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On the approximate controllability of semilinear fractional differential systems[☆]

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ABSTRACT

Fractional differential equations have wide applications in science and engineering. In this paper, we consider a class of control systems governed by the semilinear fractional differential equations in Hilbert spaces. By using the semigroup theory, the fractional power theory and fixed point strategy, a new set of sufficient conditions are formulated which guarantees the approximate controllability of semilinear fractional differential systems. The results are established under the assumption that the associated linear system is approximately controllable. Further, we extend the result to study the approximate controllability of fractional systems with nonlocal conditions. An example is provided to illustrate the application of the obtained theory.

1. Introduction

The concept of controllability plays an important role in the analysis and design of control systems. Controllability of the deterministic and stochastic dynamical control systems in infinite-dimensional spaces is well developed using different kinds of approaches, and the details can be found in various papers (see [1–5] and the references therein). Several authors [6–8] studied the concept of exact controllability for systems represented by nonlinear evolution equations, in which the authors effectively used the fixed point approach. Most of the controllability results in infinite-dimensional control system concern the so-called semilinear system that consists of a linear part and a nonlinear part.

From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. Approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications [2,9–11]. Therefore, it is important, in fact necessary to study the weaker concept of controllability, namely approximate controllability for nonlinear systems. In the recent literature, there have been few papers on the approximate controllability of the nonlinear evolution systems under different conditions [9,12–15]. Fu and Mei [16] investigated the approximate controllability of semilinear neutral functions differential systems with finite delay. The conditions are established with the help of semigroup theory and fixed point technique under the assumption that the linear part of the associated nonlinear system is approximately controllable.

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On the other hand, nowadays the concept of non-integral derivative and integral is used increasingly to model the behavior of real world problems in various fields. Several researchers studied the existence results of the initial and boundary value problem for fractional differential equations see ([17–21] and the references therein). The motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various field of sciences such as physics, chemistry, aerodynamics, electrodynamics of complex medium, and so on [22,23]. More recently, Zhou and Jiao [24,25] discussed the existence of mild solutions to fractional evolution and neutral evolution equations in an arbitrary Banach space in which the mild solution is introduced based on the probability density function and semigroup theory. Moreover, Wang and Zhou [26] introduced a new mild solution for a class of fractional evolution equations and further the existence of optimal control for the considered problem has been discussed.

Recently, Tai and Wang [27] studied the exact controllability of fractional-order impulsive neutral functional systems with infinite delay in Banach spaces. However, in order to establish the results, the assumption made in [27] were that the semigroup associated with linear part is compact and subsequently the invertibility of a controllability operator is imposed. In view of the observation made in [28], the result in [27] holds only in finite-dimensional spaces. Thus, in infinite-dimensional spaces the concept of complete controllability is usually too strong and, indeed, has limited applicability. However, it should be emphasized that to the best of our knowledge, the approximate controllability of semilinear fractional system in Hilbert spaces has not been investigated yet and it is also the motivation of this paper. In order to fill this gap, in this paper, we study the approximate controllability of semilinear fractional control systems under the assumption that the associated linear system is approximately controllable. In fact our results in the present paper are motivated by the recent work of [14] and the fractional differential equations studied in [24,25]. In particular, the controllability problem is transformed to a fixed point problem for an appropriate nonlinear operator in a function space.

2. Problem formulation and preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. In this paper, we assume that X is a Hilbert space with norm $\|\cdot\|$. Let $J=[0,b]\subset R$ and $C(J,X)$ be the Banach space of continuous functions from J into X with the norm $\|x\|=\sup_{t\in J}\|x(t)\|$, here $x\in C(J,X)$. The purpose of this paper is to establish sufficient conditions for the approximate controllability of certain classes of abstract fractional evolution equations with control of the form

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + Bu(t) + f(t, x(t)), & t \in J, \\ x(0) = x_0, \end{cases} \tag{1}$$

where the state variable $x(\cdot)$ takes values in the Hilbert space X; ${}^cD^q$ is the Caputo fractional derivative of order $0<q<1$; A is the infinitesimal generator of a C_0 semigroup $T(t)$ of bounded operators on the Hilbert space X; the control function $u(\cdot)$ is given in $L_2(J,U)$, U is a Hilbert space; B is a bounded linear operator from U into X; $f:J\times X\to X$ is a given function satisfying some assumptions and x_0 is an element of the Hilbert space X.

