Error Bounds for the Rayleigh-Ritz-Galerkin Method

MARTIN H. SCHULTZ

Department of Mathematics, California Institute of Technology,
Pasadena, California, 91109

Submitted by Richard Bellman

1. INTRODUCTION

Error bounds, in terms of approximation theory, for the Rayleigh-Ritz-Galerkin method for approximating the solution of boundary value problems for linear and quasilinear elliptic partial differential equations have been given in a number of recent papers, cf. [1], [2], [3], [5], [9], [11], and [12]. The above problems and approximation method may be viewed in a Hilbert space setting.

If $H$ is the appropriate Hilbert space, then the above error bounds have the following form. If $\varphi \in H$ is the solution of the problem, $H_M$ is any finite dimensional subspace of $H$, and $\varphi_M$ is the approximation to $\varphi$ in $H_M$, then there exists a positive constant, $K$, independent of $H_M$ such that

$$\| \varphi_M - \varphi \|_H \leq K (\inf_{y \in H_M} \| y - \varphi \|_H).$$

(1.1)

In Section 2, we show that for a wide class of self-adjoint, linear problems, including those discussed in [1], [2], [5], [7], [11], and [12], the constant $K$ of (1.1) can be taken to be 1. In Section 3, we show that for a large class of quasi-linear problems, including those discussed in [3], [4], [9], and [11], the constant $K$ of (1.1) can be taken to be of the form $1 + \epsilon_M$, where $\epsilon_M$ is a nonnegative quantity, which can be made arbitrarily small by choosing $H_M$ to be sufficiently "large" in an appropriate sense.

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2. LINEAR PROBLEMS

In this section, we define and study a Rayleigh-Ritz method for symmetric, positive definite, densely defined, linear mappings in a real Hilbert space. Analogous results hold for such mappings in complex Hilbert spaces, but the details are left to the reader. An example of the Dirichlet problem for a linear, elliptic partial differential equation is given.
Throughout this section, let $H$ be a real Hilbert space with inner-product $(\cdot, \cdot)$ and norm $\| \cdot \|$. A linear mapping $\mathcal{L}: \mathcal{D} \subset H \to H$ is said to be positive definite if and only if its domain $\mathcal{D}$ is dense in $H$ and there exists a positive constant, $\gamma$, such that

$$\gamma \| u \|^2 \leq (\mathcal{L}u, u), \quad \text{for all } u \in \mathcal{D}, \quad (2.1)$$

and symmetric if and only if

$$(\mathcal{L}u, v) = (u, \mathcal{L}v), \quad \text{for all } u \text{ and } v \in \mathcal{D}. \quad (2.2)$$

Throughout this section, we assume that $\mathcal{L}$ is symmetric and positive definite.

Corresponding to the equation

$$\mathcal{L}[u] = f \in H, \quad (2.3)$$

we consider the functional

$$F[u] = (\mathcal{L}u, u) - 2(u, f), \quad (2.4)$$

which is defined for all $u \in \mathcal{D}$. Mikhlin, cf. [7], has proved the following variational principle for Eq. (2.3).

**Theorem 2.1.** $\varphi$ is a classical solution of (2.3), i.e., $\varphi \in \mathcal{D}$ and $\mathcal{L}[\varphi] = f$, if and only if $\varphi$ yields a strict, global minimum of $F[u]$ over $\mathcal{D}$, i.e., if $\vartheta \in \mathcal{D}$ and $0 \neq \varphi$ then $F[\varphi] > F[\vartheta]$.

Theorem 2.1 enables us to replace the problem of solving (2.3) by the problem of minimizing $F[u]$. However, it may happen that (2.3) does not have a classical solution, but that the minimization problem has a solution if $\mathcal{D}$ is suitably extended. Mikhlin has shown, cf. [7], that for a symmetric, positive definite mapping, $\mathcal{L}$, it is always possible to extend $\mathcal{D}$ in a well-defined manner so that the problem of minimizing $F[u]$ has a unique solution.

