Stability of the intersection of solution sets of semi-infinite systems

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Abstract

Many mathematical programming models arising in practice present a block structure in their constraint systems. Consequently, the feasibility of these problems depends on whether the intersection of the solution sets of each of those blocks is empty or not. The existence theorems allow to decide when the intersection of non-empty sets in the Euclidean space, which are the solution sets of systems of (possibly infinite) inequalities, is empty or not. In those situations where the data (i.e., the constraints) can be affected by some kind of perturbations, the problem consists of determining whether the relative position of the sets is preserved by sufficiently small perturbations or not. This paper focuses on the stability of the non-empty (empty) intersection of the solutions of some given systems, which can be seen as the images of set-valued mappings. We give sufficient conditions for the stability, and necessary ones as well; in particular we consider (semi-infinite) convex systems and also linear systems. In this last case we discuss the distance to ill-posedness.

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1. Introduction

Many mathematical programming models present a block structure in their constraint systems, for instance the linear semi-infinite programming (LSIP in brief) formulation of certain functional approximation problems [10, Chapter 2]. For this reason, some well-known LSIP numerical methods which were initially conceived for analytic constraint systems (e.g., the purification, the feasible directions, and the hybrid methods in [1,14,9]) have been extended to LSIP problems with blocks of analytic systems (see [13,18,7] and references therein). To analyze the feasibility of these problems is equivalent to study whether the intersection of the solution sets of each of those blocks is empty or not. A second motivation is the study of the relative position of two sets, which poses the need to study the containment problem (term introduced by Mangasarian in [16]) as well as determining whether their intersection is at least non-empty. In the case of solution sets of inequality systems with fixed constraints both problems have been intensively...
analyzed, the former by means of existence theorems and the latter by establishing dual conditions [6,17,12]. However, the maintaining of the relative position of both sets through sufficiently small perturbation of the data has only been studied for the containment of solution sets of linear systems [8]. This paper deals with the problem of deciding whether the intersection of the solutions of two given systems, which can be seen as the images of set-valued mappings \( \mathcal{F} \) and \( \mathcal{G} \), is empty or not in the proximity of these (nominal) parameters. All the results have obvious extensions to the case of any finite intersection.

In general, we consider given two set-valued mappings \( \mathcal{F} : Y \Rightarrow \mathbb{R}^n \) and \( \mathcal{G} : Z \Rightarrow \mathbb{R}^n \), where \( (Y, \rho_Y) \) and \((Z, \rho_Z)\) are pseudometric spaces (i.e., \( \rho_Y : Y^2 \Rightarrow \mathbb{R} \cup \{+\infty\} \) and \( \rho_Z : Z^2 \Rightarrow \mathbb{R} \cup \{+\infty\} \) are pseudometrics) and a couple \((y_0, z_0) \in Y \times Z\). We will discuss the stability of the intersection mapping of \( \mathcal{F} \) and \( \mathcal{G} \), i.e., the mapping \( \mathcal{I} : \mathbb{R}^n \Rightarrow \mathbb{R}^n \), where \( X = Y \times Z \) and \( \mathcal{I}(y, z) := \mathcal{F}(y) \cap \mathcal{G}(z) \); in brief \( \mathcal{I} = \mathcal{F} \cap \mathcal{G} \). If \( x_i = (y_i, z_i), i = 1, 2 \), we put

\[
\rho_X(x_1, x_2) := \sup \{ \rho_Y(y_1, y_2), \rho_Z(z_1, z_2) \},
\]

which defines a pseudometric on \( \mathcal{X} \).

We say that the non-empty intersection \( \mathcal{F}(y_0) \cap \mathcal{G}(z_0) \neq \emptyset \) is stable (for short, \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) stably) at \((y_0, z_0)\) if there exists \( \varepsilon > 0 \) such that \( \mathcal{F}(y) \cap \mathcal{G}(z) \neq \emptyset \) for all \((y, z) \in Y \times Z\) whenever \( \rho_Y(y, y_0) < \varepsilon \) and \( \rho_Z(z, z_0) < \varepsilon \).

Analogously, the empty intersection \( \mathcal{F}(y_0) \cap \mathcal{G}(z_0) = \emptyset \) is stable (abbreviated as \( \mathcal{F} \cap \mathcal{G} = \emptyset \) stably) at \((y_0, z_0)\) if there exists \( \varepsilon > 0 \) such that \( \mathcal{F}(y) \cap \mathcal{G}(z) = \emptyset \) for all \((y, z) \in Y \times Z\) such that \( \rho_Y(y, y_0) < \varepsilon \) and \( \rho_Z(z, z_0) < \varepsilon \).

It is clear that \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) stably at \((y_0, z_0)\) if and only if \((y_0, z_0)\) lies in the interior of the domain of \( \mathcal{I} \) (dom \( \mathcal{I} := \{ (y_0, z_0) \in Y \times Z | \mathcal{F}(y_0) \cap \mathcal{G}(z_0) \neq \emptyset \} \) and, similarly, \( \mathcal{F} \cap \mathcal{G} = \emptyset \) stably at \((y_0, z_0)\) belongs to the exterior of the domain of \( \mathcal{F} \). In both cases we say that \( \mathcal{F} \cap \mathcal{G} \) is well-posed at \((y_0, z_0)\). Thus, \( \mathcal{F} \cap \mathcal{G} \) is ill-posed at \((y_0, z_0)\) if only if \((y_0, z_0)\) belongs to the boundary of the domain of \( \mathcal{I} \).

In the case of systems, we consider given \( y_0 = \{ f_0^t (x) \leq 0, t \in T \} \) and \( z_0 = \{ g_0^s (x) \leq 0, s \in S \} \), where the index sets \( T \) and \( S \) are arbitrary (possibly infinite) and \( f_0^t, g_0^s : \mathbb{R}^n \rightarrow \mathbb{R} \) for all \( t \in T \) and for all \( s \in S \). We denote by \( Y \) the class of all systems of the form \( y = \{ f_t^r (x) \leq 0, t \in T \}, f_t : \mathbb{R}^n \rightarrow \mathbb{R}, t \in T \) (i.e., those systems which have the same space of variables, \( \mathbb{R}^n \), and the same index set, \( T \)). As in [15], we define a pseudometric \( \rho_Y \) as follows: given \( y_t = \{ f_t^r (x) \leq 0, t \in T \} \in Y, i = 1, 2, \)

\[
\rho_Y(y_1, y_2) := \sup_{t \in T} \delta(f_t^1, f_t^2),
\]

where

\[
\delta(f_t^1, f_t^2) := \sum_{k=1}^{\infty} 2^{-k} \frac{\delta_k(f_t^1, f_t^2)}{1 + \delta_k(f_t^1, f_t^2)},
\]

with

\[
\delta_k(f_t^1, f_t^2) := \sup_{\| x \| \leq k} |f_t^1(x) - f_t^2(x)|, \quad k = 1, 2, \ldots .
\]

If a sequence \( \{ y_r \} \subset Y \) is \( \rho_Y \)-convergent to \( y_0 \) (in brief \( y_r \rightarrow y_0 \)), with \( y_r = \{ f_r^t (x) \leq 0, t \in T \} \in Y, r = 1, 2, \ldots , \) then for all \( x \in \mathbb{R}^n, f_r^t (x) \rightarrow f_0^t (x) \) uniformly on \( T \); moreover each \( f_r^t \) converges uniformly on compact sets to \( f_t \).