Further, we introduce some basic definitions and properties of fractional calculus which will be used throughout this paper. Let $E(X)$ be the space of all bounded linear operators from X to X with the norm $\|Q\|_{E(X)}=\sup\{\|Q(x)\|:\|x\|=1\}$, where $Q\in E(X)$ and $x\in X$. Throughout this paper, let A be the infinitesimal generator of C_0 semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X. Clearly, $M:=\sup_{t\in[0,\infty)}\|T(t)\|<\infty$.

Definition 2.1. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\mathrm{d}s, \quad t>0,\ \alpha>0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.2. Riemann–Liouville derivative of order α with the lower limit 0 for a function $f:[0,\infty)\to R$ can be written as

$$^LD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\mathrm{d}s, \quad t>0,\ n-1<\alpha<n.$$

Definition 2.3. The Caputo derivative of order α for a function $f:[0,\infty)\to R$ can be written as

$$^cD^\alpha f(t) = {}^LD^\alpha\left(f(t) - \Sigma_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0,\ n-1<\alpha<n.$$

Remark 2.4. (1) If $f(t)\in C^n[0,\infty)$ then

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\mathrm{d}s = I^{n-\alpha}f^{(n)}(t), \quad t>0,\ n-1<\alpha<n.$$

(2) The Caputo derivative of a constant is equal to zero.
(3) If f is an abstract function with values in X then the integrals which appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

For basic facts about fractional integrals and fractional derivatives one can refer to [23].

In order to define the concept of mild solution for the problem (1), by comparison with the ordinary fractional equations given in [24,25], we associate problem (1) to the integral equation

$$x(t) = \hat{T}_q(t)x_0 + \int_0^t (t-s)^{q-1}T_q(t-s)f(s,x(s))\mathrm{d}s + \int_0^t (t-s)^{q-1}T_q(t-s)Bu(s)\mathrm{d}s, \tag{2}$$

where $\hat{T}_q(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)\mathrm{d}\theta$, $T_q(t) = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)\mathrm{d}\theta$, $\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\bar{w}_q(\theta^{-\frac{1}{q}}) \geq 0$, $\bar{w}_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q)$, $\theta \in (0,\infty)$, ξ_q is a probability density function defined on $(0,\infty)$, that is $\xi_q(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \xi_q(\theta)\mathrm{d}\theta = 1$.

Lemma 2.5 ([24]). *For any fixed $t \geq 0$, the operators $\hat{T}_q(t)$ and $T_q(t)$ are linear and bounded operators, i.e., for any $x \in X$, $\|\hat{T}_q(t)\| \leq M\|x\|$ and $\|T_q(t)\| \leq \frac{Mq}{\Gamma(1+q)}\|x\|$.*

Let $x_b(x_0; u)$ be the state value of (1) at terminal time b corresponding to the control u and the initial value x_0. Introduce the set $\Re(b, x_0) = \{x_b(x_0; u)(0) : u(\cdot) \in L_2(J, U)\}$, which is called the reachable set of system (1) at terminal time b, its closure in X is denoted by $\overline{\Re(b, x_0)}$.

Definition 2.6. A function $x(\cdot; x_0, u) \in C(J, X)$ is said to be a mild solution of (1) if for any $u(\cdot) \in L_2(J, U)$ the integral equation (2) is satisfied.

Definition 2.7. The system (1) is said to be approximately controllable on J if $\overline{\Re(b, x_0)} = X$, that is, given an arbitrary $\epsilon > 0$ it is possible to steer from the point x_0 to within a distance ϵ from all points in the state space X at time b.