For this purpose, we define a new inner-product in $\mathcal{D}$ by taking

$$[u, v] \equiv (\mathcal{L}u, v), \quad \text{for all } u, v \in \mathcal{D}. \quad (2.5)$$

The corresponding norm is denoted by

$$|| u || \equiv [u, u]^{1/2}, \quad \text{for all } u \in \mathcal{D}. \quad (2.6)$$

and it follows from (2.1) that

$$\gamma^{1/2} || u || \leq || u ||, \quad \text{for all } u \in \mathcal{D}. \quad (2.7)$$

Defining $H_{\mathcal{D}}$ to be the completion of $\mathcal{D}$ with respect to $|| \cdot ||$, we easily have

**Lemma 2.1.** $\mathcal{D}$ is dense in $H_{\mathcal{D}}$, $H_{\mathcal{D}} \subset H$, and inequality (2.7) holds for all $u \in H_{\mathcal{D}}$. 

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If we rewrite the functional \( F \) as \( F[u] = [u, u] - 2(f, u) \), it is clear that \( F \) is defined for all \( u \in H_\varphi \) and Mikhlin, cf. [7], has shown

**Theorem 2.2.** The problem of minimizing \( F[u] \) over \( H_\varphi \) has a unique solution for every \( f \in H \). Moreover, if \( \varphi \) is a classical solution of \( L[u] = f \) then \( \varphi \) minimizes \( F[u] \) over \( H_\varphi \).

The unique minimizing element in \( H_{sr} \) provided by the preceding theorem is called the *generalized solution* of \( L[u] = f \).

We now consider the Rayleigh-Ritz method for approximating the generalized solution. Consider any finite dimensional subspace \( H_M \) of \( H_\varphi \) and let \( \{B_i\}_{i=1}^M \) be a basis for \( H_M \). The Rayleigh-Ritz method is to find an approximation to the generalized solution, \( \varphi \), of (2.3) by determining an element \( \hat{\varphi} \in H_M \), which minimizes \( F[u] \) over \( H_M \). The following fundamental result shows that this is a well-defined method.

**Theorem 2.3.** There is a unique element \( \hat{\varphi} \in H_M \) which minimizes \( F[u] \) over \( H_M \).

**Proof.** Considering

\[
F[\beta] = F \left[ \sum_{i=1}^{M} \beta_i B_i \right] = \sum_{i=1}^{M} \sum_{j=1}^{M} \beta_i \beta_j [B_i, B_j] - 2 \sum_{i=1}^{M} \beta_i (f, B_i)
\]

as a function of \( \beta \in \mathbb{R}^M \), it is clear that \( F \in C^2(\mathbb{R}^M) \) and hence \( F \) has a minimum at \( \beta^* \) if and only if

\[
\frac{\partial F}{\partial \beta_i} [\beta^*] = 0, \quad \text{for all} \quad 1 \leq i \leq M,
\]

and the Jacobian matrix

\[
J = \left[ \frac{\partial^2 F}{\partial \beta_i \partial \beta_j} [\beta^*] \right]
\]

is positive definite.

Calculating the equations of the system (2.8), we obtain

\[
\frac{\partial F}{\partial \beta_i} [\beta^*] = 2 \sum_{j=1}^{M} \beta_j [B_i, B_j] - 2(f, B_i), \quad 1 \leq i \leq M,
\]

or in matrix form

\[
A \beta^* = k,
\]

where

\[
A = \begin{bmatrix} [B_i, B_j] \end{bmatrix},
\]
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and

\[ k \equiv [(f, B_i)]. \]  

(2.13)

Clearly \( A \) is symmetric and positive definite. In fact, if \( \beta \neq 0 \), then

\[ \beta^T A \beta = \left[ \sum_{i=1}^{M} \beta_i B_i, \sum_{j=1}^{M} \beta_j B_j \right] \geq \gamma \left\| \sum_{i=1}^{M} \beta_i B_i \right\|^2 > 0. \]

Thus, (2.8) has a unique solution \( \beta^* \).

Moreover, from (2.10), it is clear that \( J = A \) and hence \( \beta^* \) is the unique minimum of \( F \) over \( R^M \).

QED

We remark that it follows from the preceding proof that the Rayleigh-Ritz method yields a linear system of equations, whose matrix is symmetric and positive definite for any finite dimensional subspace of \( H_\varphi \). Given a particular finite dimensional subspace, \( H_M \), of \( H_\varphi \), it is natural to ask whether or not \( \phi \) is a "good" approximation to \( \varphi \) in some reasonable sense. The following result shows \( \phi \) is the "best" approximation to \( \varphi \) in the subspace \( H_M \).