The system \( y_0 = \{ f_0^t (x) \leq 0, t \in T \} \) is said to be convex (linear) if for any fixed \( t \in T \) the function \( f_0^t (x) \) is convex (linear) in \( x \). In this case we associate with \( y_0 \) the subclass of \( Y \) formed by the convex (linear) systems, which we denote by \( Y_C (Y) \), respectively). In the convex case, since the convex real-valued functions are continuous, the supremum in (1) is attained. In the linear case, where \( y_0 = \{ a_0^t x \leq b_0^t, t \in T \} \), it is easy to prove that the topology induced by \( \rho_Y \) on \( Y_L \) coincides with the topology associated with the pseudometric of the uniform convergence on \( T \): given \( y_t = \{ a_t^i x \leq b_t^i, t \in T \} \in Y_L, i = 1, 2, \)

\[
d_{Y_L}(y_1, y_2) := \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 - a_t^2 \\ b_t^1 - b_t^2 \end{pmatrix} \right\|,
\]

where \( \| \cdot \| \) represents the Chebyshev norm. The fact that this topology of \( Y_L \) \( (Y_C) \) is the topology induced by \( Y_C \) \( (Y) \) yields that all the sufficient conditions for the stability properties of multivalued mappings defined on \( Y_C \) \( (Y) \) are
inherited by their restrictions to \( Y_L \) (\( Y_C \)). Nonetheless, the necessary conditions may require a direct argument. We indistinctly represent by \( \mathcal{F} \) the feasible set mapping for \( Y_L \), or \( Y_C \), or \( Y \).

In a similar way, the space of parameters associated with \( z_0 \), say \( Z \), is equipped with the pseudometric \( \rho_Z \) of the uniform convergence on the closed balls centered at the origin. We denote by \( Z_C \) (\( Z_L \)) the space of convex (linear) systems with index set \( S \) and space of variables \( \mathbb{R}^p \) and by \( \mathcal{G} \) the corresponding feasible set mapping.

The paper is organized as follows. Section 2 contains some preliminary results; in particular, it presents a characterization of the lower semicontinuity of convex-valued mappings. In Section 3, as an application of this result, we analyze those convex systems for which sufficiently small perturbations provide inconsistent and strongly inconsistent systems (i.e., systems which contain some finite inconsistent subsystem). Sections 4 and 5 deal with the stability of the intersection for convex and linear systems, respectively. In the last section it is also considered the ill-posedness of the intersection and explicit formulae are given for the distance to ill-posedness.

2. Preliminaries

Let us introduce some additional notation. If \( \Omega \) is a subset of a certain topological space bd \( \Omega \), int \( \Omega \), ext \( \Omega \) and cl \( \Omega \) represent the boundary, the interior, the exterior and the closure of \( \Omega \), respectively. If \( \Omega \) is a subset of a linear space, conv \( \Omega \) denotes the convex hull of \( \Omega \) and cone \( \Omega \) the convex conical hull of \( \Omega \cup \{0\} \). \( \theta_0 \) is the null vector in \( \mathbb{R}^n \). The open ball centered at \( x \) with radius \( \varepsilon > 0 \), for the Chebychev norm \( \| \cdot \| \) is represented by \( B(x; \varepsilon) \). We denote with \( d \) the corresponding distance, extending it with \( d(x, \emptyset) = +\infty \) for all \( x \in \mathbb{R}^n \). Furthermore for any subsets \( A, B \subset \mathbb{R}^n \), \( d(A, B) \) denotes inf \( \{\|x - y\| : x \in A, y \in B\} \).

For a convex function \( f \) we denote its graph by \( gph f \), its subdifferential by \( \partial f \) and its Fenchel conjugate by \( f^* \).

For the sake of completeness, we recall the stability concepts and some basic results for set-valued mappings that we shall consider in this paper. Let \( \mathcal{M} : \Omega \rightarrow \mathbb{R}^n \) be a set-valued mapping, where \( \Omega \) is a pseudometric space with pseudometric \( \rho_{\Omega} \). Its domain is \( \text{dom} \mathcal{M} := \{w \in \Omega \mid \mathcal{M}(w) \neq \emptyset \} \).

The following semicontinuity concepts are due to Bouligand and Kuratowski (see [2, Section 1.4]).

We say that \( \mathcal{M} \) is lower semicontinuous at \( \omega_0 \in \Omega \) in the Berge sense (lsc, in brief) if, for each open set \( W \subset \mathbb{R}^n \) such that \( W \cap \mathcal{M}(\omega_0) \neq \emptyset \), there exists an open set \( V \subset \Omega \), containing \( \omega_0 \), such that \( W \cap \mathcal{M}(w) \neq \emptyset \) for each \( w \in V \). Obviously, \( \mathcal{M} \) is lsc at \( \omega_0 \in \text{dom} \mathcal{M} \) if \( \mathcal{M} \) is lsc at \( \omega_0 \in \text{dom} \mathcal{M} \).

\( \mathcal{M} \) is upper semicontinuous at \( \omega_0 \in \Omega \) in the Berge sense (usc, in brief) if, for each open set \( W \subset \mathbb{R}^n \) such that \( \mathcal{M}(\omega_0) \subset W \), there exists an open set \( V \subset \Omega \), containing \( \omega_0 \), such that \( \mathcal{M}(w) \subset W \) for each \( w \in V \). If \( \mathcal{M} \) is usc at \( \omega_0 \in \text{dom} \mathcal{M} \), then \( \omega_0 \in \text{int} (\Omega \text{dom} \mathcal{M}) \).

\( \mathcal{M} \) is closed at \( \omega_0 \in \text{dom} \mathcal{M} \) if for all sequences \( \{\omega_r\} \subset \Omega \) and \( \{x_r\} \subset \mathbb{R}^n \) satisfying \( x_r \in \mathcal{M}(\omega_r) \) for all \( r \in \mathbb{N} \), \( \omega_r \rightarrow \omega_0 \) and \( x_r \rightarrow x_0 \), one has \( x_0 \in \mathcal{M}(\omega_0) \). If \( \mathcal{M} \) is usc at \( \omega_0 \in \text{dom} \mathcal{M} \) and \( \mathcal{M}(\omega_0) \) is closed, then \( \mathcal{M} \) is closed at \( \omega_0 \). Conversely, if \( \mathcal{M} \) is closed and locally bounded at \( \omega_0 \in \text{dom} \mathcal{M} \) (i.e., if there are a neighborhood of \( \omega_0 \), say \( V \), and a bounded set \( A \subset \mathbb{R}^n \) such that \( A \) contains \( \mathcal{M}(\omega) \) for every \( \omega \in V \)), then \( \mathcal{M} \) is usc at \( \omega_0 \).

