Lemmas 2, 3 of [2].

The irreducibility of U in the cyclic component of e^0 follows from Corollary 3.3. In fact, let Q be an operator in the commutant of $U(G^X)$, then U is decomposable with respect to the decomposition (3.7) for any Cartan subalgebra t in g. The projection onto $e^{H(t)}$ in $e^{H(g)}$ is diagonalized and it follows that $e^{H(t)}$ is invariant by Q for any Cartan subalgebra t in g. Thus Qe^0 belongs to $e^{H(t)}$ for any Cartan subalgebra t in g. By the semisimplicity of G, Qe^0 is a scalar multiple of e^0 and this is just the irreducibility of $U(G^X)$ in the cyclic component of e^0.

Moreover, using Corollary 3.4, we can actually prove that e^0 is a cyclic vector. To see this we remark that the von Neumann algebra $U(G^X)''$ contains all operators

$$W(V(\psi)\, dA) = U(\psi)\, W(dA)\, U(\psi^{-1}), \tag{3.10}$$

for $\psi \in G^X$, $A \in g^X$, and the cyclicity of e^0 follows from [5] and the following Lemma.

LEMMA 3.5. *The set* $\{V(\psi)\, dA \mid \psi \in G^X, A \in g^X\}$ *is total in* $H(g)$.

Proof. Let us show first that the set of elements dA or $[B, dA]$, $A, B \in g^X$, is total in $H(g)$. If $\omega \in H(g)$ is orthogonal to these elements, one has $\langle \omega, dA \rangle = 0$ and $\langle \omega, [B, dA] \rangle = 0$. An easy calculation shows then that $[dA, \omega] = 0$ almost everywhere. From the semisimplicity of G we have then $\omega = 0$. The lemma is now a consequence of the observations that $V(\psi)\, dA = dA$ if $\psi(x) = e$ for all $x \in \operatorname{supp} A$ and of the fact that

$$\lim_{t \to 0} \frac{1}{t} [V(\exp tB)\, dA - dA] = [B, dA]. \quad \blacksquare$$

Let us now discuss the proof of Theorem 2.2. This proof is based on the following.

LEMMA 3.6. *Let* X *be an open bounded ball of* $\mathbb{R}^2$ *and assume that* $|\lambda| < \sqrt{4\pi\rho(x)}$ *for all roots* λ *of* g *and all* $x \in X$. *Let* μ *be the Gaussian measure given by* (3.9) *and let* ν *be the canonical Poisson measure on the corresponding space* Φ. *Then the measures* μ *and* $\mu * \nu$ *are equivalent.*

Proof. The proof is a consequence of Ref. [13, pp. 45, 46]. Note that μ is the transform of the free Euclidean field measure μ_0^D of [13] with Dirichlet boundary conditions on ∂X and mass zero by the transformation $\xi \rightarrow^{\tau} \xi * G_x$, $G_x(y) \equiv G(x-y)$ being the kernel of the inverse of the Laplacian operator with zero boundary conditions. One has

$$\frac{d\tau(\mu * \nu)}{d\tau(\mu)}(\xi) = \int \frac{d\mu_0^D(\xi - \alpha G_x)}{d\mu_0^D(\xi)} \rho(x)\, dx = \int_X : e^{\alpha \xi(x)} : \rho(x)\, dx$$

for ν with support on $\alpha\delta_x$,

the right-hand side being the exponential interaction on X, which is shown in [13] to be in $L^2(d\mu_0^D)$ for all $\alpha < \sqrt{4\pi\rho(x)}$. Since τ induces a unitary transformation of $L^2(d\mu)$ to $L^2(d\mu_0^D)$ this implies $d(\mu * \nu)/d\mu \in L^2(d\mu)$. The general case of ν with support on $\alpha_1 \delta_{x_1} + \cdots + \alpha_n \delta_{x_n}$ is treated by the same method. ∎

The proof of Theorem 2.2 follows immediately from Lemma 3.6, using an atlas on X. The proof of Theorem 2.3 is based on the following lemma, as we shall see in Section 4, together with the proof of the lemma.