Consider the following linear fractional differential system

$$D_t^q x(t) = Ax(t) + (Bu)(t),\ t \in [0, b], \qquad x(0) = x_0. \tag{3}$$

The approximate controllability for linear fractional control system (3) is a natural generalization of approximate controllability of linear first-order control system [10,3,9]. It is convenient at this point to introduce the controllability operator associated with (3) as

$$\Gamma_0^b = \int_0^b (b-s)^{q-1}T_q(b-s)BB^*T_q^*(b-s)\mathrm{d}s,$$

where B^* denotes the adjoint of B and $T_q^*(t)$ is the adjoint of $T_q(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator. Let $R(\alpha, \Gamma_0^b) = (\alpha I + \Gamma_0^b)^{-1}$ for $\alpha > 0$.

Lemma 2.8. *The linear fractional control system (3) is approximately controllable on J if and only if $\alpha R(\alpha, \Gamma_0^b) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.*

The proof of this lemma is a straightforward adaptation of the proof of [3, Theorem 2].

Lemma 2.9 (*Schauder's Fixed Point Theorem*). *If K is a closed bounded and convex subset of a Banach space X and $F : K \to K$ is completely continuous, then F has a fixed point in K.*

3. Fractional control systems

In this section, we formulate and prove conditions for the approximate controllability of semilinear fractional control differential systems. To do this, we first prove the existence of a fixed point of the operator F_α defined below by using Schauder fixed point theorem. Second, in Theorem 3.3, we show that under certain assumptions the approximate controllability of fractional systems (1) is implied by the approximate controllability of the corresponding linear system (3).

Before stating and proving the main results, we impose the following conditions on data of the problem:

(H_1) $T(t)$ is a compact operator.
(H_2) for each $t \in [0, b]$, the function $f(t, \cdot) : X \to X$ is continuous and for each $x \in C([0, b], X)$ the function $f(\cdot, x) : [0, b] \to X$ is strongly measurable.

(H_3) there exists a constant $q_1 \in [0, q]$ and $m \in L^{\frac{1}{q_1}}([0, b], R^+)$ such that $|f(t, x)| \le m(t)$ for all $x \in X$ and almost all $t \in [0, b]$.

(H_4) The function $f : J \times X \to X$ is continuous and uniformly bounded and there exists $N > 0$ such that $\|f(t, x)\| \le N$ for all $(t, x) \in J \times X$.

Lemma 3.1 ([24]). *If the assumption* (H_1) *is satisfied, then $\hat{T}_q(t)$ and $T_q(t)$ are also compact operators for every $t > 0$.*

In this section, it will be shown that the system (1) is approximately controllable if for all $\alpha > 0$, there exists a continuous function $x(\cdot) \in C(J, X)$ such that

$$x(t) = \hat{T}_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s)) \mathrm{d}s + \int_0^t (t-s)^{q-1} T_q(t-s) B u(s) \mathrm{d}s, \tag{4}$$

$$u(t) = B^* T_q^*(b-t) R(\alpha, \Gamma_0^b) p(x(\cdot)), \tag{5}$$

$$p(x(\cdot)) = x_b - \hat{T}_q(b)x_0 - \int_0^b (b-s)^{q-1} T_q(b-s) f(s, x(s)) \mathrm{d}s.$$

Theorem 3.2. *If the hypotheses* (H_1)–(H_3) *are satisfied, then the fractional semilinear control system* (1) *has a mild solution on J; here $M_B = \|B\|$, $a = \frac{q-1}{1-q_1} \in (-1, 0)$ and $M_1 = \|m\|_{L^{\frac{1}{q_1}}[0,a]}$.*

Proof. The main aim in this section is to find conditions for solvability of system (4) and (5) for $\alpha > 0$. In the Banach space $C(J, X)$, consider a set

$$B_k = \{x \in C(J, X) \mid x(0) = x_0, \|x\| \le k\},$$

where k is a positive constant. For $\alpha > 0$, we define the operator F_α on $C(J, X)$ as follows

$$(F_\alpha x)(t) = z(t), \tag{6}$$

where

$$z(t) = \hat{T}_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s)) \mathrm{d}s + \int_0^t (t-s)^{q-1} T_q(t-s) B v(s) \mathrm{d}s, \tag{7}$$

$$v(t) = B^* T_q^*(b-t) R(\alpha, \Gamma_0^b) p(x(\cdot)), \tag{8}$$

$$p(x(\cdot)) = x_b - \hat{T}_q(b)x_0 - \int_0^b (b-s)^{q-1} T_q(b-s) f(s, x(s)) \mathrm{d}s.$$

It will be shown that for all $\alpha > 0$ the operator F_α from $C(J, X)$ into itself has a fixed point. The proof of this theorem is long and technical. Therefore it is convenient to divide it into several steps:

Step 1: For an arbitrary $\alpha > 0$, there is a positive constant $k_0 = k(\alpha)$ such that $F_\alpha : B_{k_0} \to B_{k_0}$.