Finally, \( \mathcal{M} \) is lsc (usc, closed, locally bounded) if it is lsc (usc, closed, locally bounded) at \( \omega_0 \) for all \( \omega \in \Omega \).

Following Rockafellar and Wets [20] we consider a mapping \( \mathcal{M} : \Omega \rightarrow \mathbb{R}^n \) and, for each \( \omega_0 \in \Omega \), the associated sets

\[
\liminf_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) \quad \text{and} \quad \limsup_{\omega \rightarrow \omega_0} \mathcal{M}(\omega),
\]

where \( \liminf_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) \) and \( \limsup_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) \) are the sets called the ‘inner limit’ and the ‘outer limit’, respectively. The inner limit is the set of all the limit points of all the possible sequences \( \{x^r\} \) satisfying \( x^r \in \mathcal{M}(\omega_r) \) for any \( \omega_r \), \( \omega_r \rightarrow \omega_0 \); whereas the outer limit consists of all the possible cluster points of such sequences. When these two limit sets coincide, we say that the limit \( \lim_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) \) exists in the Painlevé–Kuratowski sense and it is

\[
\lim_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) := \liminf_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) = \limsup_{\omega \rightarrow \omega_0} \mathcal{M}(\omega).
\]

When \( \mathcal{M}(\omega_0) = \liminf_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) \), it is said that \( \mathcal{M} \) is inner semicontinuous (for short, isc) at \( \omega_0 \) and, similarly, \( \mathcal{M} \) is outer semicontinuous (osc, in brief) at \( \omega_0 \) if \( \mathcal{M}(\omega_0) = \limsup_{\omega \rightarrow \omega_0} \mathcal{M}(\omega) \); finally, \( \mathcal{M} \) is isc (osc) if it is isc (osc) at \( \omega \) for all \( \omega \in \Omega \).

In [20] it is shown that, when \( \mathcal{M} \) is closed-valued, \( \mathcal{M} \) isc at \( \omega_0 \) is equivalent to \( \mathcal{M} \) lsc at \( \omega_0 \) and that \( \mathcal{M} \) osc at \( \omega_0 \) is equivalent to the closedness of \( \mathcal{M} \) at \( \omega_0 \).
Combining (4) and (5) we get i.e., (3) for

Similarly,

\[ \{1\} \times [-1, 1] \subset \text{conv}\{x^j, j \in J\}. \]  

Combining (4) and (5) we get

\[ [-1, 1]^2 \subset \text{conv}\{x^j, j \in J\}, \]

i.e., (3) for \( n = 2 \). The complete proof for a general \( n \) is accomplished by induction.
Theorem 1. Let $\mathcal{M} : \Omega \rightarrow \mathbb{R}^n$ be convex-valued and $\omega_0 \in \Omega$ such that $\text{int} \mathcal{M}(\omega_0) \neq \emptyset$. Then $\mathcal{M}$ is lsc at $\omega_0$ if and only if for all $v \in \text{int} \mathcal{M}(\omega_0)$ there exist $\varepsilon > 0$ and $\eta > 0$ such that

$$B(v; \eta) \subset \mathcal{M}(\omega)$$

for all $\omega \in \Omega$ with $\rho_\Omega(\omega, \omega_0) < \varepsilon$.

Proof. First, we assume that $\mathcal{M}$ is lsc at $\omega_0$ and suppose, without loss of generality, that $v = 0_n$. Let $\delta > 0$ be such that $[-\delta, \delta]^n \subset \mathcal{M}(\omega_0)$. We associate with each $j = (j_1, \ldots, j_n) \in J = \{-1, 1\}^n$ the open set

$$W(j) := \prod_{i=1}^n \left[ \frac{-\delta}{2}, \frac{\delta}{2} \right] \subset \mathcal{M}(\omega_0),$$

where $I_i^j$ is defined in (2).

Since $W(j) \cap \mathcal{M}(\omega_0) \neq \emptyset$ and $\mathcal{M}$ is lsc at $\omega_0$, there exists $\varepsilon_j > 0$ such that

$$W(j) \cap \mathcal{M}(\omega) \neq \emptyset$$

if $\rho_\Omega(\omega, \omega_0) < \varepsilon_j$. Let $\varepsilon := \min_{j \in J} \varepsilon_j > 0$ and let $\omega \in \Omega$ be such that $\rho_\Omega(\omega, \omega_0) < \varepsilon$. By (6) we can take points

$$x^j \in W(j) \cap \mathcal{M}(\omega), \quad j \in J.$$  

(7)

Since $(2/\delta)x^j \in \prod_{i=1}^n I_i^j$ for all $j \in J$, by (3) we get

$$[-1, 1]^n \subset \text{conv} \left\{ \frac{2}{\delta} x^j, j \in J \right\},$$

so that

$$\left[ \frac{-\delta}{2}, \frac{\delta}{2} \right]^n \subset \text{conv} \{x^j, j \in J\} \subset \mathcal{M}(\omega)$$

by (7) and the convexity of $\mathcal{M}(\omega)$. Then $B(0_n; \delta/2) \subset \mathcal{M}(\omega)$.

Conversely, let $W$ be an open set such that $W \cap \mathcal{M}(\omega_0) \neq \emptyset$. By the accessibility lemma (see [19, Theorem 6.1]) there exists $v \in W \cap \text{int} \mathcal{M}(\omega_0)$. Take $\varepsilon > 0$ and $\eta > 0$ such that

$$B(v; \eta) \subset \mathcal{M}(\omega)$$

for all $\omega \in \Omega$ with $\rho_\Omega(\omega, \omega_0) < \varepsilon$. Then $W \cap \mathcal{M}(\omega) \neq \emptyset$. □
Proposition 2. If $\mathcal{F}$ and $\mathcal{G}$ are convex-valued, lsc at $y_0$ and $z_0$, respectively, and $\text{int}[\mathcal{F}(y_0) \cap \mathcal{G}(z_0)] \neq \emptyset$, then $\mathcal{I} = \mathcal{F} \cap \mathcal{G}$ is lsc at $(y_0, z_0)$.

Proof. Let $v \in \text{int}[\mathcal{F}(y_0) \cap \mathcal{G}(z_0)] \subset \mathcal{I}(y_0, z_0)$. By Theorem 1 there exists $\varepsilon > 0$ such that $B(v; \varepsilon) \subset \mathcal{F}(y) \cap \mathcal{G}(z) = \mathcal{I}(y, z)$ if $\rho_X((y, z), (y_0, z_0)) < \varepsilon$. □

Observe that a necessary condition for $\mathcal{F} \cap \mathcal{G} \neq \emptyset$ stably at $(y_0, z_0)$ is that $y_0 \in \text{int dom } \mathcal{F}$ and $z_0 \in \text{int dom } \mathcal{G}$, these conditions being very close to the lsc property of $\mathcal{F}$ and $\mathcal{G}$, respectively. However, this simultaneous continuity property is not a sufficient condition unless $[\text{int } \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset$. Notice that this condition is weaker than the assumption in Proposition 2.

