Note

Directed triangles in directed graphs

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Abstract


We show that each directed graph on \( n \) vertices, each with indegree and outdegree at least \( n/t \), where \( t = 5 - \sqrt{5} + \frac{1}{2} \sqrt{21 - 4 \sqrt{5}} \approx 2.8670975 \ldots \), contains a directed circuit of length at most 3.

It is an intriguing conjecture of Caccetta and Haggkvist [1] that any directed graph on \( n \) vertices, each with outdegree at least \( k \), contains a directed circuit of length at most \( \lceil n/k \rceil \). (In this paper, directed graphs have no loops and no parallel arcs (in the same or the opposite direction).)

A particularly interesting special case that is still open is: any directed graph on \( n \) vertices with minimum outdegree at least \( n/3 \) has a directed triangle. The best result along these lines is proved in [1]: any directed graph on \( n \) vertices with...
minimum outdegree at least \( s \), where
\[
s := \frac{3}{2} + \frac{1}{2} \sqrt{5} = 2.618034 \ldots,
\]
contains a directed triangle.

It is not even known whether any directed graph on \( n \) vertices, each with both indegree and outdegree equal to \( n/3 \), contains a directed triangle.

In this note we use the result of \([1]\) to show the following.

**Theorem.** Any directed graph on \( n \) vertices, each with both indegree and outdegree at least \( n/t \), where
\[
t := 5 - \sqrt{5} + \frac{1}{2} \sqrt{47 - 21\sqrt{5}} = 2.8670975 \ldots,
\]
contains a directed triangle.

**Proof.** Suppose \( D = (V, A) \) is a directed graph with \( |V| = n \), with each indegree and each outdegree at least \( n/t \), and without any directed triangle. Let \( k := \lceil n/t \rceil \).

We may assume
\[
5 - \sqrt{5} - \frac{1}{2} \sqrt{47 - 21\sqrt{5}} \leq n/k \leq 5 - \sqrt{5} + \frac{1}{2} \sqrt{47 - 21\sqrt{5}}.
\]
(We can replace any vertex \( v \) of \( D \) by \( l \) pairwise non-adjacent vertices, and any arc \((u, v)\) by \( l^2 \) arcs, from each of the \( l \) copies of \( u \) to each of the \( l \) copies of \( v \). We obtain a directed graph \( D' \) with \( n' := nl \) vertices, such that each vertex has indegree and outdegree at least \( n'/l \), and such that \( D' \) has no directed triangle. By choosing \( l \) large enough, \( n'/k = n'/\lceil n'/l \rceil \) will satisfy (3).)

Assume that deleting any arc would give a vertex of indegree or outdegree less than \( k \). We show:

there exists a vertex \( v' \) with both indegree and outdegree equal to \( k \). \( \quad (4) \)

Suppose such a vertex does not exist. Let \( W \) be the set of vertices of indegree equal to \( k \). Then there are no arcs leaving \( W \) (since any such arc could be deleted without violating the condition that each indegree and each outdegree is at least \( k \)). Since \( W \) contains at most \( k |W| \) arcs, it follows that if \( W \neq \emptyset \), \( W \) contains a vertex of outdegree at most \( k \). If \( W = \emptyset \), we apply this argument to the set of vertices of outdegree equal to \( k \) (which set should be nonempty if \( W = \emptyset \)).

For each \( v \in V \) let \( E^+_v \) and \( E^-_v \) denote the sets of outneighbours and innighbours of \( v \), respectively. For \( u, v, w \in V \) let
\[
E^+_{uv} := E^+_u \cap E^+_v, \quad E^-_{uv} := E^-_u \cap E^-_v, \\
E^+_{uvw} := E^+_u \cap E^+_v \cap E^+_w, \quad \text{and} \quad E^-_{uvw} := E^-_u \cap E^-_v \cap E^-_w.
\]
Moreover let
\[
\varepsilon^+_v := |E^+_v|, \quad \varepsilon^-_v := |E^-_v|, \quad \varepsilon^+_{uv} := |E^+_{uv}|, \\
\varepsilon^-_{uv} := |E^-_{uv}|, \quad \varepsilon^+_{uvw} := |E^+_{uvw}| \quad \text{and} \quad \varepsilon^-_{uvw} := |E^-_{uvw}|.
\]
We observe that for all \( u, v, w \in V \):