For any positive constant k and $x \in B_k$, since $x(t)$ is continuous in t, according to assumption (H_2), $f(t, x(t))$ is a measurable function on J. Direct calculation gives that $(t-s)^{q-1} \in L^{\frac{1}{1-q_1}}[0, t]$, for $t \in J$ and $q_1 \in [0, q)$. By using Holders inequality, and (H_3), according to [24,25], taking norm on (7) and (8) which yields that

$$\begin{aligned} \|z(t)\| &\le \|\hat{T}_q(t)x_0\| + \left\|\int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s)) \mathrm{d}s\right\| + \left\|\int_0^t (t-s)^{q-1} T_q(t-s) B v(s) \mathrm{d}s\right\| \\ &\le M\|x_0\| + \frac{qM}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} [M_1 + M_B \|v\|] \end{aligned}$$

and

$$\|v(t)\| = \frac{1}{\alpha} M_B M \left[\|x_b\| + M\|x_0\| + \frac{qMM_1}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}\right].$$

The last two inequalities imply that for large enough $k_0 > 0$ the following inequality holds

$$\|(F_\alpha x)(t)\| \le k_0$$

i.e., $(F_\alpha x) \in B_{k_0}$. Therefore, F_α maps B_{k_0} into itself.

Step 2: For each $0 < \alpha \le 1$, the operator F_α maps B_{k_0} into a relatively compact subset of B_{k_0}.

According to infinite-dimensional version of the Ascoli–Arzela theorem we have to prove that

(i) for any $t \in J$ the set $V(t) = \{(F_\alpha x)(t) : x(\cdot) \in B_{k_0}\}$ is relatively compact in X.

(ii) the family of functions $\{(F_\alpha x),\ x \in B_{k_0}\}$ is relatively compact. It suffices to show that the family of functions $\{(F_\alpha x),\ x \in B_{k_0}\}$ is bounded and equicontinuous.

In the case $t = 0$ is trivial. Clearly, $V(0) = \{Fx(0)|, x(\cdot) \in B_{k_0}\} = \{x_0\}$ is compact. So let t be a fixed real number, and let τ be a given real number satisfying $0 < \tau < t$. For any $\delta > 0$, define

$$\begin{aligned}(F_\alpha^{\tau,\delta}x)(t) =& \int_\delta^\infty \xi_q(\theta)T(t^q\theta)\mathrm{d}\theta x_0 + q\int_0^{t-\tau}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\,\theta)f(s,x(s))\mathrm{d}\theta\mathrm{d}s\\ &+q\int_0^{t-\tau}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\,\theta)Bv(s)\mathrm{d}\theta\mathrm{d}s\\ =& \ T(\tau^q\delta)\int_\delta^\infty \xi_q(\theta)\left(T(t^q\theta)-T(t^q\tau)\right)\mathrm{d}\theta x_0\\ &+T(\tau^q\delta)q\int_0^{t-\tau}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)\left(T((t-s)^q\,\theta)-T(\tau^q\delta)\right)f(s,x(s))\mathrm{d}\theta\mathrm{d}s\\ &+T(\tau^q\delta)q\int_0^{t-\tau}\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)\left(T((t-s)^q\,\theta)-T(\tau^q\delta)\right)Bv(s)\mathrm{d}\theta\mathrm{d}s\\ :=& \ T(\tau^q\delta)y\,(t,\tau)\,.\end{aligned}$$