Proposition 3. If $\mathcal{F}$ and $\mathcal{G}$ are lsc at $y_0$ and $z_0$, respectively, $\mathcal{F}$ is convex-valued and $[\text{int } \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset$, then $\mathcal{F} \cap \mathcal{G} \neq \emptyset$ stably at $(y_0, z_0)$.

Proof. Let $\bar{x} \in [\text{int } \mathcal{F}(y_0)] \cap \mathcal{G}(z_0)$. By Theorem 1 there exist $\varepsilon_1 > 0$ and $\eta > 0$ such that $B(\bar{x}; \eta) \subset \mathcal{F}(y)$ for all $y \in Y$ with $\rho_Y(y, y_0) < \varepsilon_1$. Since $\bar{x} \in B(\bar{x}; \eta) \cap \mathcal{G}(z_0)$ there exists $\varepsilon_2 > 0$ such that $B(\bar{x}; \eta) \cap \mathcal{G}(z) \neq \emptyset$ for all $z \in Z$, $\rho_Z(z, z_0) < \varepsilon_2$. Hence, if $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and $\rho_Y(y, y_0) < \varepsilon$ and $\rho_Z(z, z_0) < \varepsilon$, we get $\emptyset \neq B(\bar{x}; \eta) \cap \mathcal{G}(z) \subset \mathcal{F}(y) \cap \mathcal{G}(z)$. □

The result in the previous proposition is not valid replacing the lsc property of $\mathcal{G}$ by the closedness condition.

Example 1. Let $Y = Z = \mathbb{R}$, $\mathcal{F}(y) = \{x \in \mathbb{R}^2 | -1 \leq x_i \leq 1, i = 1, 2\}$ (constant), $\mathcal{G}(z) = \{x \in \mathbb{R}^2 | -1 \leq x_i \leq 1, i = 1, 2; 0 \leq z \leq 2\}$ and $y_0 = z_0 = 0$. Obviously $0 \in (\text{int } \mathcal{F}(0)) \cap (\text{int } \mathcal{G}(0))$ but $\mathcal{F}(y) \cap \mathcal{G}(z) = \emptyset$ if $z < 0$.

The following proposition provides a sufficient condition for the stability of $\mathcal{F} \cap \mathcal{G} = \emptyset$ at $(y_0, z_0)$.

Proposition 4. Let $\mathcal{F}$ and $\mathcal{G}$ be closed at $y_0$ and $z_0$, respectively, and such that at least one of them is locally bounded. If $\mathcal{F}(y_0) \cap \mathcal{G}(z_0) = \emptyset$, then $\mathcal{F} \cap \mathcal{G} = \emptyset$ stably at $(y_0, z_0)$.

Proof. The assumptions guarantee that $\mathcal{F}$ is closed and locally bounded at $(y_0, z_0)$. Hence it is usc at $(y_0, z_0)$ and so $\mathcal{F}(y, z) = \emptyset$ in a neighborhood of $(y_0, z_0)$, i.e., $\mathcal{F} \cap \mathcal{G} = \emptyset$ stably at $(y_0, z_0)$. □

The next examples show that the local boundedness and the closedness assumptions are essential in the previous proposition.

Example 2. Let $n = 2$, $Y = Z = \mathbb{R}$, $\mathcal{F}(y) = \{x \in \mathbb{R}^2 | x_2 \geq 1\}$ (fixed) and $\mathcal{G}(z) = \{x \in \mathbb{R}^2 | x_1 - k \in [-1, 1], \mathcal{F}(0) \cap \mathcal{G}(0) = \emptyset$. Although both mappings, $\mathcal{F}$ and $\mathcal{G}$, are closed at 0, $\mathcal{F}(0) \cap \mathcal{G}(z) \neq \emptyset$ for all $z \neq 0$.

Example 3. Let $n = 2$, $Y \neq \emptyset$ arbitrary, $Z$ be the space of countable linear semi-infinite systems (LSISs in brief), $\mathcal{F}(y) = \{x \in \mathbb{R}^2 | x_i = 0, i = 1, 2\} = \{0\}$ (fixed), $\mathcal{G}(z)$ be the boundary of the solution set of $z \in Z$, $y_0 \in Y$ arbitrary and $z_0 = (1/k)x_1 + (1/k)x_2 \geq -1/k$, $k \in \mathbb{N}$. $\mathcal{F}$ is trivially closed and locally bounded at $y_0$. Clearly $\mathcal{G}(z_0) = \{x \in \mathbb{R}^2 | x_1 + x_2 = -1\}$, so that $\mathcal{F}(y) \cap \mathcal{G}(z_0) = \emptyset$. Nevertheless, $\mathcal{F}(y_0) \cap \mathcal{G}(z_r) = \{0\}$ for the sequence $\{z_r\} \subset Z$, $z_r \rightarrow z_0$ given by

$$z_r = \left\{ \frac{1}{k-r+1}x_1 + \frac{1}{k-r+1}x_2 \geq \frac{-1}{k-r+1}, k = 1, \ldots, r; \frac{1}{k-r+1}x_1 \geq 0, k = r + 1; \frac{1}{k-r+1}x_2 \geq 0, k = r + 2, \ldots \right\},$$

for all $r \in \mathbb{N}$. Thus the closedness of $\mathcal{G}$ at $z_0$ is crucial for the stability of $\mathcal{F} \cap \mathcal{G} = \emptyset$ at $(y_0, z_0)$.

3. Stability of convex and linear systems revisited

Throughout this section we consider a given (nominal) system $y_0 = \{f^0_i(x) \leq 0, \; t \in T\}$ with corresponding space of parameters, denoted by $\mathcal{Y}_C$ if $y_0$ is a convex system and by $\mathcal{Y}_L$ if $y_0$ is a linear system. $\mathcal{F}$ is the feasible set mapping in both cases, i.e., $\mathcal{F}(y)$ is the solution set of $y = \{f_i(x) \leq 0, \; t \in T\}$.
As an application of Theorem 1, we give a result on the stability of inconsistent convex systems (CSISs in brief). Now we recall some well-known results on the stability of the feasible set mapping, $\mathcal{F}$, in LSISs (see [11, 10]) and CSISs (see [15]): (i) $\mathcal{F}$ is closed, (ii) $\mathcal{F}$ is use at $y_0$ whenever $\mathcal{F}(y_0)$ is bounded and (iii) $\mathcal{F}$ is lsc at $y_0$ if and only if $y_0 \in \text{int dom} \mathcal{F}$ or, equivalently, there exists a strong Slater point (i.e., $\bar{x} \in R^n$ such that $f_0^0(\bar{x}) + \varepsilon \leq 0$ for all $t \in T$, for a certain $\varepsilon > 0$). Another useful condition in terms of the constraints of $y_0$ involves the set

$$D(y_0) := \left\{ \left( \frac{u_t}{\partial f_t^0(\mathbb{R}^n), t \in T} \right) \bigg| u_t \in \partial f_t^0(\mathbb{R}^n), t \in T \right\} = \bigcup_{t \in T} \text{gph}(f_t^0)^*.$$ 

In fact, $\mathcal{F}$ is lsc at $y_0$ if and only if $0_{n+1} \notin \text{cl conv} D(y_0)$.