LEMMA 3.7. *Let U_0, U_1 be the energy representations of G^X associated with two weights ρ_0 and ρ_1 such that $\rho_0(x) \neq \rho_1(x)$ for all $x \in X$. Let μ_0, μ_1 be the Gaussian measures with Fourier transforms* (3.9) (*the norm* $\| \ \|$ *depending on ρ_0, ρ_1*). *Let ν be any bounded measure on Φ with $\nu\{0\} = 0$. Then one has that $\mu_0 * \nu$ is disjoint from $\mu_1 * \nu$ and from μ_1.*

4. Results on Gaussian Measures and Proof of the Basic Results

The main result of this paper, which yields also the proof of the Theorems 2.1, and 2.3, is the following

THEOREM 4.1. *Let B be a bounded open cube in $\mathbb{R}^d$ and let $\mathscr{D}(B)$ resp. $\mathscr{D}'(B)$ be the space of real C^∞-functions with compact support in B resp. of distributions on B. Let us consider the two bilinear forms on $\mathscr{D}(B)$ defined by*

$$(f, g)_\alpha \equiv \int_B \sum_{i,j=1}^{d} P_{ij}^\alpha(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}\, dx \tag{4.1}$$

for $\alpha = 1, 2$, where $P^\alpha(x) \equiv ((P_{ij}^\alpha(x))$ are real symmetric $d \times d$ matrices, smoothly depending on $x \in B$ and such that for some $m, M > 0$ and all $x \in B$

$$\begin{aligned} m\mathbb{1} \leqslant P^\alpha(x) &\leqslant M\mathbb{1}, \\ P^0(x) &\neq P^1(x), \end{aligned} \tag{4.2}$$

where $\mathbb{1}$ is the unit matrix.

Let μ_α be the Gaussian measures on $\mathscr{D}'(B)$ with Fourier transforms

$$\tilde{\mu}_\alpha(f) \equiv \exp\left(-\frac{(f, f)_\alpha}{2}\right). \tag{4.3}$$

Then for all $d \geqslant 2$ there exists a Borel set Q in $\mathscr{D}'(B)$ with the following properties:

(i) $\mu_0(Q) = 1$;

(ii) $\mu_0(Q + \lambda\delta_x) = 0$ *for all $x \in B$ and all real $\lambda \neq 0$ in the case $d \geqslant 3$, resp. all*

$$|\lambda| > 4\sqrt{2\pi}\left(\frac{M^2}{m^{3/2}}\right) \text{ in the case } d = 2;$$

(iii) $\mu_1(Q + \lambda\delta_x) = 0$ *for all $x \in B$, and all real λ in the case $d \geqslant 3$, resp. all*

$$|\lambda| > 4\sqrt{2\pi}\,\frac{M^2}{m^{3/2}} \text{ or } \lambda = 0 \text{ in the case } d = 2.$$

Before coming to the proof of this Theorem we will show how this result implies Lemma 3.2 and Lemma 3.7. For this we shall derive Lemma 4.2 and Lemma 4.3, using Theorem 4.1.

Let μ_X be the Gaussian measure on the space $\mathscr{D}'(X)$ of distributions on X with Fourier transform

$$\tilde{\mu}_X(f) = \exp\left(-\tfrac{1}{2}\int |df(x)|^2\, \rho(x)\, dx\right), \tag{4.4}$$

where $\rho(x)$ is a C^∞ density with respect to the volume measure dx on X. For any open subset B of X we denote by π_B the projection from $\mathscr{D}'(X)$ onto $\mathscr{D}'(B)$ obtained by restricting the distributions to B. Let μ_B be the image of μ_X by π_B.

LEMMA 4.2. *There exists for $d \geqslant 2$ a Borel subset Q in $\mathscr{D}'(X)$ and a countable basis B_n for all open sets in X such that*

(i) $\mu_X(Q) = 1$

(ii) $\mu_{B_n}(\pi_{B_n}(Q) - \pi_{B_n}(\varphi)) = 0$ *for any $\varphi \in \mathscr{D}'(X)$*

such that $\pi_{B_n}(\varphi) = \lambda\delta_x$ for all $x \in B_n$ and all real $\lambda \neq 0$, in the case $d \geqslant 3$, resp. all $|\lambda| > 4\sqrt{2\pi\rho(x)}$ in the case $d = 2$.