**THEOREM 2.4.** If \( \varphi \) is the generalized solution of (2.3), \( H_M \) is any finite-dimensional subspace of \( H_\varphi \), and \( \phi \) is the Rayleigh-Ritz approximation to \( \varphi \) in \( H_M \), then

\[ |\phi - \varphi| = \inf_{y \in H_M} |y - \varphi|. \]  

(2.14)

**PROOF.** If \( \eta \in H_\varphi \),

\[ F[\alpha] = F[\varphi + \alpha \eta] = [\varphi, \varphi] + \alpha^2 [\eta, \eta] + 2\alpha [\varphi, \eta] - 2(\varphi, f) - 2\alpha (\eta, f) \geq F[\varphi] \]

for all real \( \alpha \) and hence \( F[\alpha] \in C^1(R) \) and \( (dF/d\alpha)(0) = 0 \). Calculating this latter expression, we obtain

\[ [\varphi, \eta] - (f, \eta) = 0. \]  

(2.15)

Taking \( \eta = \varphi \) in (2.15), we obtain for all \( \xi \in H_\varphi \)

\[ F[\xi] - F[\varphi] = [\xi, \xi] - [\varphi, \varphi] + 2(f, \varphi - \xi) = [\xi, \xi] + [\varphi, \varphi] - 2(f, \xi). \]  

(2.16)

Taking \( \eta = \xi \) in (2.15), we obtain from (2.16)

\[ F[\xi] - F[\varphi] = [\xi, \xi] - 2[\varphi, \xi] + [\varphi, \varphi] = |\xi - \varphi|^2. \]  

(2.17)

Thus, if \( y \in S \),

\[ |\phi - \varphi|^2 = F[\phi] - F[\varphi] \leq F[y] - F[\varphi] = |y - \varphi|^2 \]

and

\[ \inf_{y \in S} |y - \varphi| \leq |\phi - \varphi| \leq \inf_{y \in S} |y - \varphi|. \quad \text{QED.} \]
As a concrete example of a symmetric, positive definite mapping, we consider the following problem. Let $\Omega$ be an open, bounded set in $\mathbb{R}^N$ with boundary $\partial\Omega$, which is sufficiently regular to guarantee that Green's Theorem holds in $\Omega$.

Using standard multi-index notation, cf. [13], we consider linear elliptic boundary value problems of the form

$$L^\alpha u = \sum_{|\alpha| \leq m, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u) (x) = f(x), \quad x \in \Omega, \quad (2.18)$$

subject to the boundary conditions

$$D^j u(x) = 0, \quad x \in \partial\Omega, \quad 0 \leq j \leq m - 1, \quad (2.19)$$

where $a_{\alpha\beta}(x) \in C^{|\alpha|}(\Omega)$, $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$, $x \in \Omega$, $f \in L^2(\Omega)$, and there exists a positive constant $\gamma$ such that

$$\int \Omega u L^\alpha [u] \, dx = \sum_{|\alpha| \leq m, |\beta| \leq m} \int \Omega a_{\alpha\beta}(x) D^\beta u \, D^\alpha u \, dx \geq \gamma^2 \int \Omega u^2(x) \, dx \quad (2.20)$$

for all $u \in \mathcal{D}[\varphi] = \{u \in C^{2m}(\Omega) | u \text{ satisfies the boundary conditions } (2.19)\}$.

In this special case $H = L^2(\Omega)$, $\mathcal{D}[\varphi]$ is dense in $H$ and the functional $F$ has the form

$$F[u] = \sum_{|\alpha| \leq m, |\beta| \leq m} \int \Omega a_{\alpha\beta}(x) D^\beta u \, D^\alpha u \, dx - 2 \int \Omega f(x) u(x) \, dx. \quad (2.21)$$