If \( (u, v), (v, w), (u, w) \in A \)

\[
\varepsilon_{uw}^- + \varepsilon_{vw}^- \geq \varepsilon_{uv}^- + \varepsilon_{uw}^- + \varepsilon_{vw}^- - n \geq 4k - n. \quad (5)
\]

Indeed, as \( D \) has no directed triangles, \( (E_u^- \cup E_v^-) \cap (E_v^+ \cup E_w^+) = \emptyset \). So

\[
|E_u^- \cup E_v^-| + |E_v^+ \cup E_w^+| \leq n. \quad \text{Now}
\]

\[
\varepsilon_{uv}^- = |E_u^-| = |E_u^- \cap E_v^-| = |E_u^-| + |E_v^-| - |E_u^- \cup E_v^-| = \varepsilon_{uv}^- + \varepsilon_{vw}^- - |E_u^- \cup E_v^-|.
\]

Similarly, \( \varepsilon_{vw}^- = \varepsilon_{vw}^- + \varepsilon_{uw}^- - |E_v^+ \cup E_w^+| \). This gives the first inequality in (5). The second inequality follows from the assumption that each indegree and each outdegree is at least \( k \).

We next show:

For each arc \( (u, v) \) of \( D \): \( \varepsilon_{uv}^- \geq (3k - n)s \) and \( \varepsilon_{uv}^+ \geq (3k - n)s \),

(6)

where \( s \) is as defined in (1).

To prove this, we may assume by symmetry that \( \varepsilon_{uv}^+ \geq \varepsilon_{uv}^- \). First we show \( \varepsilon_{uv}^- > 0 \), i.e., \( E_{uv}^- \neq \emptyset \). If \( E_{uv}^- \) would be empty, then \( E_u^- \cup E_v^+ \subseteq V \setminus E_u^- \), since there is no directed triangle. Hence \( |E_u^- \cup E_v^-| \leq n - k \). As \( |E_u^-| \geq k \) and \( |E_v^+| \geq k \) and as \( n/k \leq t < 3 \), we know \( E_v^- \cap E_u^- \neq \emptyset \), implying that there is a directed digon, contradicting our assumption.

Applying Caccetta and Haggkvist’s result [1] to the subgraph induced by \( E_{uv}^- \neq \emptyset \) we obtain the existence of a \( w \in E_{uv}^- \) so that \( \varepsilon_{uw}^- < \varepsilon_{uw}^-/s \). By (5):

\[
\varepsilon_{uv}^- \geq \varepsilon_u^- + \varepsilon_v^- + \varepsilon_{vw}^- - n - \varepsilon_{uw}^- \geq 3k - n + \varepsilon_{vw}^- - \varepsilon_{uw}^-.
\]

(7)

Since \( \varepsilon_{uw}^- + \varepsilon_{vw}^- \geq |E_{uw}^- \cap E_{vw}^-| + |E_{uw}^+ \cup E_{vw}^+| = \varepsilon_{uw}^- + \varepsilon_{vw}^- \), (7) implies

\[
\varepsilon_{uv}^- \geq 3k - n + \varepsilon_{vw}^- - \varepsilon_{uw}^- > 3k - n + (1 - s^{-1})\varepsilon_{uv}^-.
\]

(8)

This implies (6).