Since $T(\tau^q\delta)$ is compact and $y\,(t,\tau)$ is bounded on B_{k_0}, the set

$$V_\tau(t) = \{(F_\alpha^{\tau,\delta}x)(t) : x(\cdot) \in B_{k_0}\}$$

is relatively compact set in X. On the other hand,

$$\begin{aligned}\|(F_\alpha x)(t)-(F_\alpha^{\tau,\delta}x)(t)\| =& \ q\Bigg|\int_0^t\int_0^\delta \theta(t-s)^{q-1}\xi_q(\theta)(T\,(t-s)^q\,\theta)[Bv(s)+f(s,x(s))]\mathrm{d}\theta\mathrm{d}s\\ &+\int_{t-\tau}^t\int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)(T\,(t-s)^q\,\theta)[Bv(s)+f(s,x(s))]\mathrm{d}\theta\mathrm{d}s\Bigg|\\ \le& \ \frac{qMb^{(1+a)(1-q_1)}}{(1+a)^{1-q_1}}\Bigg(M_1+\frac{1}{\alpha}M_BM\Bigg[\|x_b\|+M\|x_0\|\\ &+\frac{qMM_1}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}\Bigg]\Bigg)\int_0^\delta \theta\xi_q(\theta)\mathrm{d}\theta\\ &+\frac{qM}{\Gamma(1+q)(1+a)^{1-q_1}}\Bigg(M_1+\frac{1}{\alpha}M_BM\Bigg[\|x_b\|\\ &+M\|x_0\|+\frac{qMM_1}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}\Bigg]\Bigg)\tau^{(1+a)(1-q_1)}.\end{aligned}$$

This implies that, there are relatively compact sets arbitrarily close to the set $V(t)$ for each $t \in (0, b]$. Hence $V(t),\ t \in (0, b]$ is relatively compact in X.

(ii) Next we show that $V = \{(F_\alpha x)(.) \mid x(\cdot) \in B_{k_0}\}$ is an equicontinuous family of functions on $[0, b]$. For any $x \in B_{k_0}$ and $0 \le t_1 \le t_2 \le b$, we have

$$\begin{aligned}\|z(t_2)-z(t_1)\| \le& \left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}T_q(t_2-s)f(s,x(s))\mathrm{d}s\right\|\\ &+\left\|\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]T_q(t_2-s)f(s,x(s))\mathrm{d}s\right\|\\ &+\left\|\int_0^{t_1}(t_1-s)^{q-1}[T_q(t_2-s)-T_q(t_1-s)]f(s,x(s))\mathrm{d}s\right\|\\ &+\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}T_q(t_2-s)Bv(s)\mathrm{d}s\right\|\\ &+\left\|\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]T_q(t_2-s)Bv(s)\mathrm{d}s\right\|\end{aligned}$$

$$+\left\|\int_0^{t_1}(t_1-s)^{q-1}[T_q(t_2-s)-T_q(t_1-s)]Bv(s)\mathrm{d}s\right\|$$
$$\leq I_1+I_2+I_3+I_4+I_5+I_6. \quad (9)$$

By using Holders inequality and assumption (H_3), we can obtain

$$I_1\leq\frac{qMM_1(t_2-t_1)^{(1+a)(1-q_1)}}{\Gamma(1+q)(1+a)^{1-q_1}},\qquad I_2\leq\frac{qMM_1(t_2-t_1)^{(1+a)(1-q_1)}}{\Gamma(1+q)(1+a)^{1-q_1}},$$
$$I_4\leq\frac{qMM_1(t_2-t_1)^{(1+a)(1-q_1)}}{\Gamma(1+q)(1+a)^{1-q_1}}M_B\|v\|,\qquad I_5\leq\frac{qMM_1(t_2-t_1)^{(1+a)(1-q_1)}}{\Gamma(1+q)(1+a)^{1-q_1}}M_B\|v\|.$$

For $t_1=0$, $0<t_2\leq b$, it can be easily seen that $I_3=I_6=0$. For $t_1>0$ and $\epsilon>0$ small enough, we obtain