Proof. For $d \geqslant 3$ one can take an arbitrary countable basis B_n for all open subsets in X and charts ψ_n such that $\psi_n(B_n)$ are bounded cubes in $\mathbb{R}^d$.

Applying Theorem 4.1 to $\psi_n(B_n)$ one obtains a countable number of sets Q_n, having the properties of the set Q of Theorem 4.1. Setting then

$$Q = \bigcap_n \pi_{B_n}^{-1}(\psi_n^{-1}(Q_n)) \tag{4.5}$$

we see that Q has properties (i), (ii) of the lemma.

In the case $d = 2$ using dilations in $\mathbb{R}^2$ and the invariance of the Dirichlet form

$$f \to \int \sum_{i=1}^{2} \left(\frac{\partial f}{\partial x_i}\right)^2 dx$$

under dilations, one can suppose that B_n is replaced by $B_{n,p}$ with $B_{n,p} \subset B_n$ and such that the oscillation of the C^∞ function ρ on $B_{n,p}$ is less than $1/p$. We can then procede further as in the case $d \geqslant 3$. ∎

Let now Φ_λ for $\lambda > 0$ be the set of all linear combinations of δ_x for $x \in X$ with coefficients in $\lambda\mathbb{Z}$. Φ_λ is equipped with a Borel structure similar to the one of Φ, replacing $\mathscr{R}$ by $\{\lambda\}$ in the definition given in Section 3.

LEMMA 4.3. *Let Q be as in Lemma 4.2. Define for $\varphi \in \Phi_\lambda - \{0\}$ the subset Q^φ of Q by*

$$Q^\varphi = Q \Bigg| \bigcup_{\substack{x \in \operatorname{supp} \varphi \\ p \in \mathbb{Z} - \{0\}}} (\pi_{B_n}^{-1}(\pi_{B_n}(Q)) - p\lambda\delta_x),$$

where the union on n is over all n such that $B_n \cap \operatorname{supp} \varphi = \{x\}$. Then for $d \geqslant 3$ and all $\lambda > 0$ or for $d = 2$ and $\lambda > 4\sqrt{2\pi\rho(x)}$ for all $x \in X$, we have

(i) $\mu_x(Q^\varphi) = 1$,

(ii) $Q^{\varphi_1} + \varphi_1 \cap Q^{\varphi_2} + \varphi_2 = \varnothing$ *if* $\varphi_1 \neq \varphi_2$,

(iii) *if E is a Borel set in $\Phi_\lambda - \{0\}$, then the set $Q^E = \bigcup_{\varphi \in E} Q^\varphi + \varphi$ is universally measurable in $\mathscr{D}'(X)$.*

Proof. Property (i) is trivial. Property (iii) follows from the fact that if M is the Borel set in $\mathscr{D}'(X) \times (\Phi_\lambda - \{0\}) \times (Z - \{0\}) \times X \times N$ defined by

$$M = \{a, \varphi, \lambda, x, n \mid x \in \operatorname{supp} \varphi, x \in B_n, a - \varphi \in Q \mid \pi_{B_n}^{-1}(\pi_{B_n}(Q)) - p\delta_x\}, \text{ then}$$
$$Q^E = pr_1(M \cap pr_2^{-1}(E))$$

and the result is a consequence of the theorem on the universal measurability of analytic sets (e.g., [6]).

Suppose now (ii) is false, then there exist φ_1, φ_2, $\varphi_1 \neq \varphi_2$, $a_1 \in Q^{\varphi_1}$, $a_2 \in Q^{\varphi_2}$ with $a_1 + \varphi_1 = a_2 + \varphi_2$. As $\varphi_1 \neq \varphi_2$, there exists x such that δ_x

appears with different coefficients λp_1, λp_2 with $p_1, p_2 \in Z$. Taking now n such that $B_n \cap \operatorname{supp} \varphi = \{x\}$ one gets