In the linear case we can write $y_0 = \{a'_ix \leq b_t, t \in T\}$ and so

$$D(y_0) := \left\{ \left( \begin{array}{c} a_0' \\ b_0' \end{array} \right) \bigg| t \in T \right\}.$$ 

Then

$$\{a'_ix \leq b, \left( \begin{array}{c} a \\ b \end{array} \right) \in D(y_0)\}$$

is a linear representation of $\mathcal{F}(y_0)$ whose second moment cone is $\mathcal{M}(y_0) := \text{cone} D(y_0)$, so that this linear system is strongly inconsistent if and only if $(0_{n+1}) \in \text{cone} D(y_0)$ [10, Theorem 4.4], in which case $y_0$ is also strongly inconsistent.

We say that a convex system is stably (strongly) inconsistent if it belongs to the interior of the class of (strongly) inconsistent systems. Recall that a strongly inconsistent system is one which contains some finite inconsistent subsystem.

**Lemma 5.** The mapping $\mathcal{M} : Y_C \Rightarrow \mathbb{R}^n$ defined by $\mathcal{M}(y) := \text{cone} D(y)$, where $D(y) = \cup_{t \in T} \text{gph} f_t^*$, is lsc.

**Proof.** Let $y_0 = \{f_t^0(x) \leq 0, t \in T\} \in Y_C$. Suppose that $W \cap \text{cone} D(y_0) \neq \emptyset$, $W$ being an open set. Take $w \in W \cap \text{cone} D(y_0)$, and let $\bar{\varepsilon} > 0$ be such that $B(w; \bar{\varepsilon}) \subset W$. We can write

$$w = \sum_{i=1}^p \lambda_i d^i, \quad d^i = \left( \begin{array}{c} u_{t_i} \\ (f_{t_i}^0)^*(u_{t_i}) \end{array} \right),$$

for certain finitely many $u_{t_i} \in \partial f_{t_i}^0(\bar{x}^i), \bar{x}^i \in \mathbb{R}^n, t_i \in T, \lambda_i \geq 0, i = 1, \ldots, p$.

Suppose that there exists $\{y_t\} \subset Y_C$ such that $y_r \rightarrow y_0$ and $W \cap \text{cone} D(y_r) = \emptyset$. Since $f_t^r \rightarrow f_t^0$ and all these functions are convex finite-valued, an application of Corollary 8.47 (b) in [20] (also [4, Theorem 8.3.9]) guarantees the existence of sequences $\{x_r^{i,i}\}, \{u_{t_i}^r\}$ such that $u_{t_i}^r \in \partial f_{t_i}^r(x_r^{i,i}), x_r^{i,i} \rightarrow \bar{x}^i$ and $u_{t_i}^r \rightarrow u_{t_i}$, as $r \rightarrow \infty$. Then

$$(f_{t_i}^r)^*(u_{t_i}^r) = (u_{t_i}^r)'x_r^{i,i} - f_{t_i}^r(x_r^{i,i}) \rightarrow u_{t_i}'\bar{x}^i - f_{t_i}^0(\bar{x}^i) = (f_{t_i}^0)^*(u_{t_i}),$$

as $r \rightarrow \infty$, for $i = 1, \ldots, p$, which implies that

$$(d^i)^r = \left( \begin{array}{c} u_{t_i}^r \\ (f_{t_i}^r)^*(u_{t_i}^r) \end{array} \right) \rightarrow \left( \begin{array}{c} u_{t_i} \\ (f_{t_i}^0)^*(u_{t_i}) \end{array} \right) = d^i.$$ 

Hence, it follows that $w^r := \sum_{i=1}^p \lambda_i (d^i)^r \rightarrow w$; thus, for $r$ large enough, we have $w^r \in W \cap \text{cone} D(y_r)$ which is a contradiction. \qed

**Theorem 6.** Let $y_0 = \{f_t^0(x) \leq 0, t \in T\}$ be convex. Then:

(i) If $(0_{n+1}) \in \text{int cone} D(y_0)$, then $y_0$ is stably strongly inconsistent.

(ii) If $y_0$ is stably strongly inconsistent and, for each $t \in T$, $(f_t^0)^*$ is bounded from below on its effective domain, then $(0_{n+1}) \in \text{int cone} D(y_0)$.

(iii) If $y_0$ is stably inconsistent and $(f_t^0)^* \in \text{int cone} D(y_0)$ is uniformly bounded from below, then $(0_{n+1}) \in \text{int cone} D(y_0)$. 
Proof. (i) It is a consequence of Theorem 1 and Lemma 5.

(ii) Now we assume that $y_0$ is stably strongly inconsistent and $(f_1^0)^*$ is bounded on its effective domain for all $t \in T$. From Theorem 3.1 in [8] a convex system $y$ is inconsistent if and only if $(\frac{0_n}{1}) \in \text{cl} \, \text{cone} \, (y)$. We will show that if $\left(\frac{0_n}{1}\right) \in \text{bd} \, \text{cone} \, (D(y))$ then $y_0$ cannot be stably strongly inconsistent by following a similar reasoning to the linear case (see, e.g., [11]; notice that in the linear case the cardinality of $\text{gph}(f_1^0)^*$ is just 1). By the supporting hyperplane theorem there exists $(\frac{w}{x}) \in \mathbb{R}^{n+1}\setminus \{0_{n+1}\}$ such that $(\frac{w}{x})' \left(\frac{0_n}{1}\right) = 0$ and $(\frac{w}{x})'x \geq 0$ for all $x \in \text{cone} \, (D(y_0))$. Then, $x = 0$ and $w'w \geq 0$ for all $w \in \hat{c} f_1^0(\mathbb{R}^n)$, with $t \in T$.

