On the Cauchy problem for the evolution $p$-Laplacian equations with gradient term and source

Songzhe Lian a, Hongjun Yuan a,∗, Chunling Cao a, Wenjie Gao a, Xiaojing Xu a, b

a Institute of Mathematics, Jilin University, Changchun, Jilin 130012, PR China
b Institute of Applied Physics and Computational Mathematics, Beijing 100088, PR China

Received 12 June 2006; revised 7 November 2006
Available online 9 January 2007

Abstract

In this paper, the authors study the equation $u_t = \text{div}(|Du|^{p-2} Du) + |u|^{q-1} u - \lambda |Du|^l$ in $\mathbb{R}^N$ with $p > 2$. We first prove that for $1 \leq l \leq p - 1$, the solution exists at least for a short time; then for $\frac{p}{2} \leq l \leq p - 1$, the existence and nonexistence of global (in time) solutions are studied in various situations.

© 2006 Elsevier Inc. All rights reserved.

Keywords: Cauchy problem; $p$-Laplacian equation; Gradient term; Blow-up

1. Introduction

Consider the Cauchy problem

\begin{align*}
  u_t &= \text{div}(|Du|^{p-2} Du) + |u|^{q-1} u - \lambda |Du|^l \quad \text{in } S_T, \\
  u(x, 0) &= u_0(x) \in L^1_{\text{loc}}(\mathbb{R}^N),
\end{align*}

where $S_T = \mathbb{R}^N \times (0, T]$, $p > 2$, $q \geq 1$, $l \geq 1$ and $\lambda$ are constants.

* Corresponding author.
E-mail addresses: hjy@mail.jlu.edu.cn, hjy@jlu.edu.cn (H. Yuan).

0022-0396/$ – $ see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2006.11.014
For the case $p = 2$, this equation was introduced by Chipot and Weissler [3] in order to investigate the effect of a damping term on the global existence or nonexistence. Also, Souplet in [13] proposed a model in population dynamics, where the equations describe the evolution of the population density of a biological species under the effect of certain natural mechanism.

Several authors have studied the existence of nonglobal positive solutions and have given various sufficient conditions for blowing-up under certain assumptions on $q, l, \lambda$, etc. (see for instance [2, 9, 14] and [15]).

For the case $p > 2$ without the gradient term (i.e. $\lambda = 0$), the problem (1.1)–(1.2) was also investigated by many authors (see [7, 8, 10, 12, 17] and [18]). To the best of our knowledge, the first result on blow-up in $p$-Laplacian equations was obtained in [16].

Motivated by the ideas in [18] and [1], we in this paper study the problem with $p > 2$, $q, l$ and $\lambda$ in various situations. We first state the definition of solutions to the problem as follows.

**Definition 1.1.** A measurable function $u(x, t)$ defined in $S_T$ is called a weak solution of (1.1)–(1.2) if for every bounded open set $\Omega$ with smooth boundary $\partial \Omega$,

$$u \in C\left(0, T : L^1(\Omega)\right) \cap L^p_\text{loc}\left(0, T : W^{1,p}_\text{loc}(\Omega)\right) \cap L^\infty_\text{loc}(S_T), \quad (1.3)$$

and

$$\int_\Omega u(x, t) \phi(x, t) \, dx + \int_0^t \int_\Omega \left[-u \phi_t + |Du|^{p-2} D u \cdot D \phi \right] \, dx \, d\tau = \int_0^t \int_\Omega |u|^{q-1} u \phi \, dx \, d\tau - \lambda \int_0^t \int_\Omega |Du|^l \phi \, dx \, d\tau + \int_\Omega u(x, t_0) \phi(x, t_0) \, dx, \quad (1.4)$$

for all $0 \leq t_0 < t \leq T$ and all testing function $\phi \in C^1(\overline{\Omega} \times [0, T])$, $\phi = 0$ near $\partial \Omega \times (0, T)$. Moreover

$$\lim_{t \to 0} \int_{B_r} |u(x, t) - u_0(x)| \, dx = 0, \quad \forall r > 0. \quad (1.5)$$

Weak subsolutions (respectively supersolutions) are defined in the same way except that the = in (1.4) is replaced by $\leq$ (respectively $\geq$) and $\phi$ is taken to be nonnegative.

If (1.5) is replaced by

$$\lim_{t \to 0} \int_{R^N} u(x, t) \eta(x) \, dx = \int_{R^N} \eta(x) \, d\mu, \quad \forall \eta \in C^1_0(R^N), \quad (1.6)$$

where $\mu$ is a $\sigma$-finite Borel measure in $R^N$, then we say that $u$ is a weak solution of (1.1) with initial datum

$$u(x, 0) = \mu \quad \text{on } R^N. \quad (1.7)$$
We use $\gamma(a_1, a_2, \ldots, a_n)$ to denote positive constants depending only on specified quantities $a_1, a_2, \ldots, a_n$. Set

$$
\text{sgn}_\eta s = \begin{cases} 
1 & \text{if } s > \eta, \\
\frac{s}{\eta} & \text{if } -\eta \leq s \leq \eta, \\
-1 & \text{if } s < -\eta.
\end{cases}
$$

For $f \in L_{\text{loc}}(R^N)$, we define

$$
\|f\|_h = \sup_{x \in R^N} \left( \frac{1}{|B_1(x)|} \int_{B_1(x)} |f(y)|^h \, dy \right)^{1/h}. 
$$

(1.8)

If $\mu$ is a locally finite Borel measure in $R^N$, we set

$$
\|\mu\|_1 = \sup_{x \in R^N} \frac{1}{|B_1(x)|} \int_{B_1(x)} d|\mu|,
$$

(1.9)

where $d|\mu|$ denotes the variation of $\mu$.

Space of functions with finite norm $\|f\|_h$ has been investigated in [1].

Let us now state our main results.

**Theorem 1.1.** Let $1 \leq l \leq p - 1$. Assume

$$
\|u_0\|_h < \infty,
$$

(1.10)

where

(A) $h = 1$ if $1 \leq q < p - 1 + p/N$ and $h > (N/p)(q - p + 1)$ if $q \geq p - 1 + p/N$.

Then there exist a constant $\gamma = \gamma(N, p, q) \geq 1$ and a positive constant $T_0$ defined by

$$
T_0 + T_0 \|u_0\|_h^{p-2} + T_0^{1-(N(q-p+1)/(ph))} \|u_0\|_h^{q-1} = \gamma^{-1}
$$

(1.11)

such that there exists a weak solution $u$ to (1.1)–(1.2) in the strip $S_T$ satisfying, for all $0 < t < T_0$,

$$
\left\|u(\cdot, t)\right\|_h \leq \gamma \left(\|u_0\|_h + 1\right),
$$

(1.12)

$$
\left|u(x, t)\right| \leq \gamma t^{-\kappa_h/\kappa} \left(\|u_0\|_h^{\kappa_h/\kappa} + 1\right),
$$

(1.13)

$$
\left|Du(x, t)\right| \leq \gamma t^{-\kappa_h/(\kappa + \kappa_h)} \left(\|u_0\|_h^{(1+h(p-2)/\kappa_h)} + 1\right), \quad \forall x \in R^N,
$$

(1.14)

$$
\int_0^t \int_{B_1(x_0)} |Du|^{p-1} \, dx \, d\tau \leq \gamma t^{\kappa_h/\kappa} \times G(t)^{1+h(p-2)/\kappa_h} + \gamma t G(t),
$$

(1.15)

where $\kappa = N(p-2) + p$, $\kappa_h = N(p-2) + ph$ and $G(t) = \sup_{0 < \tau < t} (\int_{B_2(x_0)} u^h(x, \tau) \, dx)^{1/h}$. 

Remark 1.1. The proof of Theorem 1.1 (see Section 2) shows that Theorem 1.1 holds in the following cases:

(1) For \(1 \leq q < p - 1 + p/N\), \(u_0\) can be replaced by a \(\sigma\)-finite Borel measure \(\mu\) in \(\mathbb{R}^N\), satisfying
\[
\|\mu\|_1 < \infty;
\] (1.16)

(2) \(|u|^{q-1}u - \lambda|Du|^l\) in Eq. (1.1) can be replaced by \(F(x,t,u,Du)\) with
\[
|F| \leq \psi(x,t) + \gamma|u|^q + \gamma|Du|^l,
\] (1.17)
where \(\psi(x,t) \in L^\infty(S_T)\) and \(\gamma \geq 0\) is a constant.

We say that a solution \(u : S_T \to \mathbb{R}^+\) satisfying (1.12)–(1.14) is a solution of class \(\mathcal{R}\), if it also fulfills the requirements:

\[
\begin{align*}
\|u(\cdot,t)\|_1 &\leq \gamma, \quad (1.12') \\
|u(x,t)| &\leq \gamma t^{-\delta}, \quad \forall x \in \mathbb{R}^N, \quad (1.13') \\
|Du(x,t)| &\leq \gamma t^{-\delta_1}, \quad \forall x \in \mathbb{R}^N, \quad (1.14')
\end{align*}
\]
where
\[
\delta = N/\kappa, \quad \delta_1 = (N+1)/\kappa.
\]

Theorem 1.2. Assume \(l \geq p/2\). If \(u, v\) in \(\mathcal{R}\) are two solutions of (1.1)–(1.2) corresponding to the same initial datum \(u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)\). Then \(u_1 = u_2\) in \(S_T\).

Generally, there exists no global (in time) solution to problem (1.1)–(1.2) (see Corollary 2.1 in [18] for \(\lambda = 0\)).

Example. If \(\frac{p}{2} \leq l \leq p - 1\), take the initial value function \(u_0(x) \equiv \epsilon\), such that \(\|u_0\|_h = \epsilon < \infty\) (\(\forall h \geq 1\)) and then there exists a solution \(u\) to (1.1)–(1.2) satisfying \(T^* < \infty\), where \(\epsilon > 0\) is a constant.

Let \(v(x,t) = (\epsilon^{1-q} - (q - 1)t)^{(-1)/(q-1)}\). By comparison principle (Lemma 4.4) we have \(u(x,t) \geq v(x,t) > 0\). But \(v(x,t)\) (also \(\|v\|_1\)) \(\to \infty\) at a finite time.

Denote
\[
T^* = \sup\{T \mid \text{there exists a solution } u \text{ to (1.1)–(1.2) in } 0 < t < T\}.
\]

Theorem 1.3. Assume \(\frac{p}{2} \leq l \leq p - 1\) and \(q < p - 1 + p/N\). If there exists a solution \(u\) such that \(0 < T^* < \infty\), then
\[
\lim_{t \to T^*} \|u\|_1 = \infty.
\]
Theorem 1.4. Assume $p/2 \leq l \leq p - 1$, $p - 1 < q < p - 1 + p/N$. Then for any initial function $\psi \in C_0^1(\mathbb{R}^N)$ with $\psi \geq 0, \psi \neq 0$.

(i) There exists a $\mu_0 > 0$ (depending on $\psi$) such that for all $\mu > \mu_0$, the solution of (1.1)–(1.2) in $\mathfrak{R}$ with initial data $u_0 = \mu \psi$ blows up in finite time, i.e. $T^* < \infty$, and

$$\lim_{t \to T^*} \|u\|_1 = \infty.$$

(ii) There is a $C(\psi) > 0$ such that

$$T^*[\mu \psi] \leq C(\psi)/\mu^{q-1}, \quad \mu \gg 1.$$ 

Remark 1.2. In [8], the authors proved that when the solution blows up near the origin, one has $u(x, T_0) \simeq |x|^{\frac{p}{q} - \frac{p}{q} - \frac{1}{p - 1}} = f(x)$ ($q > p - 1$), where $T_0$ is the blowing-up time in the following sense:

$$\lim_{t \to T_0} \sup_{x \in \mathbb{R}^N} u(x, t) = +\infty.$$ 

By direct calculation, we know that for $p - 1 < q < p - 1 + p/N$, $\|f(x)\|_1 = +\infty$, and then we have $T^* \leq T_0$. Similarly, we may have $T^* \geq T_0$. Therefore $T^* = T_0$. We have in fact proved that the blowing-up time for the two different definitions of blowing-up is the same. Certainly, in further study of the problem for $\lambda \neq 0$ we hope to have some precise estimates for the solutions as that in [8].

Next we consider the Cauchy problem in the case $\lambda \geq 0$.

Theorem 1.5. Assume $u_0 \geq 0, \lambda \geq 0$ and $1 \leq l \leq p - 1$. Then for any $q > p - 1 + p/N, h > (N/p)(q - p + 1)$, there exists a constant $\gamma = \gamma(N, p, q, h)$ such that the Cauchy problem (1.1)–(1.2) has a solution defined for all positive time, provided that the initial datum satisfies

$$\|u_0\|_{h, \mathbb{R}^N} + \|u_0\|_{1, \mathbb{R}^N} < \gamma.$$ 

1.8)

2. Fundamental lemmas

We first prove some estimates which are the main tools in the proof of Theorem 1.1 as well as in the characterization of solutions of (1.1).