$$\pi_{B_n}(a_1) + \lambda(p_1 - p_2)\,\delta_x = \pi_{B_n}(a_2),$$

which, however, contradicts the definition of Q^{φ_2}, thus proving the Lemma. ∎

We shall now prove the Lemma 3.2. Taking an orthonormal basis in t, one can suppose that dim $t = 1$, so that $t^X = \mathscr{D}(X)$, $(t^X)' = \mathscr{D}'(X)$, and that Φ is replaced by some Φ_λ with $\lambda > 0$ (see [12, pp. 122–136]). Let E_1, E_2 be Borel subsets on Φ_λ such that $\nu_1(E_1) = 1$, $\nu_2(E_2) = 1$ and $E_1 \cap E_2 = \varnothing$. If $E_1 = \{0\}$ then the function $\varphi \to \mu(Q^\psi - \varphi)$ is ν_2 – almost everywhere zero for some $\psi \neq 0$, as a consequence of Lemma 4.3ii). This allows us to suppose that E_1 and E_2 are in $\Phi_\lambda - \{0\}$. One has

$$(\mu * \nu_i)(Q^{E_i}) = \int_{E_i} \mu\left(\bigcup_{\varphi \in E_i} Q^{\varphi_i} + \varphi_i - \psi\right) d\nu_i(\psi)$$

$$\geqslant \int_{E_i} \mu(Q^\psi + \psi - \psi)\, d\nu_i(\psi) = \int_{E_i} \mu(Q^\psi)\, d\nu_i(\psi) = \nu_i(E_i) = 1.$$

Hence $\mu * \nu_i$, $i = 1, 2$ are disjoint and Lemma 3.2 is proven.

The proof of Lemma 3.7 is obtained in an analogous way. Using Theorem 4.1, one constructs a set Q_0 such that

$$\mu_0(Q_0) = 1, \mu_{1,B_n}(\pi_{B_n}(Q_0) - \pi_{B_n}(\varphi)) = 0$$

for all n such that $\pi_{B_n}(\varphi) = \lambda\delta_x$, with the restriction $|\lambda| = 0$ or $|\lambda| > \sqrt{32\pi\rho(x)}$ in the case $d = 2$. In a symmetric way, there is a set Q_1 such that

$$\mu_1(Q_1) = 1, \mu_{0,B_n}(\pi_n(Q_1) - \pi_n(\varphi)) = 0.$$

Moreover one can construct the sets, for $\varphi \in \Phi_\lambda - \{0\}$,

$$Q_0^\varphi = Q_0 \Bigg| \bigcup_{\substack{n, x \in \operatorname{supp} \varphi \\ p \in Z}} \pi_{B_n}^{-1}(\pi_{B_n}(Q_1)) - p\delta_x$$

$$Q_1^\varphi = Q_1 \Bigg| \bigcup_{\substack{n, x \in \operatorname{supp} \varphi \\ p \in Z}} \pi_{B_n}^{-1}(\pi_{B_n}(Q_0)) - p\delta_x,$$

where the unions are taken over n such that $B_n \cap \operatorname{supp} \varphi = \{x\}$. Then one shows as in Lemma 4.3 that

$$\bigcup_{\varphi \in \Phi_1 - \{0\}} Q_0^\varphi + \varphi \qquad \text{and} \qquad \bigcup_{\varphi \in \Phi_1 - \{0\}} Q_1^\varphi + \varphi$$

are mutually disjoint, measurable and of measure 1 for $\mu_0 * \nu$ and $\mu_1 * \nu$, respectively, if $\nu\{0\} = 0$, and this ends the proof of Lemma 3.7.

It remains now to prove the main Theorem 4.1. The main idea of the proof is a refinement of the one used for the study of the exponential interaction in [9]. It will be useful to introduce the Gaussian random field $\xi(x)$ on B with zero mean and covariance

$$E(\xi(x)\,\xi(y)) = G_{P\alpha}(x, y), \tag{4.6}$$

where, for $d = 3$, $G_{P\alpha}$ is the kernel of the inverse of the operator $-\Delta_{P\alpha}$ with

$$\Delta_{P\alpha}\varphi = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(P_{ij}^{\alpha}\frac{\partial}{\partial x_j}\varphi\right) \tag{4.7}$$