Let $y_r := \{f_i^r(x) \leq 0, t \in T\}$, where $f_i^r(x) := f_i^0(x)+(1/r)w'x$, $r = 1, 2, \ldots$. Obviously, $\{y_r\} \subseteq Y_C$ and $y_r \to y_0$. Assume that $y_r$ contains a finite inconsistent subsystem $\{f_i^r(x) \leq 0, t \in T_r\}$, with $T_r \subset T$. Thus,

\[
\left(\begin{array}{c}
0_n \\
-1
\end{array}\right) = \lim_{k \to \infty} \sum_{t \in T_r} \left(\begin{array}{c}
\nu_{t}^{r,k} \\
(v_t^{r,k})'x_t^{r,k} - f_t^r(x_t^{r,k})
\end{array}\right),
\]

for some $\lambda_{t}^{r,k} \geq 0$, $v_{t}^{r,k} \in \mathcal{C} f_t^r(x_t^{r,k})$, $x_t^{r,k} \in \mathbb{R}^n$ for all $t \in T_r$ and $k \in \mathbb{N}$. We can write each $v_t^{r,k}$ as $u_t^{r,k} + (1/r)w$, with $u_t^{r,k} \in D(y_0) \cup \text{cl} \, \text{cone} \, (f_t^0(x_t^{r,k}))$, so that

\[
w'v_t^{r,k} = w'w_t^{r,k} + \frac{1}{r}w^2 \geq \frac{1}{r}w^2.
\]

Then, by (8),

\[
0 = \lim_{k \to \infty} \sum_{t \in T_r} \lambda_{t}^{r,k} (w'v_t^{r,k}) \geq \frac{1}{r}w^2 \lim_{k \to \infty} \sum_{t \in T_r} \lambda_{t}^{r,k},
\]

which implies that $\lim_{k \to \infty} \sum_{t \in T_r} \lambda_{t}^{r,k} = 0$. Since the expression $(v_t^{r,k})'x_t^{r,k} - f_t^r(x_t^{r,k})$ in (8) is just $(f_t^r)^*(v_t^{r,k})$, with $(f_t^r)^*$ bounded from below and $T_r$ being a finite set, we have obtained the contradiction $-1 = 0$.

(iii) It follows in a similar fashion as the case (ii) by noting that the set $T_r$ in (8) is not fixed, indeed it depends on $k$.

That is the reason for asking the uniform boundedness from below of $\{(f_t^0)^*_t\}_{t \in T}$.

Corollary 7. Let $y_0 = \{a_i'x \leq b_i, t \in T\}$ be a linear system. Then:

(i) $(\frac{0_n}{1}) \in \text{int} \, \text{cone} \, (D(y_0))$ if and only if $y_0$ is stably strongly inconsistent.

(ii) If $y_0$ is stably inconsistent and $\inf_{t \in T} b_t > -\infty$, then $(\frac{0_n}{-1}) \in \text{int} \, \text{cone} \, (D(y_0))$ and so $y_0$ is stably strongly inconsistent.

Remark 1. For the linear case there is a straightforward proof of the fact that $(\frac{0_n}{-1}) \in \text{int} \, \text{cone} \, (D(y_0))$ implies that $y_0$ is stably strongly inconsistent. Indeed, if $\{e_i, i \in I\} \subseteq \mathbb{R}^p$ and $e \in \text{int} \, \text{cone} \, (e_i, i \in I)$, then there exists some $\varepsilon > 0$ such that $e \in \text{int} \, \text{cone} \, (e_i, i \in I)$ whenever $\sup_{i \in I} \|e_i - e_i\| < \varepsilon$. Now, in our particular case $y_0 = \{a_i'x \leq b_i, t \in T\}$, $\{e_i, i \in I\} = \{(\frac{0_n}{1})_t, t \in T\}$, $D(y_0) = \{(\frac{0_n}{1})_t \mid t \in T\}$ and $d(y, y_0) = \sup_{t \in T} \|((\frac{0_n}{1})_t - (\frac{0_n}{1})_t)\|$. Hence, for $(\frac{0_n}{-1}) \in \text{int} \, \text{cone} \, (\{(\frac{0_n}{1})_t \mid t \in T\})$, if the distance $d(y, y_0)$ is small enough we get that $(\frac{0_n}{-1}) \in \text{int} \, \text{cone} \, (\{(\frac{0_n}{1})_t \mid t \in T\})$ and so $y_0$ is stably strongly inconsistent.

4. Convex systems

In this section, we consider given two CSISs $y_0 = \{f_i^0(x) \leq 0, t \in T\}$ and $z_0 = \{g_i^0(x) \leq 0, s \in S\}$. First we show that for such class of systems the stable non-empty intersection is closely related to the lower semicontinuity property of $\mathcal{F}$ and $\mathcal{G}$.
We say that the family of the constraint functions of $y_0$ is equilipschitzian if there is some positive $M$ such that $|f^0_t(x^1) - f^1_t(x^2)| \leq M|x^1 - x^2|$ for any $x^1, x^2$ in $\mathbb{R}^n$ and for all $t$ in $T$.

**Proposition 8.** Let $y_0$ and $z_0$ be convex systems.

(i) If $\mathcal{F}$ and $\mathcal{G}$ are lsc at $y_0$ and $z_0$, respectively, and $[\text{int } \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset$, then $\mathcal{F} \cap \mathcal{G} \neq \emptyset$ stably at $(y_0, z_0)$.

(ii) If $\mathcal{F} \cap \mathcal{G} \neq \emptyset$ stably at $(y_0, z_0)$, then $\mathcal{F}$ and $\mathcal{G}$ are lsc at $y_0$ and $z_0$, respectively. Moreover if the family of constraint functions of $y_0$ is equilipschitzian then $[\text{int } \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset$.

**Proof.** (i) It follows from Proposition 3.

(ii) First, observe that $\mathcal{F} \cap \mathcal{G} \neq \emptyset$ stably at $(y_0, z_0)$ yields that $y_0 \in \text{int dom } \mathcal{F}(y_0)$ and $z_0 \in \text{int dom } \mathcal{G}(z_0)$, which for consistent convex systems is equivalent to $\mathcal{F}$ and $\mathcal{G}$ being lsc at $y_0$ and $z_0$, respectively. Assume that the family of constraint functions of $y_0$, $\{f_t^0, t \in T\}$, is equilipschitzian. Then any strong Slater point of $y_0$ is an interior point of $\mathcal{F}(y_0)$, so $\text{int } \mathcal{F}(y_0) \neq \emptyset$. If $[\text{int } \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) = \emptyset$, there exists a hyperplane separating properly $\mathcal{F}(y_0)$ and $\mathcal{G}(z_0)$ which contains some $v \in \text{bd } \mathcal{F}(y_0) \cap \text{bd } \mathcal{G}(z_0)$. Let $\omega \in \mathbb{R}^n$, $\|\omega\| = 1$, such that

$$
\omega'(x - v) \leq 0 \text{ for all } x \in \mathcal{F}(y_0) \text{ and } \omega'(x - v) \geq 0 \text{ for all } x \in \mathcal{G}(z_0).
$$

Now, observe that the sequence $\{y_r\} \subset Y$,

$$
y_r := \{f_t^0 \left( x + \frac{\omega}{r} \right) \leq 0, t \in T \}, \quad r \in \mathbb{N},
$$

verifies that $y_r \to y_0$ and

$$
\mathcal{F}(y_r) \cap \mathcal{G}(z_0) = \emptyset \quad \text{for all } r \in \mathbb{N},
$$

because if $x_r \in \mathcal{F}(y_r) \cap \mathcal{G}(z_0)$, then $x_r \in \mathcal{G}(z_0)$ and $x_r + \omega/r \in \mathcal{F}(y_0)$. Thus by (9)

$$
0 \geq \omega'(x_r + \frac{\omega}{r} - v) = \omega'(x_r - v) + \frac{1}{r} \|\omega\|^2 \geq \frac{1}{r} > 0,
$$

we have a contradiction. \(\square\)

**Remark 2.** For any finite intersection of $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$, the condition $[\text{int } \mathcal{F}(y_0)] \cap G(z_0) \neq \emptyset$ should be replaced by $[\text{int } 1_{\{y_0\}}] \cap \cdots \cap [\text{int } \mathcal{F}_{k-1}(y_0^{k-1})] \cap \mathcal{F}_k(y_0^k) \neq \emptyset$ in (i) and by $[\text{int } \mathcal{F}_1(y_0^1)] \cap \mathcal{F}_2(y_0^2) \cap \cdots \cap \mathcal{F}_k(y_0^k) \neq \emptyset$ in (ii).