Proposition 2.1. Let $u$ be any locally bounded continuous weak solution of (1.1) in $S_T$ for some $0 < T < \infty$. Then for fixed $h \geq 1$ there exists a constant $\gamma$ depending only on $N, p, q, h$ such that for every ball $B_{2\rho}(x_0)$ and for all $0 < t < T$,

$$\|u(\cdot, \tau)\|_{\infty, B_{2\rho}(x_0)}^{p-2} + \sup_{x \in B_{2\rho}(x_0)} |u|^q \leq \tau^{-1}, \quad \tau \in (0, t).$$ 

(2.1)
Furthermore, the following estimate holds

$$\|u(\cdot, t)\|_{\infty, B_\rho(x_0)} \leq \gamma t^{-(N+p)/\kappa_h} \left( \int_0^t \int_{B_{2\rho}(x_0)} |u|^h \, dx \, d\tau \right)^{p/\kappa_h} + \gamma,$$

where $\kappa_h = N(p - 2) + ph$.

**Proof.** Let $\rho > 0$, $\sigma \in (0, \frac{1}{2}]$ be fixed, let $k > 0$ to be chosen and for $n = 0, 1, 2, \ldots$, set

$$\rho_n = \rho + \frac{\sigma}{2^n} \rho, \quad t_n = \frac{t}{2} - \left( \frac{\sigma}{2^{n+1}} \right)^p t, \quad k_n = k - \frac{k}{2^{n+1}},$$

$$B_n = B_{\rho_n}(x_0), \quad Q_n = B_n \times (t_n, t), \quad 0 < t_n < t \leq T.$$

Let $\xi_n(x, t)$ be a smooth cutoff function in $Q_n$ such that

$$\xi_n = 1 \text{ on } Q_{n+1}, \quad 0 \leq \frac{\partial \xi_n}{\partial t} \leq \gamma \frac{2^{np}}{\sigma \rho t}, \quad |D\xi_n| \leq \gamma \frac{2^{n+1}}{\sigma \rho}.$$

In Definition 1.1, we take the testing function $\phi = (u - k_{n+1})^\frac{h}{\xi_n} \xi_n^p = \max\{0, u - k_{n+1}\}^\frac{h}{\xi_n} \xi_n^p$ to obtain

$$\frac{1}{h + 1} \int_{B_n(t')} (u - k_{n+1})^\frac{h+1}{\xi_n} \xi_n^p \, dx$$

$$+ \frac{1}{h + 1} \int_{B_n(t')} \int_{t_n}^{t'} |Du|^{p-2} Du \cdot D(u - k_{n+1})^\frac{h}{\xi_n} \xi_n^p \, dx \, d\tau$$

$$+ \frac{1}{h + 1} \int_{B_n(t')} \int_{t_n}^{t'} (u - k_{n+1})^\frac{h}{\xi_n} \xi_n^p \, dx \, d\tau$$

$$= \frac{1}{h + 1} \int_{B_n(t')} \int_{t_n}^{t'} |Du|^l (u - k_{n+1})^\frac{h}{\xi_n} \xi_n^p \, dx \, d\tau,$$

where $t_n < t' < t$. By Schwarz’s inequality
(a) \[
\left| \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})^h \xi_n^{p-1} |Du|^{p-2} Du \cdot D\xi_n \, dx \, d\tau \right|
\leq \frac{h}{2} \int_{t_n}^{t'} \int_{B_n} |Du|^{p-2} Du \cdot D(u - k_{n+1})_+ (u - k_{n+1})_+^{h-1} \xi_n^p \, dx \, d\tau
\]
\[+ \gamma \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_+^{h+p-1} |D\xi_n|^p \, dx \, d\tau,\]

(b) \[
\left| \int_{t_n}^{t'} \int_{B_n} \lambda |Du|^l (u - k_{n+1})^h \xi_n^p \, dx \, d\tau \right|
\leq \frac{h}{2} \int_{t_n}^{t'} \int_{B_n} |Du|^p (u - k_{n+1})_+^{h-1} \xi_n^p \, dx \, d\tau
+ \gamma \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_+^{h+l/(p-l)} \xi_n^p \, dx \, d\tau.
\]

If \(1 \leq l/(p-l) \leq p-1\) (i.e. \(l \geq p/2\)), by (2.1)

(b') \[
\gamma \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_+^{h+l/(p-l)} \xi_n^p \, dx \, d\tau
\leq \gamma \left( \frac{1}{l} \right)^{(l/(p-l)-1)/(p-2)} \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_+^{h+1} \, dx \, d\tau
\]
\[\leq \gamma \frac{1}{l} \int_{t_n}^{t'} \int_{B_n} (u - k_{n})_+^{h+1} \, dx \, d\tau.
\]

If \(l/(p-l) < 1\) then for

(b0) \[k \geq 1,\]

(b'') \[
\gamma \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_+^{h+l/(p-l)} \xi_n^p \, dx \, d\tau
\leq \gamma \left( \frac{1}{k_{n+1} - k_n} \right)^{1-l/p-l} \int_{t_n}^{t'} \int_{B_n} (u - k_n)_+^{h+1} \, dx \, d\tau
\]
\[\leq \gamma 2^n \int_{t_n}^{t'} \int_{B_n} (u - k_n)_+^{h+1} \, dx \, d\tau.
\]
Notice that if \( u/2 > k_n \),

\[
(u - k_n)^{h+1}_+ \geq \frac{u}{2}(u - k_n)^h_+ \geq C(u - k_{n+1})^h_+ u;
\]

if \( k_{n+1} \leq u \leq 2k_n \)

\[
(u - k_n)^{h+1}_+ \geq (u - k_n)^h_+ (k_{n+1} - k_n) \geq 2^{-n-3}(u - k_{n+1})^h_+ u.
\]

Thus we have

\[
\left(\int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})^h_+ u^{q/p} \xi_n^p \, dx \, d\tau \right) \\
\leq \gamma 2^n \left(\int_{t_n}^{t'} \int_{B_n} u^{q-1}(u - k_n)^{h+1} \, dx \, d\tau \right).
\]

Substituting (a), (b), (b'), (b''), (c) into (2.3), we obtain

\[
\text{ess sup}_{t_n < \tau < t} \int_{B_n(\tau)} (u - k_n)^{h+1}_+ \xi_n^p \, dx \\
+ \int_Q \left| D((u - k_{n+1})^{(p-1)/p} \xi_n^p) \right|^p \, dx \, d\tau \\
\leq \gamma 2^n \sigma p I (1 + M) \int_Q (u - k_n)^{h+1}_+ \, dx \, d\tau,
\]

where

\[
M = \sup_{0 < \tau < t} \left\{ \|u(\cdot, \tau)\|_{\infty, B_2(x_0)}^{p-2} \rho^{-p} + \sup_{x \in B_2(x_0)} |u|^{q-1} \right\}.
\]

By Nirenberg–Golovkin inequality [11, p. 64],

\[
\int_Q \xi_n^d (u - k_{n+1})^b_+ \, dx \, d\tau \\
\leq \gamma \int_Q \left| D((u - k_{n+1})^{(h+p-1)/p} \xi_n^p) \right|^p \, dx \, d\tau \\
\times \left( \text{ess sup}_{t_n < \tau < t} \int_{B_n(\tau)} (u - k_{n+1})^{h+1}_+ \xi_n^p \, dx \right)^{p/N},
\]

where \( b = p + h - 1 + p(h + 1)/N \) and \( d \) is large enough.
Set \( A_n = \{(x, t) \in Q_{n-1}: u(x, t) \geq k_n\}, n = 1, 2, \ldots \), and observe that
\[
\int \int_{Q_n} (u - k_n)^{h+1}_+ \, dx \, d\tau \geq \gamma 2^{-(h+1)n} |A_{n+1}| k^{h+1}.
\] (2.6)

Hence
\[
\int \int_{Q_{n+1}} (u - k_{n+1})^{h+1}_+ \, dx \, d\tau
\leq \int \int_{Q_n} (u - k_{n+1})^{h+1}_+ \xi_n^{(h+1)d/b} \, dx \, d\tau
\leq |A_{n+1}|^s \times \left( \int \int_{Q_n} (u - k_{n+1})^{b}_+ \xi_n^d \, dx \, d\tau \right)^{(h+1)/b}
\leq \gamma |A_{n+1}|^s \times \left( \int \int_{Q_n} (u - k_{n+1})^{h+1}_+ \, dx \, d\tau \right)^{(1+p/N)(h+1)/b}
\leq \gamma k^{-(b-h-1)(h+1)/b} C_0^n (\sigma p t)^{-(1+p/N)(h+1)/b} (1 + M)^{(1+p/N)(h+1)/b}
\times \left( \int \int_{Q_n} (u - k_{n+1})^{h+1}_+ \, dx \, d\tau \right)^{1+p(h+1)/b N},
\]
where
\[
s = \left( N(p - 2) + (h + 1)p \right) / \left( N(p + h - 1) + p(h + 1) \right) \quad \text{and} \quad C_0 = 2^{(b-h-1+p+p^2/N)(h+1)/b}.
\]

If \( k \) is chosen to satisfy
\[
\int \int_{Q_0} u^{h+1}_+ \, dx \, d\tau \leq \gamma k^{N(b-h-1)/p} \left( \frac{1 + M}{\sigma p t} \right)^{(1+N/p)},
\]
then
\[
\int \int_{Q_n} (u - k_n)^{h+1}_+ \, dx \, d\tau \rightarrow 0 \quad \text{as} \ n \rightarrow \infty,
\]
i.e., \( \|u_+\|_{\infty, Q_\infty} \leq k \). Take \( k = \gamma \left\{ \left( \frac{1+M}{\sigma p t} \right)^{(1+N/p)} \int \int_{Q_0} u^{h+1}_+ \, dx \, d\tau \right\}^{p/(N(b-h-1))} + 1 \). It follows from this and Schwarz’s inequality that
\[
\|u_+\|_{\infty, Q_0} \leq \gamma \left( \frac{1 + M}{\sigma \rho_t} \right)^{(N+p)/N(b-h-1)} \left( \|u_+\|_{\infty, Q_0} \int_{Q_0} u^h dx dt \right)^{p/N(b-h-1)} + 1
\]

\[
\leq \frac{1}{2} \|u_+\|_{\infty, Q_0} + \gamma \left( \frac{1 + M}{\sigma \rho_t} \right)^{(N+p)/\kappa_h} \left( \int_{Q_0} u^h dx dt \right)^{p/\kappa_h} + 1.
\]

Similar to [1, p. 393], we obtain

\[
\|u_+\|_{\infty, B_1(x_0) \times (t/2, t)} \leq \gamma \left( \frac{1 + M}{\sigma \rho_t} \right)^{(N+p)/\kappa_h} \left( \int_{Q_0} u^h dx dt \right)^{p/\kappa_h} + \gamma. \tag{2.7}
\]

Also, taking \( \phi = (-u - k_{n+1})_{+} h \xi_n^p \), we get (2.7) for \( u_- \). This implies (2.2). \( \square \)

**Remark 2.1.** If \( \psi \in L^\infty(S_T) \), then by \( k \geq 1 \)

\[
\int_{t_n}^{t'} \int_{B_n} \psi (u - k_{n+1})_{+}^h \xi_n^p \, dx \, d\tau
\]

\[
\leq \gamma \frac{1}{k_{n+1} - k_n} \int_{t_n}^{t'} \int_{B_n} (u - k_n)_{+}^{h+1} \, dx \, d\tau
\]

\[
\leq \gamma 2^n \int_{t_n}^{t'} \int_{B_n} (u - k_n)_{+}^{h+1} \, dx \, d\tau.
\]

Hence (2.2) holds for \( F(x, t, u, Du) \).