Moreover, if $\varphi$ is the generalized solution of (2.18)-(2.19), $H_M$ is any finite dimensional subspace of $H[\varphi]$ and $\tilde{\phi}$ is the Rayleigh-Ritz approximation to $\varphi$ in $H_M$ then by Theorem 2.4

$$\left( \sum_{|\alpha| \leq m, |\beta| \leq m} \int \Omega a_{\alpha\beta}(x) D^\beta (\tilde{\phi} - \varphi) \, D^\alpha (\tilde{\phi} - \varphi) \, dx \right)^{1/2} \leq \inf_{\phi \in S} \left( \sum_{|\alpha| \leq m, |\beta| \leq m} \int \Omega a_{\alpha\beta}(x) D^\beta (\phi - \varphi) \, D^\alpha (\phi - \varphi) \, dx \right)^{1/2}. \quad (2.22)$$

We remark that in the special case of $N = 1$ and $m = 1$ these results have been previously obtained in [12].
3. Nonlinear Problems

In this section, we define and study the Galerkin method for completely continuous, nonlinear mappings in a real Hilbert space. Analogous results hold for such mappings in complex Hilbert space, but the details are left to the reader. An example of a class of boundary value problems for nonlinear elliptic partial differential equations is given.

Throughout this section let \( H \) be a real Hilbert space with inner product \((\cdot, \cdot)\) and norm \( \| \cdot \| \). For each positive number, \( R \), let

\[
S_R(H) = \{ x \in H : \| x \| = R \} \quad \text{and} \quad B_R(H) = \{ x \in H : \| x \| \leq R \}.
\]

Furthermore, let \( \{ H_n \}_{n=1}^{\infty} \) be a sequence of finite dimensional subspaces of \( H \) such that

\[
\lim_{n \to \infty} \inf_{x \in H_n} \| y - \varphi \| = 0, \quad \text{for all} \quad \varphi \in H,
\]

(3.1)

and \( \{ P_n \}_{n=1}^{\infty} \) be the sequence of linear orthogonal projection mappings of \( H \) such that

\[
P_n H = H_n, \quad \text{for all} \quad n \geq 1,
\]

(3.2)

and

\[
\| P_n \| = 1, \quad \text{for all} \quad n \geq 1.
\]

(3.3)

For each \( n \geq 1 \), we consider the approximate problem

\[
x_n = P_n T x_n \quad \text{in} \quad H_n.
\]

(3.4)

In this situation we have the following existence and convergence result.

**Theorem 3.1.** If \( T \) is a completely continuous mapping of \( B_R(H) \) into itself, \( P_n T \) has a fixed point, \( x_n \), in \( B_R(H_n) \) for all \( n \geq 1 \) and there exists a subsequence \( \{ x_{n_k} \}_{k=1}^{\infty} \) and \( x_\infty \in B_R(H) \) such that \( \| x_{n_k} - x_\infty \| \to 0 \) as \( k \to \infty \) and \( x_\infty \) is a fixed point of \( T \). Moreover, if \( x_\infty \) is the unique fixed point of \( T \) in \( B_R(H) \), then the entire sequence \( \{ x_n \}_{n=1}^{\infty} \) converges to \( x_\infty \).

**Proof.** Clearly, for all \( n \geq 1 \), \( P_n T \) maps \( B_R(H_n) \) into itself continuously. Hence, by the Brouwer Fixed-Point Theorem, there exists \( x_n \in B_R(H_n) \) such that \( P_n T x_n = x_n \). Since \( \{ x_n \}_{n=1}^{\infty} \) is a bounded sequence and \( T \) is completely continuous, there exists a subsequence \( \{ x_{n_k} \}_{k=1}^{\infty} \) and \( x_\infty \in H \) such that \( T x_{n_k} \to x_\infty \) as \( k \to \infty \). But \( x_{n_k} = P_{n_k} T x_{n_k} \to x_\infty \) and hence \( T x_\infty = \lim_{k \to \infty} T x_{n_k} = x_\infty \). Thus, there exists a subsequence of \( \{ x_n \}_{n=1}^{\infty} \) which converges to a fixed point of \( T \).