Now consider vertex \( v' \) described in (4). Since the subgraph induced by \( E_{uw}^- \neq \emptyset \) contains no loops or directed digons, the number of arcs contained in \( E_v^- \) is at most \( \varepsilon_v^- (\varepsilon_v^- - 1)/2 < \frac{1}{2} k^2 \). That is,

\[
\sum_{u \in E_{v}} \varepsilon_{uv}^- < \frac{1}{2} k^2.
\]

(9)

Similarly,

\[
\sum_{w \in E_v^-} \varepsilon_{vw}^- < \frac{1}{2} k^2.
\]

(10)

Let \( u' \) be a vertex of minimum indegree in the subgraph induced by \( E_{v}^- \) and let \( w' \) be a vertex of minimum outdegree in the subgraph induced by \( E_{v}^- \). So \( \varepsilon_{u'v'} \leq \varepsilon_{uv}^- \) for all \( u \in E_{v}^- \) and \( \varepsilon_{w'v'} \leq \varepsilon_{wv}^- \) for all \( w \in E_{v}^- \).

First assume

\[
\varepsilon_{u'v'}^- + \varepsilon_{w'v'}^- \geq 4k - n.
\]

(11)
Then (9) and (10) imply \(k^2 > (4k - n)k\), i.e., \(n/k > 3\), a contradiction. So we know
\[
\varepsilon_{u,v}^- + \varepsilon_{v,w}^+ \leq 4k - n. \tag{12}
\]
On the other hand, by (5) we know that for all \(w \in E_{u,v}^-\) one has \(\varepsilon_{u,v}^- + \varepsilon_{v,w}^+ \geq 4k - n\). This gives:
\[
\sum_{w \in E_{u,v}^-} \varepsilon_{u,v}^- = \sum_{w \in E_{u,v}^-} \varepsilon_{v,w}^+ + \sum_{w \in E_{v,w}^-\setminus E_{u,v}^-} \varepsilon_{v,w}^+ \\
\geq \varepsilon_{u,v}^-(4k - n - \varepsilon_{u,v}^-) + (\varepsilon_{v,w}^- - \varepsilon_{u,v}^-)\varepsilon_{v,w}^+. \tag{13}
\]
Similarly:
\[
\sum_{w \in E_{u,v}^-} \varepsilon_{u,w}^- \geq \varepsilon_{v,w}^-(4k - n - \varepsilon_{v,w}^-) + (\varepsilon_{v,w}^- - \varepsilon_{u,w}^-)\varepsilon_{v,w}^- \tag{14}
\]
Combining (9), (10), (13) and (14) gives:
\[
k^2 > \varepsilon_{u,v}^-(4k - n - \varepsilon_{u,v}^-) + (\varepsilon_{v,w}^- - \varepsilon_{u,v}^-)\varepsilon_{v,w}^+ + \varepsilon_{v,w}^-(4k - n - \varepsilon_{v,w}^+) \\
+ (\varepsilon_{v,w}^- - \varepsilon_{v,w}^+)\varepsilon_{v,w}^- \\
= \varepsilon_{u,v}^-\varepsilon_{v,w}^- + \varepsilon_{v,w}^+\varepsilon_{v,w}^- + (\varepsilon_{u,v}^- + \varepsilon_{v,w}^-)(4k - n - \varepsilon_{u,v}^- - \varepsilon_{v,w}^+) \\
\geq k(\varepsilon_{u,v}^- + \varepsilon_{v,w}^+) + 2(3k - n)s(4k - n - \varepsilon_{u,v}^- - \varepsilon_{v,w}^+) \\
= 2(3k - n)(4k - n)s + (k - 2(3k - n)s)(\varepsilon_{u,v}^- + \varepsilon_{v,w}^-) \\
\geq 2(3k - n)(4k - n)s + (k - 2(3k - n)s) \cdot 2(3k - n)s \\
= 2(3k - n)(5k - n - 2(3k - n)s)s.
\]
So
\[
(4s^2 - 2s)(n/k)^2 - (24s^2 - 16s)(n/k) + (36s^2 - 20s + 1) > 0, \tag{16}
\]
i.e.,
\[
(11 + 5\sqrt{5})(n/k)^2 - (60 + 28\sqrt{5})(n/k) + (82 + 39\sqrt{5}) > 0. \tag{17}
\]
This contradicts (3). □

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**References**