**Remark 2.2.** If \( \lambda \geq 0 \), we can replace (2.3) by the following (2.3')

\[
\frac{1}{h + 1} \int_{B_n(t')} (u - k_{n+1})_{+}^{h+1} \xi_n^p \, dx
\]

\[
+ h \int_{t_n}^{t'} |Du|^{p-2} Du \cdot D(u - k_{n+1})_{+} (u - k_{n+1})^{h-1} \xi_n^p \, dx \, d\tau
\]

\[
+ p \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_{+}^h \xi_n^{p-1} |Du|^{p-2} Du \cdot D\xi_n \, dx \, d\tau
\]

\[
\leq p \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_{+}^{h+1} \xi_n^{p-1} \xi_n \, dx \, d\tau + \int_{t_n}^{t'} \int_{B_n} (u - k_{n+1})_{+}^h \xi_n^p u^q \, dx \, d\tau, \tag{2.3'}
\]
and we can obtain (2.4) without \( k \geq 1 \). Thus for the non-negative solution, (2.2) can be replaced by

\[
\| u(\cdot, t) \|_{\infty, B_\rho(x_0)} \leq \gamma t^{-(N+p)/\kappa_h} \left( \int_0^t \int_{B_{2\rho}(x_0)} u^h \, dx \, d\tau \right)^{p/\kappa_h}.
\]

**Proposition 2.2.** Let the assumptions of Proposition 2.1 hold and set

\[
G(t) = \sup_{0 < \tau < t} \left( \int_{B_2(x_0)} |u(x, \tau)|^h \, dx \right)^{1/h} < \infty.
\]

Then there exists a constant \( \gamma = \gamma(N, p, q) \) such that for every ball \( B_2(x_0) \), \( 0 < t < T \), \( u \) satisfies (2.1)

\[
\int_0^t \int_{B_1(x_0)} |Du|^{p-1} \, dx \, d\tau \leq \gamma t^{h/\kappa_h} G(t)^{1+h(p-2)/\kappa_h} + \gamma G(t) t, \quad \kappa_h = N(p - 2) + ph.
\]

**Proof.** Let \( \xi(x) \) be a piecewise smooth cutoff function in \( B_{3/2}(x_0) \) such that \( \xi = 1 \) on \( B_1(x_0) \) and \( |D\xi| \leq 2 \). The calculations to follow are formal in which \( u_+(-u_-) \) is required to be strictly positive. The calculations can be made rigorous by replacing \( u_+(-u_-) \) with \( u_+ + \epsilon(-u_- + \epsilon) \) and letting \( \epsilon \to 0 \). By Hölder’s inequality

\[
\int_0^t \int_{B_1(x_0)} |Du|^{p-1} \xi^{p-1} \, dx \, d\tau
\]

\[
= \int_0^t \int_{B_1(x_0)} \tau^{p-1} u_+^{1-p} |Du|^{p-1} \tau^{-\beta} u_+^{1/2} \, dx \, d\tau
\]

\[
\leq \left( \int_0^t \int_{B_1(x_0)} \tau^{p/(p-1)} \frac{|Du|^p}{u_+^{1/(p-1)}} \, dx \, d\tau \right)^{(p-1)/p} \left( \int_0^t \int_{B_1(x_0)} \tau^{-\beta p} u_+ \, dx \, d\tau \right)^{1/p}.
\]

By (2.2), we can take the testing function as

\[
\phi = t^{\beta p/(p-1)} u_+^{1-1/(p-1)} \xi^p(x)
\]

in (1.4) to obtain

\[
\int_0^t \int_{B_1(x_0)} \tau^{\beta p/(p-1)} \frac{|Du|^p}{u_+^{1/(p-1)}} \, dx \, d\tau
\]
\[
\leq \gamma \int_0^t \int_{B_{3/2}(x_0)} \tau^{\beta p/(p-1)} u_+^{p-1/(p-1)} \, dx \, d\tau \\
+ \gamma \int_0^t \int_{B_{3/2}(x_0)} \tau^{\beta p/(p-1)-1} u_+^{2-1/(p-1)} \, dx \, d\tau \\
+ \int_0^t \int_{B_{3/2}(x_0)} \tau^{\beta p/(p-1)} u_+^{2-1/(p-1)} u_+^{q-1} \, dx \, d\tau \\
+ \gamma \int_0^t \int_{B_{3/2}(x_0)} \tau^{\beta p/(p-1)} u_+^{(1-(p-l)/p(p-1)) \frac{1}{1-\frac{1}{p}}} \, dx \, d\tau.
\]

Since \((1 - (p - l)/p(p - 1)) \frac{1}{1-\frac{1}{p}}\) is increasing with respect \(l\), by Schwarz's inequality

\[
u_+^{(1-(p-l)/p(p-1)) \frac{1}{1-\frac{1}{p}}} \leq \gamma u_+^{p-1/(p-1)} + \gamma u_+.
\]

Hence

\[
\int_0^t \int_{B_1(x_0)} \tau^{\beta p/(p-1)} \frac{|Du_+|^p}{u_+^{1/(p-1)}} \, dx \, d\tau \\
\leq \gamma (1 + M) \int_0^t \int_{B_{3/2}(x_0)} \tau^{\beta p/(p-1)-1} u_+^{2-1/(p-1)} \, dx \, d\tau \\
+ \gamma \int_0^t \int_{B_{3/2}(x_0)} \tau^{\beta p/(p-1)} u_+ \, dx \, d\tau \\
\leq \gamma (1 + M) G(t) \left( \int_0^t \tau^{\beta p/(p-1)-1} \|u_+(\cdot, \tau)\|_{L^p(B_{3/2}(x_0))} \, d\tau + \int_0^t \tau^{\beta p/(p-1)} \, d\tau \right), \tag{2.11}
\]

where

\[
M = \sup_{0 < \tau < t} \tau \left\{ \|u(\cdot, \tau)\|_{L^p(B_2(x_0))}^{p-2} \rho^{-p} + \sup_{x \in B_2(x_0)} |u|^q l^{1-1/(p-1)} \right\}.
\]

Notice that by (2.2),

\[
\|u(\cdot, \tau)\|_{L^p(B_{3/2}(x_0))} \leq \gamma t^{-(N+p)/\kappa_h} \left( \int_0^t \int_{B_2(x_0)} |u|^h \, dx \, d\tau \right)^{p/\kappa_h} + \gamma
\]
\[ \leq \gamma t^{-N/\kappa} G(t)^{\beta p/\kappa} + \gamma, \]

and

\[ \int_0^t \int_{B_1(x_0)} \tau^{-\beta p} u_+ \, dx \, d\tau \leq \int_0^t \int_{B_2(x_0)} \tau^{-\beta p} u_+ \, dx \, d\tau \leq \gamma t^{1-\beta p} G(t). \]

From (2.10) and (2.11), we get

\[ \int_0^t \int_{B_1(x_0)} |Du|^{p-1} |\xi|^{p-1} \, dx \, d\tau \leq \gamma G(t)^{1+h(p-2)/\kappa} \left( t^{\beta p/(p-1)-N(p-2)/(\kappa p)} \right)^{(p-1)/p-1} t^{-\beta} + \gamma G(t) t \]

\[ = \gamma G(t)^{1+h(p-2)/\kappa} t^{(\kappa-2 N(p-2))/\kappa p} + \gamma G(t) t. \]

**Remark 2.3.** If \( h = 1 \), for all constant \( \rho \geq 1 \) we can prove

\[ \int_0^t \int_{B_{\rho}(x_0)} |Du|^{p-1} \, dx \, d\tau \leq \gamma t^{1/\kappa} G(t)^{1+(p-2)/\kappa} + \gamma G(t) t, \]

where \( \kappa = N(p-2) + p \).

**Remark 2.4.** If \( u^q - \lambda |Du|^l \) replaced by \( F(x, t, u, Du) \), then we have

\[ \int_0^t \int_{B_1(x_0)} |Du|^{p-1} \, dx \, d\tau \leq \gamma t^{1/\kappa} G(t)^{1+h(p-2)/\kappa} + \gamma G(t) t + \gamma. \]

**Remark 2.5.** Assume \( \lambda \geq 0 \). Combining Remark 2.2, for nonnegative solution \( u \), we have

\[ \int_0^t \int_{B_1(x_0)} |Du|^{p-1} \, dx \, d\tau \leq \gamma t^{h/\kappa} G(t)^{1+h(p-2)/\kappa}, \quad \kappa = N(p-2) + ph. \]  

(2.12)

**Remark 2.6.** The estimates in Propositions 2.1–2.2 hold for weak solutions of the following boundary value problem:

\[
\begin{align*}
\{ u_t &= \text{div}(|Du|^{p-2} Du) + |f(u)|^{q-1} f(u) - \lambda (f(|Du|^2))^{1/2} \quad \text{in } \Omega_T = \Omega \times (0, T), \\
u(x, 0) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x) \in L^1(\Omega),
\end{align*}
\]
where $\Omega$ is bounded with smooth boundary $\partial \Omega$, $f(x) \in C^\infty(R)$, $0 \leq \frac{f(x)}{x} \leq \gamma$. Indeed we can regard $u$ as a function defined in the whole $R^N \times (0, T)$ by setting it zero outside $\Omega$.

### 3. The short time existence

Define a sequence of functions $\{f_n\}$ satisfying

$$f_n \in C^\infty(R), \quad f_n(r) = r \quad \text{for } r \in [-n, n], \quad f_n(r) = n + 1 \quad \text{for } |r| \geq n + 1 \text{ and } |f'| \leq \gamma.$$  

Consider the family of approximating problem

$$
\begin{aligned}
(u_n)_t &= \operatorname{div}(|Du_n|^{p-2}Du_n) + |f_n(u_n)|^q - \lambda f_n(|Du_n|^2)^{l/2} \quad \text{in } Q_n, \\
Q_n &= B_n \times R^+, \quad B_n = \{|y| < n\}, \\
u_n(y, t) &= 0, \quad \text{for } |y| = n, \\
u_n(y, 0) &= u_{0n}(y),
\end{aligned}
$$

where $u_{0n} \in C^\infty_0(B_n)$ satisfies

$$\lim_{n \to \infty} \int_{B_\rho} |u_{0n} - u_0| \, dy = 0, \quad \forall \rho > 0,$$

and

$$\|u_{0n}\|_h \leq \|u_0\|_h.$$  

By the results of [4,6] and [5], there exists a solution $u_n \in C(S_T^c) \cap L^\infty(S_T)$, $Du_n \in C^{\beta, \beta/2}(\Omega \times (\epsilon, T))$, $u_{nt} \in L^2(0, T : L^\infty_\text{loc}(R^N))$ to (3.1), where $\Omega$ can be any bounded open set, $\epsilon, T > 0$ and some $\beta \in (0, 1)$. Therefor for all $t \in R$

$$\sup_{0 < \tau < t} \sup_{y \in R^N} \left\{|u_n|^p - 2(y, \tau) + |u_n|^{q-1}(y, \tau)\right\} \leq C(n)$$

for a qualitative constant $C(n)$ depending on Theorem 1.1 will follow by a standard limiting process via the compactness results of [4] and [6] whence we show estimates (1.11)–(1.15) with $u$ and $u_0$ replaced by $u_n$ and $u_{0n}$ with constant independent of $n$. To prove these estimates we will work with (3.1) and drop the subscript $n$.

Let $B_\rho(x)$ denote the ball with center $x$ and radius $\rho$. Let $\tilde{t}$ be the largest time satisfying that for all $t \in (0, \tilde{t})$

$$\sup_{x \in R^N} \left\{\|u(\cdot, \tau)\|_{\infty, B_2(x)}^{p-2} + \sup_{y \in B_2(x)} |u|^{q-1}(y, t)\right\} \leq t^{-1}.$$  

By (3.2) we have $\tilde{t} > 0$. Thus by Proposition 2.1, there exists a constant $\gamma = \gamma(N, p, q)$ independent of $n$ such that
\[ \|u(\cdot, \tau)\|_{\infty, B_1(x)} \leq \gamma t^{-(N+p)/\kappa_h} \left( \int_0^t \int_{B_2(x)} |u|^h(y, \tau) \, dy \, d\tau \right)^{p/\kappa_h} + \gamma, \]

\[ \kappa_h = N(p-2) + ph, \quad \forall 0 \leq t < \bar{t}. \]  

(3.4)

Set

\[ \psi(t) = \sup_{0 < \tau < t} \|u(\cdot, \tau)\|_h \]

and observe that \( \psi(t) \) is finite.

Next we assume (A) holds. It follows from (3.4) that for all \( 0 < t < \bar{t} \)

\[ t \sup_{x \in \mathbb{R}^N} |u|^{p-2} \leq \gamma t^{1-N(p-2)/\kappa_h} \psi^{p(p-2)/\kappa_h}(t) + \gamma t, \]

\[ t \sup_{x \in \mathbb{R}^N} |u|^{q-1} \leq \gamma t^{1-N(q-1)/\kappa_h} \psi^{p(q-1)/\kappa_h}(t) + \gamma t. \]  

(3.5)

Also for \( \delta > 0 \) to be chosen, we define

\[ t^* = \sup \{ t > 0 : t^h \psi^{p-2} + t^{(1/p)(\kappa_h-N(q-1))} \psi^{q-1}(t) \leq \delta \}. \]  

(3.6)

Notice that \( \kappa_h - N(q-1) > 0 \). Therefore for all \( 0 < t < \min\{\bar{t}, t^*\} \),

\[ t \sup_{x \in \mathbb{R}^N} |u|^{p-2}(x, t) + t \sup_{x \in \mathbb{R}^N} |u|^{q-1}(x, t) \leq \gamma \delta^{p/\kappa_h}. \]

It follows that \( \delta = \delta(p, q, N) \) can be chosen small enough such that \( t^* \leq \bar{t} \). Let \( \xi(x) \) be non-negative smooth cutoff function in \( B_2(x) \) such that \( \xi = 1 \) on \( B_1(x) \), \( |D\xi| \leq \gamma \). We use \((u_{\pm})^{h-1} \xi^p\) as a testing function in (3.1). If \( h > 1 \), we get

\[
\int_{B_2(x)} u^h(y, t) \xi^p \, dy + h(h-1) \int_0^t \int_{B_2(x)} u^{h-2} |Du|^p \xi^p \, dy \, d\tau \\
\leq \int_{B_2(x)} u_0^h(y) \xi^p \, dy + ph \int_0^t \int_{B_2(x)} u^{h-1} |Du|^{p-1} |D\xi|^p \xi^{p-1} \, dy \, d\tau \\
+ \int_0^t \int_{B_2(x)} u^{q+h-1} \, dy \, d\tau + \gamma \int_0^t \int_{B_2(x)} |Du|^h u^{h-1} \xi^p \, dy \, d\tau.
\]

Hence by Schwarz’s inequality
\[
\int_{B_1(x)} u^h(y, t) \, dy \\
\leq \int_{B_2(x)} u^h_0(y) \, dy + \int_0^t \int_{B_2(x)} u^{p+h-2} \, dy \, d\tau \\
+ \int_0^t \int_{B_2(x)} u^{q-1} u^h \, dy \, d\tau + \gamma \int_0^t \int_{B_2(x)} u^{h-1+l/(p-l)} \, dy \, d\tau \\
\leq \int_{B_2(x)} u^h_0(y) \, dy + \int_0^t \int_{B_2(x)} u^{p+h-2} \, dy \, d\tau \\
+ \int_0^t \int_{B_2(x)} u^{q-1} u^h \, dy \, d\tau + \gamma,
\] 
(3.7)

here we used that \(0 < h - 1 + l/(p-l) \leq h + p - 2\).