Finally, if \( T \) has a unique fixed point, \( x_\infty \), in \( R_R(H) \) and \( \{ x_{n_k} \}_{k=1}^{\infty} \) is any subsequence of \( \{ x_n \}_{n=1}^{\infty} \) which does not converge to \( x_\infty \) then by the above argument there is a subsequence \( \{ x_{n_{k_l}} \}_{l=1}^{\infty} \) of \( \{ x_{n_k} \}_{k=1}^{\infty} \) which converges to a fixed point of \( T \) which by hypothesis must be \( x_\infty \).
Hence, every subsequence of \( \{x_n^*\}_{n=1}^{\infty} \) converges to \( x_\infty \) and thus \( \{x_n\}_{n=1}^{\infty} \) converges to \( x_\infty \).

**COROLLARY** (Schauder Fixed Point Theorem). If \( T \) is a completely continuous mapping of \( B_R(H) \) into itself, then \( T \) has a fixed point in \( B_R(H) \), i.e., there is an \( x \in B_R(H) \) such that \( Tx = x \).

**THEOREM 3.2.** Let \( T \) be a completely continuous mapping of \( B_R(H) \) into itself with a fixed point \( x_\infty \in B_R(H) \). If \( T \) has a Fréchet derivative, \( F \), in a neighborhood, \( \mathcal{N} \), of \( x_\infty \) and 1 is not in the spectrum of \( F \), then \( x_\infty \) is the unique fixed point of \( T \) in \( \mathcal{N} \) and the approximate problem \( x_n = P_nT x_n \) has a solution \( x_n \in H_n \) for all \( n \geq 1 \), which is unique for all sufficiently large \( n \) and

\[
\|x_\infty - x_n\| \leq (1 + \epsilon_n) \|x_\infty - P_n x_\infty\| = (1 + \epsilon_n) \inf_{y \in H_n} \|x_\infty - y\| \quad (3.5)
\]

for all \( n \geq 1 \) where \( \epsilon_n \to 0 \) as \( n \to \infty \).

**PROOF.** If \( y_\infty \in \mathcal{N} \) is a fixed point of \( T \) in \( \mathcal{N} \), then

\[
x_\infty - y_\infty = F(x + \theta y)(x - y)
\]

for some \( 0 \leq \theta \leq 1 \) and hence \( x_\infty = y_\infty \). Thus \( x_\infty \) is the unique fixed point \( T \) of \( \mathcal{N} \).

By Theorem 3.1, the fixed points \( x_n \) exist, \( x_n \to x_\infty \) as \( n \to \infty \), and hence \( x_n \in \mathcal{N} \) for all sufficiently large \( n \). Moreover, since linear mappings are their own Fréchet derivatives, the Fréchet derivative of \( P_n T \) is \( P_n F \) and 1 is not in the spectrum of \( P_n F \) for all sufficiently large \( n \). Thus, \( x_n \) is the unique fixed point of \( P_n T \) for all sufficiently large \( n \).

Moreover, there exists an \( N > 0 \) such that

\[
(I - F)(x_n - x_\infty) + (F - P_n F)(x_n - x_\infty) + P_n(FP_n - F)(x_n - x_\infty)
\]

\[
= (I - F)(P_n x_\infty - x_\infty) + (F - P_n F)(P_n x_\infty - x_\infty), \quad \text{for all} \quad n \geq N. \quad (3.6)
\]

Let \( \|(I - F)^{-1}\| = R \), \( \|F - P_n F\| = \alpha_n \), and \( \|FP_n - F\| = \beta_n \). Since \( F \) is completely continuous, cf. [6, Lemma 4.1, p. 135], it is easily verified that \( \alpha_n \to 0 \) and \( n \to \infty \) and \( \beta_n \to 0 \) as \( n \to \infty \). Furthermore,

\[
\gamma_n = \frac{\|Tx_n - Tx_\infty - F(x_n - x_\infty)\|}{\|x_n - x_\infty\|} \to 0 \quad \text{as} \quad n \to \infty.
\]

Applying \((I - F)^{-1}\) to both sides of (3.6) and taking norms, we have

\[
\|x_n - x_\infty\| = R(\alpha_n + \beta_n + \gamma_n) \|x_n - x_\infty\|
\]

\[
\leq \|P_n x_\infty - x_\infty\| + R\alpha_n \|P_n x_\infty - x_\infty\| \quad (3.7)
\]
and hence
\[ \| x_n - x_\infty \| \leq (1 + \epsilon_n) \| P_n x_\infty - x_\infty \|, \]
where
\[ \epsilon_n \equiv R(2\alpha_n + \beta_n + \gamma_n) \left( 1 - R\alpha_n - R\beta_n - R\gamma_n \right)^{-1}. \quad \text{QED} \]