$$\begin{aligned}I_3 &\leq \int_0^{t_1-\epsilon}(t_1-s)^{q-1}\|T_q(t_2-s)-T_q(t_1-s)\|\,\|f(s,x(s))\|\mathrm{d}s\\ &\quad+\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}\|T_q(t_2-s)-T_q(t_1-s)\|\,\|f(s,x(s))\|\mathrm{d}s,\\ &\leq \frac{M_1(t_1^{1+a}-\epsilon^{1+a})^{1-q_1}}{\Gamma(1+q)(1+a)^{1-q_1}}\sup_{s\in[0,t_1-\epsilon]}\|T_q(t_2-s)-T_q(t_1-s)\|+\frac{2qMM_1}{\Gamma(1+q)(1+a)^{1-q_1}}\epsilon^{(1+a)(1-q_1)},\\ I_6 &\leq \int_0^{t_1-\epsilon}(t_1-s)^{q-1}\|T_q(t_2-s)-T_q(t_1-s)\|\,\|Bv(s)\|\mathrm{d}s\\ &\quad+\int_{t_1-\epsilon}^{t_1}(t_1-s)^{q-1}\|T_q(t_2-s)-T_q(t_1-s)\|\,\|Bv(s)\|\mathrm{d}s,\\ &\leq \frac{M_1(t_1^{1+a}-\epsilon^{1+a})^{1-q_1}M_B\|v\|}{\Gamma(1+q)(1+a)^{1-q_1}}\sup_{s\in[0,t_1-\epsilon]}\|T_q(t_2-s)-T_q(t_1-s)\|+\frac{2qM_1MM_B\|v\|}{\Gamma(1+q)(1+a)^{1-q_1}}\epsilon^{(1+a)(1-q_1)}.\end{aligned}$$

Since the assumption (H_1) and Lemma 3.1 imply the continuity of $T_q(t)(t>0)$ in t in the uniform operator topology, it can be easily seen that I_3 and I_6 tend to zero independently of $x\in B_{k_0}$ as $t_2-t_1\to 0$, $\epsilon\to 0$. It is clear that $I_1, I_2, I_4, I_5\to 0$ as $t_2-t_1\to 0$. Thus the right-hand side of (9) does not depend on particular choices of $x(\cdot)$ and tends to zero as $t_2-t_1\to 0$, which means that $\{(F_\alpha x), x\in B_{k_0}\}$ is equicontinuous. Thus $F_\alpha[B_{k_0}]$ is equicontinuous and also bounded. By the Ascoli–Arzela theorem, $F_\alpha[B_{k_0}]$ is relatively compact in $C(J,X)$. On the other hand, it is easy to see that for all $\alpha>0$, F_α is continuous on $C(J,X)$. Hence, for all $\alpha>0$, F_α is completely continuous operator on $C(J,X)$. Thus from the Schauder's fixed point theorem, F_α has a fixed point. Therefore, the fractional control system (1) has a mild solution on J. The proof is complete. □

Theorem 3.3. *Assume that assumptions of Theorem 3.2 hold and, in addition, hypothesis* (H_4) *holds and the linear system (3) is approximately controllable on* $[0,b]$. *Then the semilinear fractional system (1) is approximately controllable on* $[0,b]$.

Proof. Let $\hat{x}_\alpha(\cdot)$ be a fixed point of F_α in B_{k_0}. By Theorem 3.2, any fixed point of F_α is a mild solution of (1) under the control

$$\hat{u}_\alpha(t)=B^*T_q^*(b-t)R(\alpha,\Gamma_0^b)p(\hat{x}_\alpha)$$

and satisfies the inequality

$$\hat{x}_\alpha(b)=x_b+\alpha R(\alpha,\Gamma_0^b)p(\hat{x}_\alpha). \quad (10)$$

By the condition (H_4)

$$\int_0^b\|f(s,\hat{x}_\alpha(s))\|^2\mathrm{d}s\leq N^2b.$$

Consequently, the sequence $\{f(s,\hat{x}_\alpha(s))\}$ is bounded in $L_2(J,X)$. Then there is a subsequence denoted by $\{f(s,\hat{x}_\alpha(s))\}$, that converges weakly to say $f(s)$ in $L_2(J,X)$. Define

$$w=\hat{T}_q(b)x_0+\int_0^b(b-s)^{q-1}T_q(b-s)f(s)\mathrm{d}s-x_b.$$

It follows that

$$\begin{aligned}\|p(\hat{x}_\alpha)-w\| &= \left\|\int_0^b(b-s)^{q-1}T_q(b-s)[\|f(s,\hat{x}_\alpha(s))-f(s)\|]\mathrm{d}s\right\|\\ &\leq \sup_{0\leq t\leq b}\left\|\int_0^t(t-s)^{q-1}T_q(t-s)[f(s,\hat{x}_\alpha(s))-f(s)]\mathrm{d}s\right\|. \quad (11)\end{aligned}$$