Combining (3.5), (3.6), we have

\[
\int_{B_1(x)} u^h(y, t) \, dy \\
\leq \int_{B_2(x)} u^h_0(y) \, dy + \gamma \left\{ \sup_{0<\tau<t} \sup_{x \in \mathbb{R}^N} \int_{B_2(x)} u^h(y, \tau) \, dy \right\} \\
\times \left\{ \int_0^t \tau^{-N(p-2)/\kappa_h \psi} (p-2)/\kappa_h (\tau) \, d\tau + \int_0^t \tau^{-N(q-1)/\kappa_h \psi} (q-1)/\kappa_h (\tau) \, d\tau \right\} + \gamma \\
\leq \gamma \| u_0 \|^h_{h \psi} + \gamma \psi(t) \left\{ t^h \psi^{p-2}(t) + t(1/p)(\kappa_h-N(q-1)) \psi^{q-1}(t) \right\}^{p/\kappa_h} + \gamma \\
\leq \gamma \| u_0 \|^h_{h \psi} + \gamma \delta^{p/\kappa_h \psi}(t) + \gamma.
\] 
(3.8)

If \(h = 1\), take \(\text{sgn}_\eta u \xi^P\) as a testing function in (3.1). After a Steklov averaging process and standard calculations, we get

\[
\int_{B_2(x)} \int_0^t \text{sgn}_\eta u \, ds \xi^P \, dy + \int_0^t \int_{B_2(x)} |\nabla u|^p \text{sgn}_\eta u \xi^P \, dy \, d\tau \\
+ p \int_0^t \int_{B_2(x)} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi^{p-1} \, dx \, d\tau,
\]
\[
\begin{align*}
\int_{B_1(x)} |u(y,t)| \, dy &
\leq \int_{B_2(x)} |u_0(y)| \, dy + \gamma \int_{B_2(x)} |u|^q \, dy + \gamma \left( \sup_{0 < \tau < t} \int_{B_2(x)} |u(\tau,y)| \, dy \right) \\
& \times \left( \int_{B_2(x)} |u|^q \, dy \right) + \frac{t}{\kappa(p-2)/\psi(t) + \frac{t}{\kappa(p-2)/\psi(t)}} + \gamma
\end{align*}
\]

Let
\[
\gamma t^* \leq \frac{1}{2}.
\]
By (3.8) and (3.10), we can determine $\delta = \delta(p, q, N)$ a priori depending only on the indicated quantities so that

$$
\psi(t) \leq \gamma \|u_0\|_h^p + \gamma, \quad \forall 0 < t < t^*.
$$

(3.11)

The number $t^*$ is still only qualitatively known. A quantitative lower bound can be found by substituting (3.11) into the definition of $t^*$ in (3.6). It gives that (3.11) holds for all $0 < t < T_0$ where $T_0$ is the smallest root of

$$
T_0 + T_0 \|u_0\|_h^{p-2} + T_0^{1+N(p-1-q)/ph} \|u_0\|_h^{q-1} = \gamma^{-1}
$$

for a constant $\gamma = \gamma(p, q, N) \geq 1$. Substituting (3.11) into (3.5), we get (1.13). Inequality (1.15) follows from Proposition 2.2.

We now prove (1.14). Let $0 < T < T_0$. Consider the sequences

$$
T_n = \frac{T}{2} - \frac{T}{2^{n+2}}, \quad \rho_n = 1 + \frac{1}{2^n}, \quad \bar{\rho}_n = \frac{1}{2}(\rho_n + \rho_{n+1}) = 1 + \frac{3}{2^{n+2}}, \quad n = 0, 1, 2, \ldots.
$$

Set

$$
B_n = B_{\rho_n}(x_0), \quad \bar{B}_n = B_{\bar{\rho}_n} Q_n = B_n \times (T_n, T), \quad \bar{Q}_n = \bar{B}_n \times (T_{n+1}, T).
$$

By (1.13), Eq. (3.1) on $\mathbb{R}^N \times (T/4, T)$ can be written as

$$
ut_t = \text{div}(|Du|^{p-2} Du) + u^q - \lambda \left(f_n(|Du|^2)^{1/2} \right), \quad (3.12)
$$

if $n$ is large enough. Thus we can discuss (3.12) instead of (3.1).

Take the $x_i$-derivative in (3.12) to obtain formally

$$
\frac{\partial}{\partial t} u_{xi} = \text{div}\left(|Du|^{p-2} Du_{xi} + (p-2)|Du|^{p-3}\left(\frac{\partial}{\partial x_i}|Du|\right) Du\right)
$$

$$
+ qu^{q-1} u_{xi} - \lambda \frac{l}{2} (f_n)^{1/2-1} f_n'(|Du|^2)_{xi}, \quad (3.13)
$$

Let $\xi_n(x, t)$ be a piecewise smooth cutoff function in $Q_n$ satisfying $\xi_n = 1, (x, t) \in \bar{Q}_n, |D\xi_n| \leq 2^{n+1}, 0 \leq \partial \xi_n / \partial t \leq 2^n / T$.

Multiply (3.13) by the testing functions

$$
\eta_n = 2u_{xi} (v - k)^{\beta} \xi_{n}^2, \quad \beta > 0, \quad v = |Du|^2
$$

and integrate over $Q_n$, where $k > 1$ will be chosen later. Proceeding formally, we have (repeated indices denote summation over those indices)
(a) \[
2 \int_{T_n} \int_{B_n} \frac{\partial}{\partial \tau} u_{x_i} u_{x_i} (v - k)^\beta \varepsilon^2_n \, dx \, d\tau
\]
\[
= \frac{1}{\beta + 1} \int_{T_n} \int_{B_n} \frac{\partial}{\partial \tau} (v - k)^{\beta + 1} \varepsilon^2_n \, dx \, d\tau
\]
\[
\geq \frac{1}{\beta + 1} \int_{B_n(t)} (v - k)^{\beta + 1} \, dx
\]
\[
- \frac{2}{\beta + 1} \int_{T_n} \int_{B_n} (v - k)^{\beta + 1} \xi_n \xi_n \, dx \, d\tau, \quad \forall T_{n+1} < t < T,
\]

(b) \[
- \int \int_{Q_n} \text{div} \left\{ |Du|^{p-2} Du_{x_i} + (p - 2)|Du|^{p-3} \left( \frac{\partial}{\partial x_i} |Du| \right) Du \right\} \eta_n \, dx \, d\tau
\]
\[
= \int \int_{Q_n} |Du|^{p-2} \left\{ \beta |Dv|^2 (v - k)^{\beta - 1} \varepsilon^2_n \right\} \, dx \, d\tau
\]
\[
+ 2 \sum_{i=1}^N |Du_{x_i}|^2 (v - k)_{+}^\beta \varepsilon^2_n + 2 (v - k)_{+}^\beta \xi_n Dv \cdot D\xi
\]
\[
+ (p - 2) \int \int_{Q_n} |Du|^{p-2} \sum_{i=1}^N \left( \frac{\partial}{\partial x_i} |Du| \right)^2 (v - k)_{+}^{\beta - 1} \varepsilon^2_n \, dx \, d\tau
\]
\[
+ 2\beta (p - 2) \int \int_{Q_n} |Du|^{p-2} (D(|Du|) \cdot Du)^2 (v - k)_{+}^{\beta - 1} \varepsilon^2_n \, dx \, d\tau
\]
\[
+ 4(p - 2) \int \int_{Q_n} |Du|^{p-3} D(|Du|) \cdot Du (v - k)_{+}^\beta Du \cdot D\xi_n \xi_n \, dx \, d\tau
\]
\[
\geq \frac{\beta}{2} \int \int_{Q_n} |Du|^{p-2} |Dv|^2 (v - k)_{+}^{\beta - 1} \varepsilon^2_n \, dx \, d\tau
\]
\[
+ \frac{2\beta}{(1 + \beta)^2} \int \int_{Q_n} |Du|^{p-2} (D(v - k)_{+}^{(1+\beta)/2})^2 \, dx \, d\tau
\]
\[
- \frac{\gamma 2^{2n}}{\beta} \int \int_{Q_n} |Du|^{p-2} (v - k)_{+}^{1+\beta} \, dx \, d\tau.
\]

Since \( k > 1 \), by Schwarz’s inequality
Combination of (a), (b) and (c) yields

\[
\sup_{T_{n+1} < t < T} \int_{B_n(t)} (v - k)^{\beta + 1} dx + \int \int_Q |Du|^{p - 2} |D(v - k)^{\beta + 1/2}|^2 dxd\tau \\
\leq \gamma 4^n \int \int_Q |Du|^{p - 2} (v - k)^{\beta + 1} dx d\tau + \frac{\gamma 4^n}{T} \int \int_Q (v - k)^{\beta + 1} dx d\tau \\
+ \gamma \int \int_Q q |u|^{q - 1} v(v - k)^{\beta} dx d\tau,
\]

where \( \gamma = \gamma(p, q, N, \beta) \).

These calculations are somewhat formal. They can be rigorously justified by first writing (3.13) in terms of difference quotients in the space variable and Steklov averaging in \( t \). Then in the weak formality one takes testing functions \( \eta_n \) where the derivatives appear as difference quotients and further they are averaged in time. A standard limiting process makes the result rigorous. Also it is easily seen that the integrals involving \( (v - k)^{\beta - 1} > 0 \) are well defined even if \( \beta - 1 < 0 \).

Let \( k_n, n = 0, 1, 2, \ldots \), be the increasing sequence

\[
k_n = k - \frac{k}{2n+1}, \quad n = 0, 1, 2, \ldots.
\]

Define

\[
A_n = \{(x, t) \in Q_n : v(x, t) > k_n\}, \quad v = |Du|^2
\]
and observe that on the set $A_n$

$$|Du|^{p-2} \geq \left( \frac{1}{2} \right)^{(p-2)/2}. $$

Also define

$$\phi(t) = \sup_{0 < \tau < t} \tau^{h(N+1)/\kappa_h} \sup_{x_0 \in \mathbb{R}^N} \|Du\|_{\infty, B_2(x_0)}(\tau), \quad \kappa_h = N(p - 2) + ph. $$

In view of (1.13) and the results in [6], it is easily seen that $\phi(t)$ is well defined. Then for all $t \in (T/4, T)$

$$|Du|^{p-2} \leq \gamma T^{-(p-2)(N+1)/k}(\phi(T))^{p-2}. $$

Without loss of generality, we may assume $\phi \geq 1$. We write (3.14) with $k$ replaced by $k_{n+1}$ and set

$$w_n = (v - k_{n})^{(\beta + 1)/2}, \quad \beta > 0. $$

Notice that if $v/2 > k_n$,

$$(v - k_n)^{\beta + 1/2} \geq \left( \frac{v}{2} \right)^{1/2} (v - k_n)^{\beta} \geq C v^{1/2} (v - k_{n+1})^{\beta};$$

and if $k_{n+1} \leq v \leq 2k_n$,

$$(v - k_n)^{\beta + 1/2} \geq (v - k_n)^{\beta}(k_{n+1} - k_n)^{1/2} \geq C2^{-n/2} v^{1/2} (v - k_{n+1})^{\beta}. $$

In either case

$$|Du| (v - k_{n+1})^{\beta} \leq C2^n (v - k_n)^{\beta + 1/2}. $$

Hence we get from (3.14) and (1.13) that

$$\sup_{T_{n+1} < t < T} \int_{B_{n+1}(t)} w_{n+1}^2 \, dx + k^{(p-2)/2} \int \int_{Q_n} |Dw_{n+1}|^2 \, dx \, d\tau$$

$$\leq \gamma 4^n \left( T^{-(N+1)(p-1)/\kappa_h} \phi^{p-1} + T^{-(N+1)/\kappa_h - 1} \phi 