and, for $d = 2$, $G_{P\alpha}$ is the kernel of $-\Delta_{P\alpha}(-\Delta_{P\alpha} + 1)^{-2}$. Note that for $P = \mathbb{1}$, $G_{\mathbb{1}}(x, y)$ is a function, also denoted by $G_{\mathbb{1}}$, of $x - y$. The measure μ_α corresponding to process (4.4) has $\exp(-\frac{1}{2}\iint \varphi(x)\, G_{P\alpha}(x, y)\,\varphi(y)\,dx\,dy)$ as Fourier transform and is the image of μ^α by the measurable transformation $\xi \to \xi * H_{P\alpha}$ acting on $\mathscr{D}'(B)$, where $H_{P\alpha}(x, y)$ is $G_{P\alpha}(x, y)$ if $d \geqslant 3$ and $H_{P\alpha}(x, y)$ is the kernel of $(-\Delta_{P\alpha} + 1)^{-1}$ for $d = 2$. In this description, the image of δ_x for $x \in B$ is the distribution $H_{P\alpha,x}$, where

$$H_{P\alpha,x}(y) \equiv H_{P\alpha}(x, y). \tag{4.8}$$

We shall now use regularizations and dilations of the field ξ. Let namely φ be a smooth function of compact support in $\mathbb{R}^d$ such that $\int \varphi(x)\,dx = 1$. For $k > 0$ we define

$$\varphi_k(x) \equiv k^d \varphi(kx). \tag{4.9}$$

Putting $\xi_k \equiv \varphi_k * \xi$ we obtain a Gaussian random field with mean zero and covariance $E(\xi_k(x)\,\xi_k(y)) = G_{P\alpha,k}(x, y)$, where

$$G_{P\alpha,k}(x, y) \equiv \varphi_k * G_{P\alpha} * \varphi_k. \tag{4.10}$$

It is easily shown that one has for $d \geqslant 3$

$$G_{P\alpha,k}(k^{-1}x, k^{-1}y) = k^{d-2} G_{P\alpha,k^{-1}_{,1}}(x, y), \tag{4.11}$$

where

$$P_{i,j}^{\alpha,k}(x) \equiv P_{i,j}^{\alpha}(kx), \tag{4.12}$$

and for $d = 2$ one has

$$\frac{m}{2\pi M^2}\log k - k_1 \leqslant G_{P\alpha,k}(k^{-1}x, k^{-1}y) \leqslant \frac{1}{2\pi m}\log k + k_2, \tag{4.13}$$

for some positive constants k_1, k_2. Equation (4.13) follows from (4.2) using

$$\frac{m^2}{M^2}\left(-\Delta_{\mathbb{1}}+\frac{1}{m}\right)^{-2}(-\Delta_{\mathbb{1}}) \leqslant -\Delta_{p\alpha}(-\Delta_{p\alpha}+1)^{-2} \leqslant \frac{1}{m}\left(-\Delta_{\mathbb{1}}+\frac{1}{m}\right)^{-1}, \quad (4.14)$$

together with the logarithmic singularity of $(-\Delta_{\mathbb{1}}+1)^{-1}$ in dimension $d=2$. For any $b>0$ and any integer n let us now define the following sets

$$N_{n,b} \equiv \{\xi \in \mathscr{D}'(B) \mid |\xi_{2^n}(x)| \leqslant b n^{1/2} G_{\mathbb{1},2^n}(0)^{1/2} \text{ for all } x \in B\} \quad (4.15)$$

and

$$Q_b \equiv \bigcup_N \bigcap_{n \geqslant N} N_{n,b}. \quad (4.16)$$

We shall now compute $\mu_\alpha(N_{n,b}+\lambda H_x^\alpha)$. The image of $N_{n,b}+\lambda H_x^\alpha$ under the mapping $\xi \to \xi_{2^n}$ is the translation of the image of $N_{n,b}$ by $H^\alpha_{x,2^n}$, hence we need only estimate the Radon–Nikodym derivative

$$\frac{d\mu_{\alpha,2^n}(\xi+\lambda H^\alpha_{x,2^n})}{d\mu_{\alpha,2^n}(\xi)}. \quad (4.17)$$