Let $y = \{f_t(x) \leq 0, t \in T\}$ and $z = \{g_s(x) \leq 0, s \in S\}$ be convex systems in $Y_C$ and $Z_C$, respectively. We define the disjoint union

$$
(y, z) := \{f_t(x) \leq 0, t \in T; g_s(x) \leq 0, s \in S\}
$$

(with possibly repeated constraints), so that

$$
D(y, z) := \left( \bigcup_{t \in T} \text{gph } f_t^* \right) \cup \left( \bigcup_{s \in S} \text{gph } g_s^* \right).
$$

Then, $\mathcal{F}(y_0) \cap \mathcal{G}(z_0) = \emptyset$ if and only if $(0_{-1}) \in \text{cl cone } D(y_0, z_0)$ \[8]. The stability of $\mathcal{F} \cap \mathcal{G} = \emptyset$ at $(y_0, z_0)$ is directly related to the inconsistency of the systems $(y, z)$ for systems $y$ and $z$ close enough to $y_0$ and $z_0$.

**Proposition 9.** Let $y_0$ and $z_0$ be convex systems. If $(0_{-1}) \in \text{int cone } D(y_0, z_0)$, then $\mathcal{F} \cap \mathcal{G} = \emptyset$ stably at $(y_0, z_0)$. The converse holds if $\{(f_t^0)^*, t \in T; (g_s^0)^*, s \in S\}$ is uniformly bounded from below.

**Proof.** It is a straightforward application of Theorem 6. \(\square\)
5. Linear systems

Throughout this section we consider given two LSISs \( y_0 = \{ a_t' x \leq b_t, t \in T \} \) and \( z_0 = \{ c_s x \leq d_s, s \in S \} \). By just repeating the arguments used in the proof of Proposition 8, we can obtain the following linear version:

**Proposition 10.** Suppose that the set \( \{ a_t, t \in T \} \) is bounded. Then, \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) stably at \( (y_0, z_0) \) if and only if \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \) and \( \mathcal{F} \) and \( \mathcal{G} \) are lsc at \( y_0 \) and \( z_0 \), respectively.

The condition \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \) is not necessary when the general hypothesis on the boundedness of the coefficients is not true, since for any infinite set \( T \) and any closed convex set we can find a linear representation through a linear system \( y_0 \) in such a way that the feasible set mapping \( \mathcal{F} \) remains constant in some open neighborhood of \( y_0 \) (the same for \( \mathcal{G} \)).

Taking into account that \( f^*_i = b_i \) and \( g^*_s = d_s \) (on their effective domains), we get the following linear version of Proposition 9:

**Proposition 11.** If \( (0_1) \in \text{int cone} \ D(y_0, z_0) \), then \( \mathcal{F} \cap \mathcal{G} = \emptyset \) stably at \( (y_0, z_0) \). The converse holds if \( \inf_{i \in T} b_i > -\infty \) and \( \inf_{s \in S} d_s > -\infty \).

In the particular case of finite sets of indexes, i.e., ordinary linear systems, the lsc property is equivalent to the existence of a Slater point which is always an interior point of the feasible set. Thus, for finite \( T \) and \( S \) we have the following characterization of the stability of \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) and of \( \mathcal{F} \cap \mathcal{G} = \emptyset \).

**Theorem 12.** If \( y_0 \) and \( z_0 \) are ordinary linear systems then,

(i) \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) stably at \( (y_0, z_0) \) if \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \) and \( y_0 \) does not contain the trivial inequality and \( z_0 \) possesses a Slater point.

(ii) If \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) stably at \( (y_0, z_0) \) then \( y_0 \) and \( z_0 \) share a Slater point (equivalently \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \).)

(iii) \( \mathcal{F} \cap \mathcal{G} = \emptyset \) stably at \( (y_0, z_0) \) if and only if \( (0_1) \in \text{int cone} \ D(y_0, z_0) \).

**Proof.** (i) Taking into account that for ordinary linear systems the lsc property is equivalent to the existence of a Slater point and that \( \mathcal{F}(y_0) \) is the set of the Slater points of \( y_0 \) under the assumption on it, an application of Proposition 10 gives the statement.

(ii) Assume that \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) stably at \( (y_0, z_0) \). Then neither \( y_0 \) nor \( z_0 \) contain the trivial inequality. Thus, by Proposition 10 \( \mathcal{F} \) and \( \mathcal{G} \) are lsc at \( y_0 \) and \( z_0 \), respectively, and \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \) and \( \mathcal{F}(y_0) \cap \mathcal{G}(z_0) \neq \emptyset \), because both sets of gradients \{\( a_t, t \in T \) and \( \{ c_s, s \in S \) \ are bounded. Let \( x^1 \in [\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \) and \( x^2 \in \mathcal{F}(y_0) \cap \mathcal{G}(z_0) \); if \( x^1 = x^2 \) then \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \). If \( x^1 \neq x^2 \), by the accessibility lemma, \([x^1, x^2] \subset \text{int} \mathcal{F}(y_0) \) and \([x^1, x^2] \subset \text{int} \mathcal{G}(z_0) \), hence \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \) as well.

(iii) It follows straightforwardly from Proposition 11. \( \square \)

**Remark 3.** The above statements (i) and (ii) can be rearranged into only one: (i') If \( y_0 \) and \( z_0 \) are ordinary linear systems not containing the trivial inequality then, \( \mathcal{F} \cap \mathcal{G} \neq \emptyset \) stably at \( (y_0, z_0) \) if and only if \( y_0 \) and \( z_0 \) share a Slater point (equivalently \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \).)

We also get from the Propositions 10 and 11 the following characterization of the ill-posedness of \( \mathcal{F} \cap \mathcal{G} \).