+ T^{-N(q-1)/\kappa_h - h(N+1)/\kappa_h} \phi \right) \times \int \int_{Q_n} (v - k_{n})^{\beta + 1/2} \, dx \, d\tau$$

$$\leq \gamma 4^n H(T) \int \int_{Q_n} (v - k_{n})^{\beta + 1/2} \, dx \, d\tau, \quad (3.15)$$
where
\[ H(T) = T^{-h(N+1)(p-1)/\kappa_h} \phi^{p-1} + T^{-h(N+1)/\kappa_h} \phi. \]

Let \( \eta_n(x) \) be a cutoff function in \( B_n \) which equals one on \( B_{n+1} \) and \( |D \eta_n| \leq \gamma 2^n \). Then
\[ w_n \eta_n \in L^\infty((T_{n+1}, T) : L^2(\overline{B}_n)) \cap L^2((T_{n+1}, T) : W^{1,2}_0(\overline{B}_n)), \]
and by the embedding inequality (see [11, p. 62]),
\[
\int \int_{Q_{n+1}} w_{n+1}^{2(N+2)/N} \, dx \, d\tau \\
\leq \int \int_{Q_n} (w_{n+1} \eta_n)^{2(N+2)/N} \, dx \, d\tau \\
\leq \gamma \left\{ \int \int_{Q_n} |D w_{n+1}|^2 \, dx \, d\tau + 4^n \int \int_{Q_n} w_{n+1}^2 \, dx \, d\tau \right\} \\
\times \left( \sup_{T_{n+1} < t < T} \int_{\overline{B}_n} w_{n+1}^2 \, dx \right)^{2/N}. \tag{3.16}
\]

Next we impose on \( k \) the restriction
\[ 1 < k < T^{-2h(N+1)/\kappa_h} \phi^2. \tag{3.17} \]

Then
\[
4^n \int \int_{Q_n} w_{n+1}^2 \, dx \, d\tau = \frac{4^n k^{(p-2)/2}}{k^{(p-2)/2}} \int \int_{Q_n} w_{n+1}^2 \, dx \, d\tau \\
\leq \gamma 4^n H(T) k^{-(p-2)/2} \int \int_{Q_n} w_{n+1}^{2(\beta+1)/(\beta+1)} \, dx \, d\tau.
\]

We substitute this in (3.16) and estimate the right-hand side by making use of (3.15) to obtain
\[
\int \int_{Q_{n+1}} w_{n+1}^{2(N+2)/N} \, dx \, d\tau \leq \gamma \left( 4^n H(T) \right)^{(N+2)/N} k^{-(p-2)/2} \\
\times \left( \int \int_{Q_n} w_{n}^{2(\beta+1)/(\beta+1)} \, dx \, d\tau \right)^{(N+2)/N}. \tag{3.18}
\]

By Hölder’s inequality
\[
\int \int_{Q_{n+1}} w_{n+1}^{(2\beta+1)/(\beta+1)} \, dx \, d\tau \leq \left( \int \int_{Q_{n+1}} w_{n+1}^{2(N+2)/N} \, dx \, d\tau \right)^{N(2\beta+1)/(N+2)(\beta+1)} \\
\times |A_{n+1}|^{(2\beta+2+N/2)/(N+2)(\beta+1)}.
\] (3.19)

Since
\[
|A_{n+1}| \leq 2^{(\beta+1/2)(n+2)} k^{-\beta-1/2} \int \int_{Q_n} (v - k_n)^{\beta+1/2} \, dx \, dt,
\]
we get from (3.18) and (3.19) that
\[
\int \int_{Q_{n+1}} (v - k_{n+1})_+^{\beta+1/2} \, dx \, dt
\leq \gamma b^n H(T)^{(2\beta+1)/(2\beta+2)} k^{-((p-2)N(\beta+1/2)+(2\beta+1)(2\beta+2+N/2))/2(N+2)(\beta+1)}
\times \left( \int \int_{Q_n} (v - k_n)_+^{\beta+1/2} \, dx \, dt \right)^{1+(2\beta+1)/(\beta+1)(N+2)},
\]
where
\[
b = 2^{(\beta+1/2)((2\beta+2+N/2)/(N+2)(\beta+1))+(2\beta+1)/(\beta+1)}.
\]

It follows from Lemma 5.6 of [11, p. 95] that
\[
\int \int_{Q_n} (v - k_n)_+^{\beta+1/2} \, dx \, dt \to 0 \quad \text{as } n \to \infty
\]
provided
\[
\int \int_{Q_0} (v - k/2)_+^{\beta+1/2} \, dx \, dt \leq \gamma H(T)^{-N-2}/2 k^{(N(p-1)+4(\beta+1))/4}.
\]

This condition is satisfied if we choose
\[
k = \gamma^{-4/\sigma} H(T)^{2(N+2)/\sigma} \left( \int \int_{Q_0} v_+^{\beta+1/2} \, dx \, dt \right)^{4/\sigma} + 1,
\]
\[
\sigma = N(p-1) + 4(\beta+1).
\] (3.20)

Since the choice of (3.20) is compatible with (3.17), then for all \( t \in (T/2, T) \),
\[
\| Du \|_{\infty, B_1(x_0)}(t) \leq \gamma H(T)^{(N+2)/\sigma} \left( \int \int_{T/2 B_2(x_0)} |Du|^{2\beta+1} \, dx \, dt \right)^{2/\sigma} + \gamma.
\] (3.21)
On the other hand (3.20) is not compatible with (3.17), we have
\[ T^{-h(N+1)/\kappa_h} \phi(T) \leq \gamma H(T)^{(N+2)/\sigma} \left( \int_{T/2}^{T} \int_{B_{2}(x_0)} |Du|^{2\beta + 1} \, dx \, dt \right)^{2/\sigma} + \gamma. \]

By the definition of \( \phi(t) \), this again implies (3.21). Hence (3.21) holds in either case by suitably modifying the constant \( \gamma \).

To proceed, we choose \( \beta = (p-2)/2 \) so that
\[ \sigma = N(p-1) + 2p. \]

Multiply both sides of (3.21) by \( th(N+1)/\kappa_h \). Then for all \( T/4 < t < T \)
\[ t^{h(N+1)/\kappa_h} \| Du \|_{\infty, B_1(x_0)}(t) \]
\[ \leq \gamma \phi^{(p-1)(N+2)/\sigma} t^{2h(N+1)/\sigma \kappa_h} \left( \int_{T/4}^{T} \int_{B_{2}(x_0)} |Du|^{p-1} \, dx \, dt \right)^{2/\sigma} + \gamma t^{h(N+1)/\kappa_h} \]
\[ \leq G_1 + G_2 + \gamma. \]  
(3.22)

By Proposition 2.2, we have
\[ \int_{0}^{T} \int_{B_{1}(x_0)} |Du|^{p-1} \, dx \, dt \]
\[ \leq \gamma \left( T^{h/\kappa_h} \left( \sup_{0 < t < T} \left( \int_{B_{2}(x_0)} u^{h} \, dx \right)^{1/h} \right)^{1 + h(p-2)/\kappa_h} + T \sup_{0 < t < T} \left( \int_{B_{2}(x_0)} u^{h} \, dx \right)^{1/h} \right). \]

Thus
\[ G_1 \leq \gamma \phi^{(p-1)(N+2)/\sigma} \]
\[ \times \left( \left( \sup_{0 < t < T} \int_{B_{2}(x_0)} u^{h} \, dx \right)^{1/h + (p-2)/\kappa_h} + \left( \sup_{0 < t < T} \int_{B_{2}(x_0)} u^{h} \, dx \right)^{1/h} \right)^{2/\sigma} \]
\[ \leq \frac{\phi}{4} + \gamma \left( G^{1+h(p-2)/\kappa_h} + G \right), \]
\[ G_2 \leq \gamma \phi^{(N+2)/\sigma} \times \left( G^{1+h(p-2)/\kappa_h} + G \right)^{2/\sigma} \]
\[ \leq \frac{\phi}{4} + \gamma \left( G^{2(1+h(p-2)/\kappa_h)/(\kappa + p-2)} + G^{2/(\kappa + p-2)} \right). \]
We substitute these estimates into (3.22) to get
\[ t^h(N+1)/\kappa h \| Du \|_{\infty, B_1(x_0)(t)} \leq \frac{\phi}{2} + \gamma G^{1+h(p-2)/\kappa h} + \gamma. \]

By (1.12), we have
\[ t^h(N+1)/\kappa h \| Du \|_{\infty, B_1(x_0)(t)} \leq \phi_2 + \gamma G_1 + h(p-2)/\kappa h + \gamma. \]

Taking the supremum over \( x_0 \in \mathbb{R}^N, t \in (0, T) \), we get
\[ \phi \leq \gamma \left( \| u_0 \|_h^{1+h(p-2)/\kappa h} + 1 \right). \] (3.23)

There exists a constant \( \gamma = \gamma(p, q) \) such that for all \( x \in \mathbb{R}^N, y \in B_2(x) \)
\[ \gamma^{-1} (1 + |x|) \leq 1 + |y| \leq \gamma (1 + |x|). \]

From (3.23), we get
\[ | Du(x, t) | \leq \gamma t^{-h(N+1)/\kappa h} \left( \| u_0 \|_h^{1+h(p-2)/\kappa h} + 1 \right). \]

Remark 3.1. Assume \( \lambda \geq 0 \). Combining Remarks 2.2 and 2.5, for non-negative solution \( u \), we have
\[ T_0 \| u_0 \|_h^{p-2} + T_0^{1+N(p-1-q)/ph} \| u_0 \|_h^{q-1} = \gamma^{-1}. \]

We now prove (1.5).

We first prove that for any \( r > 0 \),
\[ \int_{B_r} \left| u_{n1}(x, t) - u_{n2}(x, t) \right| \, dx \leq \gamma \int_{B_{2r}} \left| u_{0n1}(x, t) - u_{0n2}(x, t) \right| \, dx + C(t), \] (3.24)

where \( u_{n1}, u_{n2} \) are two solutions of (3.1) with initial values \( u_{0n1}, u_{0n2} \in C^\infty_0(\mathbb{R}^N) \) respectively, \( \gamma, C(t) \) are the constants independent of \( n \) and \( \lim_{t \to 0} C(t) = 0 \).

Let \( \xi(x) \in C^1_0(B_{2r}) \) be a cutoff function in \( B_{2r} \) with \( 0 \leq \xi \leq 1, \xi = 1 \) on \( B_r \). Taking \( \text{sgn}_\eta(u_{n1} - u_{n2})\xi \) as a testing function in (1.4), we can obtain
\[ \int_{B_{2r}} \xi I_\eta(u_{n1}(x, t) - u_{n2}(x, t)) \, dx \]
\[ + \int_0^t \int_{B_{2r}} \xi (|Du_{n1}|^{p-2} Du_{n1} - |Du_{n2}|^{p-2} Du_{n2}) \]
\[ \times (Du_{n1} - Du_{n2}) \text{sgn}_\eta(u_{n1} - u_{n2}) \, dx \, d\tau \]
\[
\begin{align*}
&\leq - \int_{0}^{t} \int_{B_{2r}} \left( |Du_{n1}|^{p-2}Du_{n1} - |Du_{n2}|^{p-2}Du_{n2} \right) \\
&\quad \times D\xi \, \text{sgn}_{\eta}(u_{n1} - u_{n2}) \, dx \, d\tau \\
&\quad + \int_{0}^{t} \int_{B_{2r}} \xi \left( f_{n1}^{q}(u_{n1}) - f_{n2}^{q}(u_{n2}) \right) \text{sgn}_{\eta}(u_{n1} - u_{n2}) \, dx \, d\tau \\
&\quad + \left| \lambda \right| \int_{0}^{t} \int_{B_{2r}} \xi \left( |Du_{n1}|^{l} + |Du_{n2}|^{l} \right) \text{sgn}_{\eta}(u_{n1} - u_{n1}) \, dx \, d\tau \\
&\quad + \int_{B_{2r}} \xi I_{\eta}(u_{0n1} - u_{0n2}) \, dx,
\end{align*}
\]

where \( I_{\eta}(s) = \int_{0}^{s} \text{sgn}_{\eta} \tau \, d\tau \). Letting \( \eta \to 0 \), and using Hölder inequality, we get

\[
\int_{B_{2r}} \xi \left| u_{n1}(x, t) - u_{n2}(x, t) \right| \, dx \\
\leq \int_{0}^{t} \int_{B_{2r}} \left( |Du_{n1}|^{p-2}Du_{n1} - |Du_{n2}|^{p-2}Du_{n2} \right) \left| D\xi \right| \, dx \, d\tau \\
+ \int_{0}^{t} \int_{B_{2r}} \xi \left( |u_{n1}|^{q} + |u_{n2}|^{q} \right) \, dx \, d\tau \\
+ \gamma \int_{0}^{t} \int_{B_{2r}} \xi \left( |Du_{n1}|^{p-1} + |Du_{n2}|^{p-1} \right) \, dx \, d\tau \\
+ \gamma r^{N} t + \int_{B_{2r}} \xi \left| u_{0n1} - u_{0n2} \right| \, dx. \tag{3.25}
\]