This is, however, equal to

$$\exp\left(\lambda\xi_{2^n}(x)-\frac{\lambda^2}{2}G_{p\alpha,2^n}(x,x)\right) \equiv\ :\exp\lambda\xi_{2^n}(x):, \quad (4.18)$$

the notation $:\exp\lambda\xi_{2^n}(x):$ being the one familiar from quantum field theory, [13, 9]. But for $\xi \in N_{n,b}$ we have

$$\lambda\xi_{2^n}(x)-\frac{\lambda^2}{2}G_{p\alpha,2^n}(x,x) \leqslant |\lambda|\, b n^{1/2} G_{\mathbb{1},2^n}(0)-\frac{\lambda^2}{2}G_{p\alpha,2^n}(x,x). \quad (4.19)$$

In the case $d \geqslant 3$ one has $G_{p\alpha,2^n}(x,x) \geqslant M^{-1}G_{\mathbb{1},2^n}(0)$ hence by (4.11) we have that (4.18) goes to zero as $n\to\infty$. In the case $d=2$ we can draw the same conclusion if

$$|\lambda| > \frac{2M^2 b}{m}\sqrt{\frac{2\pi}{\log 2}}, \quad (4.20)$$

as a consequence of (4.13). It follows thus that for any $b>0$, $x \in B$ and all λ if $d \geqslant 3$ and all λ satisfying (4.20) if $d=2$, one has

$$\mu_\alpha(Q_b+\lambda H_x^\alpha)=0. \quad (4.21)$$

On the other hand, let B_0 be a fixed square and let $\tilde{B}$ be any of the 2^{nd} translates of B_0 forming a covering of the cube $2^n B$. Introducing for any such $\tilde{B}$ the set

$$M_{n,b,\tilde{B}} \equiv \left\{ \xi \in \mathscr{D}'(B) \,\middle|\, \frac{|\xi_{2^n}(2^{-n}x)|}{G_{1\!\!1,2^n}(0)^{1/2}} \leqslant bn^{1/2} \text{ for all } x \in \tilde{B} \right\}, \tag{4.22}$$

we see that

$$N_{n,b} = \bigcap_{\tilde{B}} M_{n,b,\tilde{B}}. \tag{4.23}$$

The processes $x \to G_{1\!\!1,2^n}(0)^{-1/2}\, \xi_{2^n}(2^{-n}x)$ are associated with Gaussian measures coming from matrices as in (4.2) which are translations or dilations (as in (4.12)) of P^α. By (4.2) all the covariance of these processes are smaller than $1/m$, moreover they are almost everywhere bounded and continuous. It follows that the functions

$$\xi \to \exp\left(\beta \sup_{x \in B_0} \frac{|\xi_{2^n}(2^{-n}x)|^2}{G_{1\!\!1,2^n}(0)}\right) \tag{4.24}$$

are uniformly integrable for all $\beta < m/2$ (see [7a, p. 14]) and this implies for $\beta < (m/2)$.

$$\mu_\alpha\left(\complement\, M_{n,b,\tilde{B}}\right) \leqslant Ce^{-\beta b^2 n} \tag{4.25}$$

for some constant C. Hence from (4.23)

$$\mu_\alpha\left(\complement\, N_{n,b}\right) \leqslant C2^{dn} e^{-\beta b^2 n}. \tag{4.26}$$

Taking now b such that

$$b > 2\left(\frac{\log 2}{m}\right)^{1/2} \tag{4.27}$$

one can choose $\beta < m/2$ with $\gamma \equiv 2 \log 2 - \beta^{b^2} > 0$, obtaining then

$$\mu_0\left(\complement \bigcap_{n \geqslant N} N_{n,b}\right) \leqslant \frac{C}{1 - e^{-\nu}}\, e^{-\nu N}. \tag{4.28}$$

Consequently we have

$$\mu_0(Q_b) = 1. \tag{4.29}$$

From this and (4.21) we see that the set Q_b has properties (i, ii) of the set Q of Theorem 4.1, hence this part of the theorem is proven.

To prove part (iii) of Theorem 4.1 we first observe that μ_0, μ_1 as defined by (4.3) are disjoint, being Gaussian measures to different scalar products, by well known results on Gaussian measures.