We now consider in detail the application of Theorems 3.1 and 3.2 to a general class of boundary value problems of the following form cf. [10]. Let $\Omega$ be an open, bounded set in $\mathbb{R}^N$ with boundary $\partial \Omega$, which is sufficiently regular to guarantee that Green's Theorem holds in $\Omega$. Consider
\[
\mathcal{L}[u] = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u)(x) = f(x, u), \quad x \in \Omega, \tag{3.9}
\]
subject to the boundary conditions
\[
D^j x(x) = 0, \quad x \in \partial \Omega, \quad 0 \leq j \leq m - 1, \tag{3.10}
\]
where $a_{\alpha\beta}(x) \in C^{|\alpha|}(\Omega)$, $a_{\alpha\beta}(x) = \alpha_{\alpha\beta}(x)$, $x \in \Omega$, $f(x, u)$ is a measurable function of $x$ for all $-\infty < u < \infty$, there exists a positive constant, $K$, such that
\[
|f(x, u)| \leq K, \quad \text{for all} \quad x \in \Omega, \quad -\infty < u < \infty, \tag{3.11}
\]
and there exists a positive constant, $c$, such that
\[
\int_{\Omega} u \mathcal{L}[u] \, dx \geq c \int_{\Omega} u^2(x) \, dx, \tag{3.12}
\]
for all $u \in C^{2m}(\Omega)$ satisfying the boundary conditions (3.10).

As in Section 2, let $H_\mathcal{L}$ denote the Hilbert space which is the completion of the $C^{2m}(\Omega)$-functions satisfying the boundary conditions (3.10) with respect to the norm $\| u \|_{H_\mathcal{L}} \equiv [u, u]^{1/2} = (\mathcal{L}[u], u)^{1/2}$. It follows from [7, Theorem 2, p. 323] that $H_\mathcal{L} \subset L^2(\Omega)$.

From Theorem 3, p. 222 and Theorem 2, p. 461 of [7], we have

**Theorem 3.3.** If $\mathcal{L}$ satisfies (3.12) and is such that any set which is bounded in $H_\mathcal{L}$ is precompact in $L^2(\Omega)$ and $F: u \to f(x, u)$ satisfies (3.11), then $\mathcal{L}$ has an extension $\hat{\mathcal{L}}$ such that $D_\mathcal{L} \subset D_\mathcal{\hat{L}} \subset H_\mathcal{\hat{L}}$, $[u, v] = (\hat{\mathcal{L}}u, v)$ for all $u, v \in D_\mathcal{\hat{L}}$, and $\hat{\mathcal{L}}^{-1}$ is defined as a completely continuous mapping from $L^2(\Omega)$ to $H_\mathcal{\hat{L}}$, $T \equiv \hat{\mathcal{L}}^{-1}F$ is a completely continuous mapping of $H_\mathcal{\hat{L}}$ into itself, $\hat{\mathcal{L}}$ has a countable set of eigenvalues
\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \tag{3.13}
\]
with $\lambda_n \to \infty$ as $n \to \infty$, and if $R \equiv \| \hat{\mathcal{L}}^{-1} \| K$ (meas $\Omega)^{1/2}$, $\hat{\mathcal{L}}^{-1}F$ maps $B_R(H_\mathcal{\hat{L}})$ into itself.
THEOREM 3.4. If $\mathcal{L}$ satisfies (3.12) and is such that any set which is bounded in $H_\varphi$ is precompact in $L^2(\Omega)$, $F : u \to f(x, u)$ satisfies (3.11), and $\{H_n\}_{n=1}^\infty$ is a sequence of finite dimensional subspaces of $H_\varphi$ with associated orthogonal projection mappings $\{P_n\}_{n=1}^\infty$ such that $\lim_{n \to \infty} \| \varphi - P_n\varphi \|_{H_\varphi} = 0$ as $n \to \infty$ for all $\varphi \in H_\varphi$, then the problem

$$x_n = P_n \mathcal{L}^{-1} Fx_n, \quad n \geq 1,$$  \hspace{1cm} (3.14)

has a solution for all $n \geq 1$ and there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ converging to a solution of $x = \mathcal{L}^{-1} Fx$.