By the proofs of (1.12), (1.13), (1.15) and Hölder inequality we get from (3.25) that

\[
\int_{R^{N}} \xi \left| u_{n1}(x, t) - u_{n2}(x, t) \right| \, dx \\
\leq \gamma t^{h/k_{h}} \sup_{0<\tau<t} \left\{ \int_{B_{2r}} \left( |u_{n1}|^{h}(x, \tau) + |u_{n2}|^{h}(x, \tau) \right) \, dx \right\}^{1/h + (p-2)/k_{h}} \\
+ \gamma t \sup_{0<\tau<t} \left\{ \int_{B_{2r}} \left( |u_{n1}|^{h}(x, \tau) + |u_{n2}|^{h}(x, \tau) \right) \, dx \right\}^{1/h}.
\]
\[ + \gamma t^{1-N(q-1)/\kappa h} \sup_{0<\tau<t} \left\{ \int_{B_{2r}} u_{n1}(x, \tau) \, dx + \int_{B_{2r}} u_{n2}(x, \tau) \, dx \right\} \]

\[ + \gamma t^N + \int_{R^N} \xi |u_{0n1} - u_{0n2}| \, dx \]

\[ \leq Ct + \int_{R^N} \xi |u_{0n1} - u_{0n2}| \, dx. \]

We now prove (1.5). By (3.24)

\[
\int_{B_r} |u(x, t) - u_0(x, t)| \, dx \]

\[ \leq \int_{B_r} |u(x, t) - u_{n1}(x, t)| \, dx + \int_{B_r} |u_{n1}(x, t) - u_{n2}(x, t)| \, dx \]

\[ + \int_{B_r} |u_{n2}(x, t) - u_{0n2}(x, t)| \, dx + \int_{B_{2r}} |u_{0n2}(x, t) - u_0(x, t)| \, dx + C(t) \]

\[ + \int_{B_r} |u_{n2}(x, t) - u_0(x, t)| \, dx. \]

This implies (1.5) and Theorem 1.1 is proved.

4. Proof of Theorems 1.2–1.5

Denote \( \lambda_1 = \max\{q, p - 1\} \). Assume \( u, v \) be two solutions of (1.1) in \( S_T, 0 < T < \infty \). Set \( w = u - v \). Then \( w \) satisfies

\[ -w_t + (a^{ij}(x, t)w_{x_i})_{x_j} + c(x, t)Dw + b(x, t)w = 0 \quad \text{in} \ S_T, \quad (4.1) \]

where

\[
a^{ij}(x, t) = \int_{0}^{1} \left| D(su + (1-s)v) \right|^{p-2} ds \delta_{ij} \]

\[ + (p-2) \int_{0}^{1} \left| D(su + (1-s)v) \right|^{p-4} \]

\[ \times (su + (1-s)v)_{x_i} (su + (1-s)v)_{x_j} ds, \]
\[ b(x, t) = q \int_{0}^{1} |x u + (1 - s)v|^{q-1} ds, \]
\[ c(x, t) = -\lambda l \int_{0}^{1} |D(su + (1 - s)v)|^{l-2} D(u + v) ds. \]  

(4.2)

Let \( u_1, v_1 \) be subsolution and supersolution of (1.1) respectively. Then \( w_1 = u_1 - v_1 \) satisfies

\[ \int_{\Omega} w_1(x, t) \phi(x, t) dx + \int_{t}^{t_0} \int_{\Omega} [-w_1 \phi_t + a_{ij} D w_1 \cdot D\phi] dxd\tau \]
\[ \leq \int_{t}^{t_0} \int_{\Omega} b w_1 \phi dxd\tau + \int_{t}^{t_0} \int_{\Omega} c \phi dxd\tau + \int_{\Omega} w_1(x, t_0) \phi(x, t_0) dx, \]  

(4.3)

for all \( 0 \leq t_0 < t \leq T \) and all testing function \( \phi \geq 0 \).

The matrix \( (a)_{ij} \) is positive semidefinite and for all \( \xi \in \mathbb{R}^N, (x, t) \in S_T \)

\[ a_0(x, t)|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq (p-1)a_0(x, t)|\xi|^2, \]
\[ a_0(x, t) = \int_{0}^{1} |D(su + (1 - s)v)|^{p-2} ds. \]  

(4.4)

To prove Theorem 1.2, we need the following lemmas.

For \( \beta \geq \frac{k}{p(p-2)} \), set

\[ A_\beta(x) = \left( 1 + |x|^p \right)^{-\beta} \]  

(4.5)

and define for \( t \in (0, T) \)

\[ h_\beta(t) = \sup_{0 < \tau < t} \int_{\mathbb{R}^N} |u(x, \tau)| A_\beta(x) dx. \]  

(4.6)

By Corollary 4.2 of [1, Part II], Hölder inequality and (1.12')

\[ \sup_{\rho \geq 1} \rho^{-N} \left( \int_{B_\rho} |w| \, dy \right) \leq \gamma \| w(\cdot, \tau) \|_1 \leq \gamma \| w(\cdot, \tau) \|_h \leq \gamma, \quad \forall 0 < \tau < T. \]

Thus we have

\[ h_\beta(t) \leq C(\beta). \]  

(4.7)
Lemma 4.1. There exists a constant \( \gamma = \gamma(N, p, q, T) \) such that
\[
\int_0^t \int_{R^N} |Du|^{p-1} A_{\beta+1/p}(x) \, dx \, d\tau \leq \gamma t \left( 1 - \delta(\lambda_1^{-1}) \right)^{1/p}, \quad t \in (0, T).
\]

Proof. In (1.4), take the testing function
\[
\phi = (t - \epsilon)^{1/p} u^{1-2/p} A_{\beta+1/p} \xi^p,
\]
where \( \xi \) is the usual cutoff function in \( B_\rho \). After a Steklov averaging process and standard calculations, we obtain
\[
\int_\epsilon^t \int_{B_\rho} (\tau - \epsilon)^{1/p} |Du|^{p} \frac{|Du|^p}{u^{2/p}} A_{\beta+1/p} \xi^p \, dx \, d\tau \\
\leq \gamma \int_\epsilon^t \int_{B_\rho} (\tau - \epsilon)^{1/p} u^{1-2/p} |Du|^{p-1} D \left( A_{\beta+1/p} \xi^p \right) \, dx \, d\tau \\
+ \gamma \int_\epsilon^t \int_{B_\rho} (\tau - \epsilon)^{1/p} u^{p-1} A_{\beta+1/p} \xi^p \, dx \, d\tau \\
+ \int_\epsilon^t \int_{B_\rho} (\tau - \epsilon)^{1/p} u^{q+1-2/p} A_{\beta+1/p} \xi^p \, dx \, d\tau \\
+ \gamma \int_\epsilon^t \int_{B_\rho} (\tau - \epsilon)^{1/p} u^{1-2/p} |Du|^{1} A_{\beta+1/p} \xi^p \, dx \, d\tau \\
= J_\rho^{(1)} + J_\rho^{(2)} + J_\rho^{(3)} + J_\rho^{(4)}.
\]  

(4.8)

For \( J_\rho^{(2)} \), we have
\[
J_\rho^{(2)} \leq \gamma \int_\epsilon^t (\tau - \epsilon)^{(1-p-(p-2)\delta)/p} \\
\times \int_{B_\rho} \tau^{(p-2)\delta/p} u(x, t)^{(p-2)/p} u(x, \tau) A_{\beta}(x) \, dx \, d\tau.
\]

By (1.13') and (4.7)
\[
J_\rho^{(2)} \leq \gamma (t - \epsilon)^{(1-(p-2)\delta)/p}, \quad \forall \rho \geq 1.
\]
Since $|D(A^{1/p})| \leq \gamma A^{2/p}$, we have

$$J_{\rho}^{(1)} \leq \gamma \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} |u(x, t)(p-2)/p| |Du|^{p-1} A^{1/p} \xi^{p-1} dx \, d\tau$$

$$+ \gamma \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} |u(x, t)(p-2)/p| |Du|^{p-1} |D\xi| \, dx \, d\tau$$

$$\leq \frac{1}{4} \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} |Du|^{p} \frac{A^{1/p} \xi^{p}}{u^{2/p}} \, dx \, d\tau$$

$$+ \gamma \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} u^{p-2/p} A^{1/p} \, dx \, d\tau$$

$$\leq \frac{1}{4} \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} |Du|^{p} \frac{A^{1/p} \xi^{p}}{u^{2/p}} \, dx \, d\tau$$

$$+ \gamma \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} u^{p-1-2/p} A^{1/p} \, dx \, d\tau$$

$$\leq \frac{1}{4} \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} |Du|^{p} \frac{A^{1/p} \xi^{p}}{u^{2/p}} \, dx \, d\tau + \gamma (t - \epsilon)^{(1+1/p)(1-\delta(p-2))}.$$}

We now estimate $J_{\rho}^{(3)}$, $J_{\rho}^{(4)}$. Similarly to the above, we get

$$J_{\rho}^{(3)} \leq \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} u^{q-2/p} A^{1/p} \xi^{p} \, A_{\beta} u \, dx \, d\tau$$

$$\leq \gamma \int_{\frac{1}{4}}^{t} (\tau - \epsilon)^{1/p-\delta(q-2/p)} \, d\tau \leq \gamma (t - \epsilon)^{1/p-\delta(q-2/p)+1}$$

$$\leq \gamma (t - \epsilon)^{(1+1/p)(1-\delta(q-2)/(p+1))} \leq \gamma (t - \epsilon)^{(1+1/p)(1-\delta(\lambda_{1}-1))}.$$

$$J_{\rho}^{(4)} \leq \frac{1}{4} \int_{\frac{1}{4}}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} |Du|^{p} \frac{A^{1/p} \xi^{p}}{u^{2/p}} \, dx \, d\tau$$
\[ + \gamma \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} u^{p/(p-1)-2/p} A_{\beta+1/p} \, dx \, d\tau. \]

Since \( 1 \leq \lambda \leq p - 1 \), we have

\[
\gamma \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} u^{p/(p-1)-2/p} A_{\beta+1/p} \, dx \, d\tau \leq \gamma \left( \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} u^{p-2/p} A_{\beta+1/p} \, dx \, d\tau + \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} A_{\beta+1/p} \, dx \, d\tau \right).
\]

\[
\leq \gamma \left( \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} u^{p-2/p} A_{\beta+1/p} \, dx \, d\tau + (t - \epsilon)^{1/p+1} \right)
\leq \gamma \left( (t - \epsilon)^{1/p+1 - \delta(p-1)/p} + (t - \epsilon)^{1/p+1} \right)
\leq \gamma (t - \epsilon)^{1+1/p}(1-\delta(p-2)).
\]

Hence we get from (4.8)

\[
\int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{1/p} \frac{|Du|^p}{u^{2/p}} A_{\beta+1/p} \xi^p \, dx \, d\tau \leq \gamma (t - \epsilon)^{(1/p)(1-\delta(\lambda-1))}, \quad \forall \rho \geq 1, \ t \in (\epsilon, T),
\]

where we have changed \( \rho \) to be \( 2\rho \). Next, for all \( \epsilon \) in \( (0, T) \)

\[
\int_{\epsilon}^{t} \int_{B_{\rho}} |Du|^{p-1} A_{\beta+1/p} \, dx \, d\tau
\leq \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{(p-1)/p} \frac{|Du|^{p-1}}{u^{2(p-1)/p^2}} A^{(p-1)/p} \beta+1/p \, dx \, d\tau
\times (\tau - \epsilon)^{- (p-1)/p^2} u^{-2(p-1)/p^2} A_{\beta+1/p}^{1/p} \, dx \, d\tau
\leq \left( \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{(1/p) \frac{|Du|^{p}}{u^{2/p}} A_{\beta+1/p} \xi^p} \, dx \, d\tau \right)^{(p-1)/p}
\times \left( \int_{\epsilon}^{t} \int_{B_{\rho}} (\tau - \epsilon)^{- (p-1)/p} u^{(p-2)/p} A_{\beta+1/p} u A_{\beta} \, dx \, d\tau \right)^{1/p}
\leq \gamma (t - \epsilon)^{(1-\delta(\lambda-1))/p}.
Hence Lemma 4.1 follows by letting $\epsilon \to 0$. □

**Lemma 4.2.** There exists a constant $\gamma = \gamma(N, p, q)$ such that if $w(\cdot, t) \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $t \to 0$ then

$$\int_{\mathbb{R}^N} |w(x, t)| A_{\beta_1}(x) \, dx \leq \gamma t^{(1/p)(1-\delta(\lambda-1))}, \quad 0 < t < T,$$

where $\beta_1 = \beta + 1/p$.