On the other hand, for the set Q_b, whose definition is independent of the μ_α, $\alpha = 1, 2$, we have $\mu_\alpha(Q_b) = 1$, $\alpha = 1, 2$ under above choices of λ, b for $d = 2$. But for $d \geqslant 3$ or $d = 2$, $|\lambda| > 4\sqrt{2\pi}\, M^2/m^{3/2}$, we have $\mu_\alpha(Q_b + \lambda\delta_x) = 0$, as shown above. Hence the set Q_b has also property (iii) of Theorem 4.1. ∎

5. The One Dimensional Case

Let G be a semisimple compact Lie group and g its Lie algebra, equipped with the inner product $(\xi_1, \xi_2) = -K(\xi_1, \xi_2)$, where K is the Killing form. g has then a positive definitive inner product $(,)$ which is invariant under the coadjoint action of G on g.

Let $\xi(t)$ be the generalized stochastic process [15] which gives the standard white noise process on g, i.e., for any $f \in L_2(R; g)$ we have that

$$\langle \xi, f \rangle = \int \xi(t) f(t)\, dt \tag{5.1}$$

is a Gaussian random variable with mean zero and variance $\|f\|_2^2 = \int (f(t), f(t))\, dt$. The standard Wiener process starting at 0 for $t = 0$ is then given by

$$W(t) = \int_0^t \xi(\tau)\, d\tau. \tag{5.2}$$

It is well known that $W(t)$ is continuous for almost all realizations. The standard Brownian motion $\eta(t)$ on G is the stochastic process on G given by the following stochastic differential equation

$$d\eta(t)\eta^{-1}(t) = \xi(t). \tag{5.3}$$

Another way to describe the solutions of (5.3) is to observe that the Wiener process $W(t)$ has almost surely continuous realizations hence defines a measure μ_W on $C(R^+; g)$, the space of continuous functions on R^+ with values in g. Moreover the integral equation

$$\int_0^t d\eta(\tau)\eta^{-1}(\tau) = W(t) \tag{5.4}$$

has a unique continuous solution $\eta(t; h)$ such that $\eta(0; h) = h$ for any $h \in G$, for any $W \in C(R^+; g)$. (See [16].)

This gives a mapping from $C(R^+; g)$ into $C(R^+, G)$. The image μ_h of the Wiener measure μ_W by this map is the measure on $C(R^+, G)$ corresponding to the standard Brownian process $\eta(t;h)$ on G which starts at h at $t=0$. $\eta(t;h)$ is a Markov process and it is well known that $\eta(t-s;h)$ converges in probability to a homogeneous process $\eta(t)$ as $s \to \infty$, independent of h. $\eta(t)$ is called the standard Brownian motion on G and satisfies (5.3). To $\eta(t)$ there corresponds a measure μ on $C(R;G)$ which we call the standard Brownian motion measure on $C(R;G)$. Let $S=R/Z$ and let μ_0 be the measure on $C(S;G)$ obtained by conditioning the measure μ on $C(R,G)$ with respect to the condition $\eta(0)=\eta(1)$. We shall refer to this measure μ_0 as the standard Brownian motion measure on $C(S,G)$. It is well known that the measure μ does only depend on the Riemann structure on G, i.e., the left invariant Riemann structure given by the negative Killing form. That is to say for any Riemann manifold there is a unique standard Brownian motion, and in the case of a compact semisimple Lie-group with the left invariant Riemann structure given by the negative Killing form the unique standard Brownian motion is given by $\eta(t)$.

Now since the adjoint action of G, as well as the inversion $\eta \to \eta^{-1}$, leave the Riemann structure invariant we have the following proposition.

PROPOSITION 5.1. *The standard Brownian motion measures on $C(R,G)$ and $C(S,G)$ are both invariant under the adjoint action of G as well as under the inversion $\eta(t) \to \eta(t)^{-1}$.* ∎