We now consider the form of finite dimensional problems (3.14). If $H_n$ has a basis $\varphi_1, \ldots, \varphi_n$, then $x_n$ satisfies (3.14) if and only if

$$[x_n, \varphi_i] = [\mathcal{L}^{-1} Fx_n, \varphi_i], \quad 1 \leq i \leq n.$$  \hspace{1cm} (3.15)

Using the results of Theorem 3.3 we have that $x_n$ satisfies (3.14) if and only if

$$[x_n, \varphi_i] = (Fx_n, \varphi_i), \quad 1 \leq i \leq n.$$  \hspace{1cm} (3.16)

Moreover, if $x_n = \sum_{j=1}^n \alpha_j \varphi_j$, then

$$\sum_{j=1}^n \alpha_j \varphi_j = \left( \sum_{j=1}^n \alpha_j \varphi_j, \varphi_i \right) = \left( F \sum_{j=1}^n \alpha_j \varphi_j, \varphi_i \right), \quad 1 \leq i \leq N.$$  \hspace{1cm} (3.17)

Using the definition of $[\cdot, \cdot]$ in this particular case, we have

$$\sum_{j=1}^n \alpha_j \left( \sum_{|\beta| \leq m} \int_{|x| = \gamma_j} a_{\alpha\beta}(x) \, D^\beta \varphi_j \, D^\gamma \varphi_i \, dx \right) = \int_{\Omega} f(x, \sum_{j=1}^n \alpha_j \varphi_j) \varphi_i(x) \, dx,$$  \hspace{1cm} (3.18)

$1 \leq i \leq n$, which is the system of algebraic equations obtained by the usual Galerkin procedure.

THEOREM 3.5. Let $\mathcal{L}$ satisfy (3.12) and be such that any set which is bounded in $H_\varphi$ is precompact in $L^2(\Omega)$, $F : u \to f(x, u)$ satisfy (3.11) and $\partial f/\partial u$ exist for all $x \in \Omega$ and be such that either $\partial f/\partial u (x, u) \leq \gamma_1 < \gamma_1$ for all $x \in \Omega$, $-\infty < u < \infty$, or $\gamma_j \leq (\partial f/\partial u)(x, u) \leq \gamma_{j+1} < \gamma_{j+1}$ for all $x \in \Omega$, $-\infty < u < \infty$, where $\{\lambda_i\}_{i=1}^\infty$ denote the eigenvalues of $\mathcal{L}$ given by Theorem 3.3, and there exists a positive constant, $K_1$, such that $| (\partial f/\partial u) (x, u) | \leq K_1$ for all $x \in \Omega$ and $-\infty < u < \infty$. If $\{H_n\}_{n=1}^\infty$ is a sequence of finite dimensional subspaces of $H_\varphi$ with associated orthogonal projection mappings $\{P_n\}_{n=1}^\infty$ such that $\lim_{n \to \infty} \| \varphi - P_n\varphi \|_{H_\varphi} = 0$ for all $\varphi \in H_\varphi$, then the problem

$$x = \mathcal{L}^{-1} Fx,$$  \hspace{1cm} (3.16)

has a unique solution, $x$, in $H_\varphi$ and the problem

$$x_n = P_n \mathcal{L}^{-1} Fx_n$$  \hspace{1cm} (3.17)
has a solution \( x_n \in H_n \) for all \( n \geq 1 \), which is unique for all sufficiently large \( n \) and

\[
\| x - x_n \|_{H^2} \leq (1 + \epsilon_n) \inf_{y \in H_n} \| x - y \|_{H^2} \quad \text{for all} \quad n \geq 1,
\]

where \( \epsilon_n \to 0 \) as \( n \to \infty \).

**Proof.** Since linear mappings are Fréchet differentiable, cf. [6], the above hypotheses imply that \( \mathcal{L}^{-1}F \) is Fréchet differentiable and 1 is not in the spectrum of the Fréchet derivative, \( \hat{\mathcal{L}}^{-1}F \) of \( \hat{\mathcal{L}}^{-1}F \). Thus, the result follows directly from Theorem 3.2.

QED

**References**