**Proof.** In the weak formulation of (1.4), take the testing function $\text{sgn}_\eta(w) A_{\beta_1} \xi(x)$, where $\xi$ is the usual cutoff function in $B_\rho$. Using the assumptions of the lemma, we deduce

\[
\begin{align*}
\int_{B_{2\rho}} & w(x,t) \int_0^t \text{sgn}_\eta s \, ds \, A_{\beta_1} \xi(x) \, dx \\
& + \int_0^t \int_{B_{2\rho}} a^{ij} w_{xi} w_{xj} \text{sgn}_\eta'(w) A_{\beta_1} \xi(x) \, dx \, d\tau \\
& - \int_0^t \int_{B_{2\rho}} a^{ij} w_{xi} \text{sgn}_\eta(w)(A_{\beta_1} \xi(x))_{xj} \, dx \, d\tau \\
& + \int_0^t \int_{B_{2\rho}} c Dw \text{sgn}_\eta(w) A_{\beta_1} \xi(x) \, dx \, d\tau \\
& + \int_0^t \int_{B_{2\rho}} b w \text{sgn}_\eta(w) A_{\beta_1} \xi(x) \, dx \, d\tau.
\end{align*}
\]

Giving up the second term on the left-hand side, which is nonnegative and then letting $\eta \to 0$, we derive

\[
\begin{align*}
\int_{B_{2\rho}} |w(x,t)| A_{\beta_1}(x) \xi(x) \, dx & \leq \int_0^t \int_{B_{2\rho}} (|Du|^{p-1} + |Dv|^{p-1}) |D(A_{\beta_1} \xi)| \, dx \, d\tau \\
& + \int_0^t \int_{B_{2\rho}} (|u|^q + |v|^q) A_{\beta_1} \xi \, dx \, d\tau \\
& + |\lambda| \int_0^t \int_{B_{2\rho}} (|Du|^l + |Dv|^l) A_{\beta_1} \xi \, dx \, d\tau. \tag{4.9}
\end{align*}
\]
Notice that
\[ |DA_{\beta_1}| \leq \gamma A_{\beta_1+1/p}, \quad |D\xi| = 0 \quad \text{on } |x| \leq \rho/2, \quad A_{\beta_1} |D\xi| \leq \gamma A_{\beta_1+1/p}, \]
\[
\int_0^t \int_{B_\rho} (|Du|^p + |Dv|^p) A_{\beta_1} \xi dxd\tau \\
\leq \gamma \int_0^t \int_{B_\rho} (|Du|^{p-1} + |Dv|^{p-1}) A_{\beta_1} \xi dxd\tau + \gamma \int_0^t \int_{B_{2\rho}} A_{\beta_1} dxd\tau \\
\leq \gamma \int_0^t \int_{B_{2\rho}} (|Du|^{p-1} + |Dv|^{p-1}) A_{\beta_1} \xi dxd\tau + \gamma t.
\]
By (1.13′) and (4.7),
\[
\int_0^t \int_{B_{2\rho}} |u|^q A_{\beta_1} \xi dxd\tau = \int_0^t \int_{B_{2\rho}} |u|^{q-1}|u| A_{\beta_1} \xi dxd\tau \\
\leq \gamma \int_0^t \tau^{-\delta(q-1)} h_{\beta_1}(\tau) d\tau \\
\leq \gamma t^{1-\delta(q-1)}.
\]
Letting \( \rho \to \infty \) in (4.9), we have
\[
\int_{R^N} |w(x, t)| A_{\beta_1} (x) dxd\tau \\
\leq \gamma \int_0^t \int_{R^N} (|Du|^{p-1} + |Dv|^{p-1}) A_{\beta_1} dxd\tau + \gamma t^{1-\delta(q-1)}.
\]
Hence Lemma 4.2 follows from Lemma 4.1.

**Lemma 4.3.** Suppose that
\[ w(\cdot, t) \to 0 \quad \text{in } L^1_{\text{loc}}(R^N) \quad \text{as } t \to 0. \]
Then for any \( \epsilon \in (0, (1 - \delta(\lambda_1 - 1))/(p\delta)) \),
\[ w(\cdot, t) \to 0 \quad \text{in } L^{1+\epsilon}_{\text{loc}}(R^N) \quad \text{as } t \to 0. \]
**Proof.** Let $\epsilon \in (0, (1 - \delta(\lambda_1 - 1))/(p\delta))$ be fixed. Then for all $t \in (0, T)$

$$
\int_{R^n} |w(x, t)|^{1+\epsilon} A_{\beta_1}(x) \, dx \\
\leq \int_{R^n} |w(x, t)|^{\epsilon} |w(x, t)| A_{\beta_1}(x) \, dx.
$$

By (1.13')

$$
|w(x, t)|^{\epsilon} \leq \gamma t^{-\epsilon\delta},
$$

so that

$$
\int_{R^n} |w(x, t)|^{1+\epsilon} A_{\beta_1}(x) \, dx \\
\leq \gamma t^{-\epsilon\delta} \int_{R^n} |w(x, t)| A_{\beta_1}(x) \, dx
$$

$$
\leq \gamma t^{(1/p)(1-\delta(\lambda_1-1)) - \epsilon\delta}.
$$

Hence for all $\rho \geq 1$

$$
\int_{R^n} |w(x, t)|^{1+\epsilon} \, dx \\
\leq \gamma \rho^{\beta_1} \int_{R^n} |w(x, t)|^{1+\epsilon} A_{\beta_1}(x) \, dx
$$

$$
\leq \gamma (N, p, q, \rho) t^{(1/p)(1-\delta(\lambda_1-1)) - \epsilon\delta} \to 0 \quad \text{as } t \to 0. \quad \Box
$$

**Remark 4.1.** Replace the testing function in Lemma 4.2 by $\text{sgn}_\eta w_{+} A_{\beta_1} \xi(x)$, we have

$$
\int_{R^n} w_{+} A_{\beta_1}(x) \, dx \\
\leq \gamma t^{(1/p)(1-\delta(\lambda_1-1))}, \quad 0 < t < T,
$$

and then

$$
w_{+} \to 0 \quad \text{in } L^{1+\epsilon}_{\text{loc}}(R^n) \text{ as } t \to 0.
$$

**Proof of Theorem 1.2.** In the weak formulation of (4.1), we may take testing function

$$
(|w| + \eta)^\epsilon \text{sgn}_{\epsilon_1} w A_{\beta_1} \xi^2, \quad \epsilon \in (0, (1 - \delta(\lambda - 1))/(p\delta)), \quad \eta \in (0, 1),
$$

since $w(., t) \to 0$ in $L^1_{\text{loc}}(R^n)$, a standard Steklov averaging process gives that this is a admissible testing function. Hence we can deduce
\[
\int_{B_{\rho}} \int_{0}^{w(x,t)} (|s| + \eta)^\epsilon \operatorname{sgn}_{\epsilon_1} s \, ds \, A_{\beta_1} \xi^2 \, dx
\]
\[
+ \epsilon \int_{\eta}^{t} \int_{B_{\rho}} a_0(x, \tau) \frac{|\nabla w|^2}{(|w| + \eta)^{1-\epsilon}} \frac{w \operatorname{sgn}_{\epsilon_1} w}{|w|} A_{\beta_1} \xi^2 \, dx \, d\tau
\]
\[
+ \int_{\eta}^{t} \int_{B_{\rho}} a_0(x, \tau) |\nabla w|^2 (|w| + \eta)^\epsilon \operatorname{sgn}'_{\epsilon_1} w A_{\beta_1} \xi^2 \, dx \, d\tau
\]
\[
\leq \int_{B_{\rho}} \int_{0}^{w(x,\eta)} (|s| + \eta)^\epsilon \operatorname{sgn}_{\epsilon_1} s \, ds \, A_{\beta_1} \xi^2 \, dx
\]
\[
+ \gamma \int_{\eta}^{t} \int_{B_{\rho}} a_0(x, \tau) |\nabla w|(|w| + \eta)^\epsilon A_{\beta_1}^{1/2} \xi |\nabla (A_{\beta_1}^{1/2} \xi)| \, dx \, d\tau
\]
\[
+ \int_{\eta}^{t} \int_{B_{\rho}} c Dw \operatorname{sgn}_{\epsilon_1} w A_{\beta_1} \xi^2 \, dx \, d\tau
\]
\[
+ \int_{\eta}^{t} \int_{B_{\rho}} b w \operatorname{sgn}_{\epsilon_1} w A_{\beta_1} \xi^2 \, dx \, d\tau.
\]
Giving up the third term on the left-hand side, which is nonnegative, and letting \(\epsilon_1 \to 0\), we further obtain
\[
\frac{1}{1 + \epsilon} \int_{B_{\rho}(t)} (|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 \, dx + \epsilon \int_{\eta}^{t} \int_{B_{\rho}} a_0(x, \tau) \frac{|Dw|^2}{(|w| + \eta)^{1-\epsilon}} A_{\beta_1} \xi^2 \, dx \, d\tau
\]
\[
\leq \frac{1}{1 + \epsilon} \int_{B_{\rho}(\eta)} (|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 \, dx + \int_{\eta}^{t} \int_{B_{\rho}} b(x, \tau) |w|(|w| + \eta)^{\epsilon} A_{\beta_1} \xi^2 \, dx \, d\tau
\]
\[
+ \gamma \int_{\eta}^{t} \int_{B_{\rho}} a_0(x, \tau) \frac{|Dw|}{(|w| + \eta)^{(1-\epsilon)/2}} (|w| + \eta)^{(1+\epsilon)/2} A_{\beta_1}^{1/2} \xi \, D(A_{\beta_1}^{1/2} \xi) \, dx \, d\tau
\]
\[
+ \gamma \int_{\eta}^{t} \int_{B_{\rho}} \left( \int_{0}^{1} |D(su + (1-s)v)|^{l-2} D(su + (1-s)v) \cdot D(u - v) \, ds \right)
\]
\[
\times (|w| + \eta)^{\epsilon} A_{\beta_1} \xi^2 \, dx \, d\tau
\]
\[
= I_1 + I_2 + I_3 + I_4,
\]
where \(a_0(x, t)\) has been defined in (4.4). By the Schwarz inequality,

\[
I_3 = \gamma \int_{\eta}^{t} \int_{B_\rho} a_0(x, t) \frac{|Dw|}{(|w| + \eta)^{(1-\epsilon)/2}} \left( \left| \frac{A_{\beta_1}^{1/2} \xi D(A_{\beta_1}^{1/2} \xi)}{1 + \epsilon} \right| \frac{A_{\beta_1}^{1/2} \xi |D(A_{\beta_1}^{1/2} \xi)|}{|w| + \eta} \right) dx d\tau
\]

\[
\leq \frac{\epsilon}{2} \int_{\eta}^{t} \int_{B_\rho} a_0(x, t) \frac{|Dw|^2}{(|w| + \eta)^{1-\epsilon}} A_{\beta_1} \xi^2 dx d\tau
\]

\[
+ \gamma (\epsilon) \int_{\eta}^{t} \int_{B_\rho} a_0(x, t) (|w| + \eta)^{1+\epsilon} (A_{\beta_1} |D\xi|^2 + |DA_{\beta_1}^{1/2}|^2) dx d\tau;
\]

\[
I_4 \leq \gamma \int_{\eta}^{t} \int_{B_\rho} \left( \int_{0}^{1} |D(su + (1-s)v)|^{l-1} ds \right) |Dw| (|w| + \eta)^{\epsilon} A_{\beta_1} \xi^2 dx d\tau
\]

\[
\leq \frac{\epsilon}{2} \int_{\eta}^{t} \int_{B_\rho} \left( \int_{0}^{1} |D(su + (1-s)v)|^{p/2-1} ds \right)^2 \frac{|Dw|^2}{(|w| + \eta)^{1-\epsilon}} A_{\beta_1} \xi^2 dx d\tau
\]

\[
+ \gamma \int_{\eta}^{t} \int_{B_\rho} \left( \int_{0}^{1} |D(su + (1-s)v)|^{l-p/2} ds \right)^2 (|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 dx d\tau
\]

\[
\leq \frac{\epsilon}{2} \int_{\eta}^{t} \int_{B_\rho} a_0(x, t) \frac{|Dw|^2}{(|w| + \eta)^{1-\epsilon}} A_{\beta_1} \xi^2 dx d\tau
\]

\[
+ \gamma \int_{\eta}^{t} \int_{B_\rho} \left( \int_{0}^{1} |D(su + (1-s)v)|^{2l-p} ds \right) (|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 dx d\tau.
\]

Notice that

\[
A_{\beta_1} |D\xi|^2 + |DA_{\beta_1}^{1/2}|^2 \leq \gamma A_{\beta_1}^2 A_{2/p}.
\]