Consider now the Sobolev Lie groups $H(S,G) \subset C(S,G)$ and $H(R,G) \subset C(R,G)$ of maps such that

$$\int_0^1 |d\eta(\tau)\eta^{-1}(\tau)|^2\, d\tau < \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} |d\eta(\tau)\eta^{-1}(\tau)|^2\, d\tau < \infty, \tag{5.5}$$

respectively, where $|\ |^2 = (\,,)$. $H(S,G)$ and $H(R,G)$ are complete metric groups in the Sobolev metrics given by the square roots of the expressions in (5.5). See [3]. Now (5.3) gives us a map $\eta(t) \to \xi(t) = \eta^{-1}(t)\, d\eta(t)$ of $C(R,G)$ in $S'(R, g)$, the space of tempered distributions with values in g, which takes the standard Brownian motion on G into the standard white noise in g. Moreover we see that for any $\varphi \in H(R,G)$ we have under this map

$$\eta(t)\varphi(t) \to d(\eta\varphi)(t)(\eta\varphi)^{-1}(t) = \varphi(t)\, d\eta(t)\eta^{-1}(t)\varphi^{-1}(t) + d\varphi(t)\varphi^{-1}(t). \tag{5.6}$$

Hence if $\eta(t) \to \xi(t)$ then

$$\eta(t)\,\varphi(t) \to \varphi(t)\,\xi(t)\,\varphi^{-1}(t) + d\varphi(t)\,\varphi^{-1}(t). \tag{5.7}$$

That is in the space of white noise, i.e., $S'(R, g)$ we see that the right multiplication by $H(R, G)$ is realized as the action of the Sobolev Lie group on the dual of the g-valued one forms on R. The same formula holds of course for the Sobolev Lie group $H(S, G)$. Hence we get that μ as well as μ_0 are quasi invariant under right translations by elements in $H(R, G)$, respectively, $H(S, G)$ and that the unitary representation of $H(R, G)$ ($H(S, G)$) in $L_2(\mu)$ ($L_2(\mu_0)$) given by

$$(U_\varphi f)(\eta) = \left(\frac{d\mu(\eta\varphi)}{d\mu(\eta)}\right)^{1/2} f(\eta\varphi) \tag{5.8}$$

(resp. the same with μ replaced by μ_0) is unitarity equivalent to the cyclic component of the vacuum of the energy representation of the Sobolev Lie group $H(R, G)$, ($H(S, G)$). Hence we have proved the following theorem.

THEOREM 5.2. *The unitary representations of the Sobolev Lie groups $H(R, G)$ and $H(S, G)$ induced by the right translation $\eta \to \eta\varphi$ on $L_2(d\mu)$, respectively $L_2(d\mu_0)$, in the following way*:

$$(U_\varphi f)(\eta) = \left(\frac{d\mu(\eta\varphi)}{d\mu(\eta)}\right)^{1/2} f(\eta\varphi)$$

and

$$U_\varphi^0 f(\eta) = \left(\frac{d\mu_0(\eta\varphi)}{d\mu_0(\eta)}\right)^{1/2} f(\eta\varphi)$$

are unitarily equivalent to the cyclic component of the vacuum of the energy representation of the corresponding Sobolev Lie groups. Moreover the unitary equivalences are given by the mapping $\eta \to d\eta(t)\,\eta^{-1}(t)$ from $C(R, G)$ ($C(S, G)$) into the dual space of the g-valued one forms on $R(S)$. ∎

By Proposition 5.1 we have that μ and μ_0 are both invariant under $\eta(t) \to \eta(t)^{-1}$. Now since this mapping takes right translations into left translations, we get that μ as well as μ_0 are quasi invariant under left translation by elements in the corresponding Sobolev Lie groups. Hence the unitary left translation V_φ and V_φ^0 defined by

$$(V_\varphi f)(\eta) = \left(\frac{d\mu(\varphi^{-1}\eta)}{d\mu(\eta)}\right)^{1/2} f(\varphi^{-1}\eta) \tag{5.9}$$

and correspondingly for V_φ^0 are unitary representations of the Sobolev Lie groups which are unitarily equivalent to U_φ and U_φ^0, respectively, hence again unitarily equivalent to the energy representation. On the other hand it is obvious that U and V commute and therefore we get the following theorem.

THEOREM 5.3. *The energy representation of the Sobolev Lie groups $H(R, G)$ and $H(S, G)$ is reducible. In fact, in the cyclic component of the vacuum, the commutant to the algebra generated by the energy representation contains in both cases a unitary representation which is equivalent with the energy representation.*

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