**Proposition 13.** If \( D(y_0, z_0) \) is bounded, then \( \mathcal{F} \cap \mathcal{G} \) is ill-posed at \( (y_0, z_0) \) if and only if \( (0_1) \notin \text{int cone} \ D(y_0, z_0) \) and

\[
0_{n+1} \in [\text{cl conv} \ D(y_0)] \cup [\text{cl conv} \ D(z_0)]
\]

if \([\text{int} \mathcal{F}(y_0)] \cap [\text{int} \mathcal{G}(z_0)] \neq \emptyset \).
Proof. In view of the above propositions in this section and under the assumption of \( D(y_0, z_0) \) being bounded, it is clear that \( \mathcal{F} \cap \mathcal{G} \) is ill-posed at \((y_0, z_0)\) is equivalent to (1) \( \mathcal{F} \) is not lsc at \( y_0 \) or \( \mathcal{G} \) is not lsc at \( z_0 \) or \([\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) = \emptyset \) or \( \mathcal{F}(y_0) \cap \mathcal{G}(z_0) = \emptyset \); and (2) \( (\frac{y_0}{n+1}) \notin \text{int cone} \ D(y_0, z_0). \) Recall that \( \mathcal{F} \) (\( \mathcal{G} \)) is lsc at \( y_0 \) (\( z_0 \)) if and only if \( 0_{n+1} \notin \text{cl conv} \ D(y_0) \) (\( \text{cl conv} \ D(z_0) \)). So, we only need to show that

\[ [\text{int} \mathcal{F}(y_0)] \cap \mathcal{G}(z_0) \neq \emptyset \neq \mathcal{F}(y_0) \cap [\text{int} \mathcal{G}(z_0)] \]

implies that \([\text{int} \mathcal{F}(y_0)] \cap [\text{int} \mathcal{G}(y_0)] \neq \emptyset \). The argument is the same as in Theorem 12 (ii). \( \Box \)

Remark 4. Most of the results obtained in Sections 4 and 5 are symmetric in the sense that both systems, \( y_0 \) and \( z_0 \), play similar roles. This is not the case with Propositions 8, 10 and Theorem 12 part (i) where either the assumptions or the thesis are different for \( y_0 \) and \( z_0 \).

Finally, we will discuss the distance to ill-posedness. We identify \( X := Y_L \times Z_L \) with the class of LSISs on \( \mathbb{R}^n \) with index set \( T \cup S \) (replace \( T \) and \( S \) by \( T \times \{0\} \) and \( S \times \{1\} \), respectively, if they are not disjoint), so that \( \mathcal{F}(y_0, z_0) \) is the solution set of the system

\[ \{a_i'x \leq b_i, t \in T; c_i'x \leq d_s, s \in S\} \]

that we also represent by \((y_0, z_0)\). Recall that \( \mathcal{F} \cap \mathcal{G} \) is ill-posed at \((y_0, z_0)\) if and only if \((y_0, z_0) \in \text{bd dom} \mathcal{F} \). In this case, \( \rho_X((y_0, z_0), \text{bd dom} \mathcal{F}) \) is the radius of the greatest open ball centered at \((y_0, z_0)\) which is contained in \( \text{dom} \mathcal{F} \) (if \((y_0, z_0) \in \text{dom} \mathcal{F}\)) or in \( X \setminus \text{dom} \mathcal{F} \) (otherwise), i.e., the distance to ill-posedness. Since \( \rho_X \) is a pseudometric, this distance can be \( \infty \); put

\[ X^\infty := \{(y, z) \in X|\rho_X((y, z), \text{bd dom} \mathcal{F}) = +\infty\} \subset \text{ext dom} \mathcal{F}. \]

The elimination from \( X \) of this uninteresting set emphasizes the role of

\[ X^s := \{(y, z) \in X|(y, z) \text{ is strongly inconsistent}\}, \]

because

\[ (\text{bd dom} \mathcal{F}) \setminus X^\infty = (\text{bd} X^s) \setminus X^\infty \]

(see [5]). For this reason, we say that \( \mathcal{F} \cap \mathcal{G} \) is generalized ill-posed at \((y_0, z_0)\) if \((y_0, z_0) \in \text{bd} X^s \). If \(|T \cup S| < \infty\) (i.e., \( y_0 \) and \( z_0 \) are ordinary), \( X^\infty = \emptyset \) and so there is a unique concept of ill-posedness.

Let \( H(y_0, z_0) := [\text{conv} \ D(y_0, z_0)] + \mathbb{R}^n_+ (\frac{y_0}{n+1}). \) Given \((y_0, z_0) \in X\), by [5, Theorem 4],

\[ (y_0, z_0) \in \text{int} X^s \Leftrightarrow 0_{n+1} \in \text{int} H(y_0, z_0) \]

(and the same with ‘bd’ and ‘ext’ instead of ‘int’). Moreover, in [5, Theorem 6], the distance from \((y_0, z_0)\) to generalized ill-posedness can be expressed as the distance from the origin to \( H(y_0, z_0) \):

\[ \rho_X((y_0, z_0), \text{bd} X^s) = d(0_{n+1}, \text{bd} H(y_0, z_0)). \]  

(11)

The effective calculus of (11) is possible if a certain linear representation of \( \text{cl} H(y_0, z_0) \) is available because \( \text{bd} \text{cl} H(y_0, z_0) = \text{bd} H(y_0, z_0) \). Suppose that

\[ \text{cl} H(y_0, z_0) = \{x \in \mathbb{R}^{n+1}|p_i'x \leq q_i, i \in I\}, \]

where \((p_i, q_i) \neq 0_{n+2}\) for all \( i \in I \). Consider \( J = \{-1, 1\}^{n+1} \) and \( v_j := (j(1), \ldots, j(n + 1)), j \in J \). Thus \([-1, 1]^{n+1} = \text{conv} \{v_j, j \in J\}, \) with \(|J| = 2^{n+1} \).
If \((y_0, z_0) \in \text{int} X^4\), then \(0_{n+1} \in \text{int} H(y_0, z_0)\) and \(q_i > 0\) for all \(i \in I\). Hence

\[
0 < d(0_{n+1}, \text{bd} H(y_0, z_0)) = \sup\{\gamma \in \mathbb{R}_+: \{\gamma\}^{n+1} \subset H(y_0, z_0)\} = \sup\{\gamma \in \mathbb{R}_+: p'_i(\gamma v_j) \leq q_i, i \in I, j \in J\} = \inf \left\{ \frac{q_i}{p'_i v_j} \mid p'_i v_j > 0, i \in I, j \in J \right\}.
\]

In this case, \(d(0_{n+1}, \text{bd} H(y_0, z_0))\) is either the minimum of a finite set (when \(I\) is finite) or the value of a global minimization problem (otherwise).

Now we assume that \((y_0, z_0) \in \text{ext} X^4\), i.e., \(0_{n+1} \in \text{ext} H(y_0, z_0)\). Then

\[
d(0_{n+1}, \text{bd} H(y_0, z_0)) = d(0_{n+1}, \text{cl} H(y_0, z_0)) = \nu(P),
\]

where \((P)\) is the linear optimization problem

\[
\begin{align*}
(P) \quad & \min \gamma \\
\text{s.t.} \quad & -\gamma \leq x_k \leq \gamma, \quad k = 1, \ldots, n + 1, \\
& p'_i x \leq q_i, \quad i \in I.
\end{align*}
\]

Obviously, \((P)\) is a LP problem if \(|I| < \infty\) and it is a LSIP problem otherwise. Numerical methods for LSIP can be found in \([10,18,7]\) and references therein.

References