Substituting this into (4.10), we get

\[
\int_{B_\rho(t)} \left( \left| \frac{A_{\beta_1} \xi^2}{1 + \epsilon} \right| \frac{A_{\beta_1} \xi |D(A_{\beta_1} \xi)|}{|w| + \eta} \right) dx
\]

\[
\leq \int_{B_\rho(\eta)} \left( \left| \frac{A_{\beta_1} \xi^2}{1 + \epsilon} \right| \frac{A_{\beta_1} \xi |D(A_{\beta_1} \xi)|}{|w| + \eta} \right) dx
\]
\[ + \int_{\eta}^{t} \int_{B_\rho} b(x, t)(|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 \, dx \, d\tau \]
\[ + \gamma \int_{\eta}^{t} \int_{B_\rho} a_0(x, t) A_{2/p}(x)(|w| + \eta)^{1+\epsilon} A_{\beta_1}(x) \, dx \, d\tau \]
\[ + \gamma \int_{\eta}^{t} \left( \int_{0}^{1} \left| D(su + (1-s)v) \right|^{2l-p} \, ds \right)(|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 \, dx \, d\tau. \]  
(4.11)

By (1.13') and (1.14')
\[ b(x, t) \leq \gamma (|u|^{q-1} + |v|^{q-1}) \leq \gamma \tau^{-\delta(q-1)}, \]
\[ a_0(x, \tau) A_{2/p}(x) \leq \gamma \tau^{-\delta_1(p-2)}, \]
\[ \int_{0}^{1} \left| D(su + (1-s)v) \right|^{2l-p} \, ds \leq \gamma \tau^{-\delta_1(2l-p)}. \]

Carrying these estimates in (4.10), we obtain
\[ \int_{B_\rho(t)} (|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 \, dx \]
\[ \leq \int_{B_\rho(\eta)} (|w| + \eta)^{1+\epsilon} A_{\beta_1} \xi^2 \, dx \]
\[ + \gamma \int_{\eta}^{t} \left( \tau^{-\sigma} + \tau^{-\delta_1(2l-p)} \right) \int_{B_\rho} (|w| + \eta)^{1+\epsilon} A_{\beta_1}(x) \, dx \, d\tau, \]

where \( \sigma = \max\{\delta_1(p-2), \delta(q-1)\} \). Letting \( \eta \to 0, \rho \to \infty \) in turn, by Lemma 4.3, we get
\[ \int_{R^N} (|w| + \eta)^{1+\epsilon} A_{\beta_1} \, dx \leq \gamma \int_{\eta}^{t} \left( \tau^{-\sigma} + \tau^{-\delta_1(2l-p)} \right) \int_{R^N} (|w| + \eta)^{1+\epsilon} A_{\beta_1}(x) \, dx \, d\tau. \]

Since \( 0 \leq 2l - p \leq p - 2 \), so \( \tau^{-\sigma}, \tau^{-\delta_1(2l-p)} \) is integrable, this implies
\[ \int_{R^N} (|w| + \eta)^{1+\epsilon} A_{\beta_1} \, dx = 0, \quad \forall t \in (0, T), \]
by Gronwall’s Lemma, provided

\[
\int_{\mathbb{R}^N} \left( |w| + \eta \right)^{1+\epsilon} A_{\beta_1} \, dx \in L^\infty(0, T). \tag{4.12}
\]

Notice that the parameter \( \beta_1 \) in the calculation above is only restricted by

\[
\beta_1 = \beta + 1/p \geq \frac{k}{p(p-2)} + 1/p.
\]

Thus by Lemma 4.2 if we choose \( \beta_1 > \frac{k}{p(p-2)} + 1/p + \epsilon/(p-2) \), we have

\[
\int_{\mathbb{R}^N} \left( |w| + \eta \right)^{1+\epsilon} A_{\beta_1} \, dx 
\leq \gamma t^{-\delta \epsilon} |w| A_{k/p(p-2)+1/p}(x) \, dx \leq \gamma t^{-\delta \epsilon(1-\delta(\lambda_1-1))} \leq \gamma_1,
\]

i.e. (4.12) holds and Theorem 1.2 is proved. \( \square \)

**Lemma 4.4.** Assume \( l \geq p/2 \) and \( u_1, v_1 \) in \( \mathcal{X} \) are subsolution and supersolution of (1.1) respectively. If \( u_1, v_1 \) satisfies

\[
\int_{0}^{t} \int_{\mathbb{R}^N} |Du|^{p-1} A_{\beta+1/p}(x) \, dx \, d\tau \leq \gamma t^{(1-\delta(\lambda_1-1))/p}, \quad t \in (0, T), \tag{4.13}
\]

and

\[
\lim_{t \to 0} \int_{B_R} \left( u_1(x, t) - v_1(x, t) \right) \, dx = 0, \quad \forall R > 0,
\]

then

\[
u_1(x, t) \leq v_1(x, t) \quad \text{in } S_T.
\]

**Proof.** Replace the testing function in the proof of Theorem 1.2 by \((w_1)_+ + \eta \epsilon \text{ sgn}_x(w_1)_+\), then combining Remark 4.1 we get

\[
\int_{\mathbb{R}^N} \left( (w_1)_+ + \eta \right)^{1+\epsilon} A_{\beta_1} \, dx = 0, \quad \forall t \in (0, T). \quad \square
\]

**Remark 4.2.** By Remark 2.3, for the solution that is from Theorem 1.1 satisfies (4.13).

**Proof of Theorem 1.3.** Assume \( \limsup_{t \to T^*} \|u(\cdot, t)\|_h = M < \infty \). Take \( v_0 = u(x, T^* - \epsilon) \), where \( 0 < \epsilon < T^* \). By Theorem 1.1, we know that there exist a constant \( T_0(N, p, q, M) \) and a weak solution \( u \) to (1.1)–(1.2), where \( u_0 \) is replaced by \( v_0 \). Combine Theorem 1.2, we have \( T^* \geq T^* - \epsilon + T_0 \). Taking \( \epsilon \) small enough, we get a contradiction. \( \square \)
Proof of Theorem 1.4. We seek an unbounded self-similar subsolution of (1.1) on \([t_0, 1/\epsilon) \times \mathbb{R}^N, 0 < t_0 < 1/\epsilon\), of the form

\[
v(x, t) = \frac{1}{(1 - \epsilon t)^k} V \left( \frac{|x|}{(1 - \epsilon t)^m} \right),
\]

where \(V\) is defined by

\[
V(y) = \begin{cases} 
(1 - y^2)^3, & 0 \leq y \leq 1, \\
0, & y \geq 1,
\end{cases}
\]

with \(k, m, \epsilon > 0\) and \(t_0\) to be determined.

Let us note that

\[
\forall t \in [t_0, 1/\epsilon), \quad \text{Supp}(v(\cdot, t)) \subset B(0, (1 - \epsilon t_0)^m).
\]

We compute (setting \(y = |x|/(1 - \epsilon t)^m\) for convenience)

\[
Lv = v_t - \text{div}(\nabla v)^{p-2} \nabla v - v^q + \lambda |\nabla v|^l
= \frac{\epsilon}{(1 - \epsilon t)^{k+1}} (kV + myV')
- \frac{1}{(1 - \epsilon t)^{(k+m)(p-1)+m}} (-V')^{p-2} (V'(p-1)(N-1)/y + V''(p-1))
- \frac{1}{(1 - \epsilon t)^{kq}} V^q + \lambda \frac{1}{(1 - \epsilon t)^{(k+m)l}} |V'|^l.
\]

We first choose

\[
k = \frac{1}{q - 1}, \quad m = \frac{q - (p - 1)}{2(q - 1) p},
\]

so that \(kq = k + 1 > k(p - 1) + mp \geq (k + m)l\), and next we choose

\[
\alpha = \left( \frac{k}{k + 3m} \right)^{1/2}, \quad \epsilon < \frac{(1 - \alpha^2)^{3q} \gamma}{2k}.
\]

(I) If \(0 \leq y \leq \alpha\), then the function \(V\) obviously satisfies

\[
(1 - \alpha^2)^3 \leq V \leq 1, \quad |V'| \leq 6, \quad |V'/y| \leq 6, \quad |V''| \leq 30,
\]

\[
Lv \leq \frac{k \epsilon}{(1 - \epsilon t)^{k+1}} + \frac{\gamma}{(1 - \epsilon t)^{(k+m)(p-1)+m}}
- \frac{(1 - \alpha^2)^{3q}}{(1 - \epsilon t)^{kq}} + \frac{\gamma}{(1 - \epsilon t)^{(k+m)l}}.
\]

For \(t_0 \leq t < 1/\epsilon\), with \(t_0\) sufficiently close to \(1/\epsilon\), then by (4.17), (4.18) we have
\[ L v \leq \frac{\gamma}{(1 - \epsilon t)^{(k+m)(p-1)+m}} - \frac{1}{2} \frac{(1 - \alpha^2)^3q}{(1 - \epsilon t)^{kq}} + \frac{\gamma}{(1 - \epsilon t)^{(k+m)l}} < 0. \]

(II) If \( \alpha < y \leq 1 \), then the function \( V \) obviously satisfies

\[ -V' \geq 6\alpha(1 - y^2)^2 \geq 0, \quad -V'' = 6(1 - y^2)^2 - 24(1 - y^2)y^2 \leq 6(1 - y^2)^2, \]

and by (4.18) we have that

\[ kV + myV' \leq \frac{1}{2} myV'. \]

Hence,

\[
Lv \leq \frac{\epsilon}{(1 - \epsilon t)^{k+1}} (kV + myV')
+ \frac{1}{(1 - \epsilon t)^{(k+m)(p-1)+m}} (-V')^{p-2} (-V'(p-1)(N-1)/y + 6(1 - y^2)^2(p-1))
- \frac{1}{(1 - \epsilon t)^{kq}} V^q + \lambda \frac{1}{(1 - \epsilon t)^{(k+m)l}} |V'|^l
\leq \frac{1}{2} \frac{\epsilon}{(1 - \epsilon t)^{k+1}} (myV')
+ \frac{1}{(1 - \epsilon t)^{(k+m)(p-1)+m}} (\gamma |V'|^{p-1} + \gamma |V'|^{p-2}(1 - y^2)^2)
+ \gamma \frac{1}{(1 - \epsilon t)^{(k+m)l}} |V'|^l
\leq (1 - y^2)^2 \left( \frac{-3\epsilon m\alpha}{(1 - \epsilon t)^{k+1}} + \frac{\gamma |V'|^{p-2}}{(1 - \epsilon t)^{(k+m)(p-1)+m}} + \frac{\gamma |V'|^{l-1}}{(1 - \epsilon t)^{(k+m)l}} \right).
\]

Hence for \( t_0 \leq t < 1/\epsilon \), with \( t_0 \) sufficiently close to \( 1/\epsilon \), we get

\[ Lv \leq 0. \]

Of course, we also have \( Lv \leq 0 \) for \( y > 1 \).

Now, by translation, one can assume without loss of generality that \( \psi(0) > 0 \). Under the assumptions above, it follows that

\[ \forall x \in B(0, \rho), \quad \psi(x) \geq C, \quad (4.19) \]

for some ball \( B(0, \rho) \) and some constant \( C > 0 \). Taking \( t_0 \) still closer to \( 1/\epsilon \) if necessary, one can assume that \( B(0, (1 - \epsilon t_0)^m) \subset B(0, \rho) \). Combining (4.16) and (4.19) with the fact that \( \psi \geq 0 \) then yields

\[ \phi(x) = \mu \psi(x) \geq \frac{V(0)}{(1 - \epsilon t_0)^k} \geq v(x, t_0), \quad x \in R^N, \]
for all $\mu \geq \mu_0 = V(0)/(C(1 - \epsilon t_0)^{1/k})$. By the comparison principle (Lemma 4.4), it follows that

$$u(x, t) \geq v(x, t + t_0), \quad 0 < t < \min(T^*, 1/\epsilon - t_0), \quad x \in \mathbb{R}^N,$$

hence $T^* \leq 1/\epsilon - t_0$, which proves (i).

To establish (ii), it suffices to note that, by the previous calculation, $T^* = T^*[\mu \psi] \leq 1/\epsilon - T$ whenever $t_0 \leq T < 1/\epsilon$ and $\mu \geq V(0)/(C(1 - \epsilon T)^{1/k})$, or in other words,

$$T^*[\mu \psi] \leq \frac{1}{\epsilon} \left( \frac{1}{\mu C} \right)^{q-1}, \quad \text{for all} \quad \mu \geq \frac{1}{C(1 - \epsilon t_0)^{1/(q-1)}}. \quad \square$$

**Proof of Theorem 1.5.** Since $\lambda \geq 0$, the solutions of (1.1)–(1.2) are subsolutions of the following Cauchy problem

$$v_t = \text{div}(|Dv|^{p-2}Dv) + v^q \quad \text{in} \quad S_T, \quad (4.20)$$

$$v(x, 0) = u_0(x) \in L^1_{\text{loc}}(\mathbb{R}^N). \quad (4.21)$$

Combining Remarks 2.2, 2.5 and 3.1, similar to in [18] (see [18, pp. 359–362]), we can complete the proof of Theorem 1.5. \quad \square

**References**