As in the proof of Theorem 3.2 using infinite-dimensional version of the Ascoli–Arzela theorem one can show that an operator $l(\cdot) \to \int_0^{\cdot} (\cdot - s)^{q-1} T_q(\cdot - s) l(s) \mathrm{d}s : L_2(J, X) \to C(J, X)$ is compact, consequently the right-hand side of (11) tends to zero as $\alpha \to 0^+$.

Then from (10), we obtain

$$\begin{aligned} \|\hat{x}_\alpha(b) - x_b\| &\le \|\alpha R(\alpha, \Gamma_0^b)(w)\| + \|\alpha R(\alpha, \Gamma_0^b)\| \|p(\hat{x}_\alpha) - w\| \\ &\le \|\alpha R(\alpha, \Gamma_0^b)(w)\| + \|p(\hat{x}_\alpha) - w\| \to 0. \end{aligned} \tag{12}$$

Further, (12) tends to zero as $\alpha \to 0^+$ by the estimation (11) and Lemma 2.8. This proves the approximate controllability of (1). □

4. Fractional control systems with nonlocal conditions

The study on nonlocal problems are motivated by physical problems. For example, it is used to determine the unknown physical parameters in some inverse heat conduction problems [29]. The result concerning the existence and uniqueness of mild solutions to abstract Cauchy problems with nonlocal initial conditions was first formulated and proved by Byszewski, see [30]. Since the appearance of this paper, several papers have addressed the issue of existence and uniqueness results for various types of nonlinear differential equations. Mophou and Guérékata [19,20] discussed the existence of mild solution for some fractional differential equations with nonlocal conditions. More recently, [24] studied the existence of mild solutions to nonlinear fractional differential equations with nonlocal conditions. Chang et al. [31] derived a set of novel sufficient conditions for the controllability of a class of first-order semilinear differential systems with nonlocal initial conditions in Banach spaces by using noncompactness technique and the Sadovskii fixed point theorem. On the other hand, Mahmudov [32] established sufficient conditions for the approximate controllability of certain classes of abstract evolution equations with nonlocal initial conditions. However, up to now in the present literature, no work has been reported concerning the approximate controllability of fractional system with nonlocal conditions. Motivated by this consideration, in this section, we discuss the approximate controllability of the fractional system (1) with nonlocal condition of the form

$$x(0) + g(x) = x_0, \tag{13}$$

where $g : C([0, b], X) \to X$ is a given function which satisfies the following condition:

(H_5) There exists a constant $L > 0$ such that $\|g(x) - g(y)\| \le L\|x - y\|$, for $x, y \in C([0, b], X)$.

The nonlocal condition can be applied in physics with a better effect than the classical initial condition $x(0) = x_0$. For example, $g(x)$ can be written as

$$g(x) = \sum_{i=1}^{m} c_i x(t_i),$$

where $c_i (i = 1, 2, \ldots, n)$ are given constants and $0 < t_1 < \cdots < t_n \le b$. As pointed out in [24,25], the nonlocal conditions can be more useful than the standard initial condition to describe some physical phenomena.

Theorem 4.1. *Assume that all the assumptions of Theorem 3.3 hold and, in addition, hypothesis (H_5) holds. Then the semilinear fractional system (1) with nonlocal condition (13) is approximately controllable on J.*

Proof. For $\alpha > 0$, define the operator $\hat{\Phi}_\alpha$ on $C(J, X)$ as follows

$$(\hat{\Phi}_\alpha x) = z,$$

where

$$\begin{aligned} z(t) &= \hat{T}_q(t)[x_0 - g(x)] + \int_0^t (t - s)^{q-1} T_q(t - s) f(s, x(s)) \mathrm{d}s + \int_0^t (t - s)^{q-1} T_q(t - s) B v(s) \mathrm{d}s, \\ v(t) &= B^* T_q^*(b - t) R(\alpha, \Gamma_0^b) p(x(\cdot)), \\ p(x(\cdot)) &= x_b - \hat{T}_q(b)[x_0 - g(x)] - \int_0^b (b - s)^{q-1} T_q(b - s) f(s, x(s)) \mathrm{d}s. \end{aligned}$$

By employing the technique used in Theorem 3.2, one can easily show that, if for all $\alpha > 0$, the operator $\hat{\Phi}_\alpha$ has a fixed point. Then we can show that the fractional control system (1) with nonlocal condition (13) is approximately controllable by adopting the technique used in Theorem 3.3. The proof of this theorem is similar to that of Theorems 3.2 and 3.3, and hence it is omitted. □

Remark 4.2. In recent years, the theory of functional differential equations with infinite delay have been the subject of considerable activity due to its applications in science and engineering. Concerning to the theory of functional differential equations with infinite delay we refer to the book by Hino et al. [33]. If the delay is infinite, the selection of phase space plays an important role in the quantitative and qualitative studies of differential equations. The above result can be extended to study the approximate controllability of semilinear fractional differential systems with infinite delay by suitably introducing the abstract phase space defined in [27].

Example 4.3. We consider a simple example here as an application of Theorem 3.3. Consider a control system governed by the following fractional partial differential equation of the form

$$\begin{cases} \partial_t^q x(t,z) = \partial_z^2 x(t,z) + \mu(t,z) + \partial_z \hat{F}(t, x(t,z)) \quad t \in [0,1],\ z \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \\ x(0,z) = x_0(z), \end{cases} \tag{14}$$

where ∂_t^q is the Caputo fractional partial derivative of order $0 < q < 1$, $\hat{F}$ is a given continuous function.

Let us take $X = U = L^2[0,\pi]$ and define the operator A by $Aw = w''$ with the domain $D(A) = \{w(\cdot) \in L^2[0,\pi], w, w'$ are absolutely continuous, $w'' \in L^2[0,\pi]$, $w(0) = w(\pi) = 0\}$.

Then

$$Aw = -\sum_{n=1}^{\infty} n^2 \langle w, e_n \rangle e_n,\ w \in D(A),$$

where $e_n(z) = (2/\pi)^{1/2} \sin nz,\ 0 \le z \le \pi,\ n = 1, 2, \ldots$. Clearly A generates a compact semigroup $T(t)$, $t > 0$ in X and it is given by

$$T(t)w = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t} \langle w, e_n \rangle e_n,\ w \in X.$$

Clearly the assumption (H_1) is satisfied. On the other hand, it can be easily seen that the deterministic linear system corresponding to (14) is approximately controllable on [0, 1] [34].

Put $x(t) = x(t,\cdot)$, that is $x(t)(z) = x(t,z)$, $t \in [0,1]$, $z \in [0,\pi]$ and $u(t) = \mu(t,\cdot)$, here $\mu : [0,1] \times [0,\pi] \to [0,\pi]$ is continuous. Define the function $f : [0,1] \times X \to X$ by $f(t,x(t))(z) = f(t,x(t,z))$ and the bounded linear operator $B : U \to X$ by $Bu(t)(z) = \mu(t,z)$. Further, we take $q = \frac{1}{2}$ and $f(t,x(t)) = \frac{1}{t^{1/3}} \sin x(t)$, then assumptions (H_2) and (H_3) are satisfied. Therefore, the system (14) can be written to the abstract form (1) and all the conditions of the Theorem 3.3 are satisfied. Thus by Theorem 3.3, fractional control system (14) is approximately controllable on [0, 1].

Remark 4.4. The theory of impulsive differential equations is emerging as an important area of investigation since it is richer than the theory of classical differential equations [35]. Many systems in physics and biology exhibit impulsive dynamical behavior due to sudden jumps at certain instants during the dynamical process [36]. Among the previous research, little is concerned with differential equations with fractional order and impulses [17,37]. Moreover, impulsive control, which is based on the theory of impulsive differential equations has gained renewed interests recently for its promising applications toward controlling systems exhibiting chaotic behavior. By adapting the techniques and ideas established in this paper, one can prove the approximate controllability of fractional control systems with impulses.

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