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Multiplicities of interpoint distances in finite planar sets

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Abstract

What is the maximum number of unit distances between the vertices of a convex n-gon in the plane? We review known partial results for this and other open questions on multiple occurrences of the same interpoint distance in finite planar subsets. Some new results are proved for small n. Challenging conjectures, both old and new, are highlighted.

1. Introduction

We review known results and open problems for multiple occurrences of the same distance between points in finite subsets of the Euclidean plane. Arbitrary finite subsets X as well as subsets V whose points are the vertices of a convex polygon are considered. In all cases, n denotes the number of points in the subset. We prove several new results for small values of n and note that there exists V every point in which is distance 1 from three other vertices. Open problems are highlighted as numbered conjectures. Some are old, others are new, and all are challenging. Related problems in discrete geometry are discussed in [6, 15].

Let $d(x, y)$ be the distance between x and y. The *diameter* of $X = \{x_1, x_2, \dots, x_n\}$ is $\delta = \max d(x_i, x_j)$. Let $d_1, \dots, d_m$ denote the $m \leqslant \binom{n}{2}$ different positive distances between points in X. The multiplicity of d_k is

$$r_k = |\{(i,j)\colon 1 \leqslant i < j \leqslant n,\ d(x_i, x_j) = d_k\}|.$$

We arrange the m multiplicities as $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m$ with $\sum r_k = \binom{n}{2}$, without regard to comparative values of the d_k, and let $r(X) = (r_1, r_2, \dots, r_m)$.

Examples.
(1) $r(X) = (n-1, n-2, \dots, 2, 1)$ for equally-spaced points on a line;
(2) $r(X) = (18, 9, 9, 6, 3)$ for the bowling-pin arrangement of 10 points;

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(3) $r(V) = (n, n, \dots, n)$, $m = (n-1)/2$, for a regular n-gon for odd $n \geqslant 3$;
$r(V) = (n, \dots, n, n/2)$, $m = n/2$, for a regular n-gon with n even.

An old problem of Erdős's on the minimum number of interpoint distances was resolved by Altman [1,2] for convex n-gons. The minimum is realized by regular polygons.

Theorem 1. *$\min m = \lfloor n/2 \rfloor$ over V with $|V| = n$.*

Erdős [6] notes that Szemerédi conjectured $\min m = \lfloor n/2 \rfloor$ over n-sets X with no three points on a line, but could prove only that $m \geqslant \lfloor n/3 \rfloor$. The same paper reports that very little is known about min m when X has no three points on a line and no four points on a circle. Theorem 1 in [8] shows that $\min m < (3/2)\ n^{\log 3/\log 2}$ for this case, and Erdős et al. [7] improve this to $\min m < ne^{c\sqrt{\log n}}$ for some $c > 0$. It is still not known whether this min m is superlinear.

The following is an old conjecture by Erdős [5].

Conjecture 1(a). Some vertex of V has at least $\lfloor n/2 \rfloor$ distinct distances to other vertices.

Many years ago, Moser [14] proved that some vertex of V has at least $\lfloor (n+2)/3 \rfloor$ distinct distances to other vertices. We know of no substantial increase in his bound.

Let $D(V)$ be the sum, over vertices of V, of the number of distinct distances from a vertex to the other vertices. Also let $D_n = \min D(V)$ over all convex n-gons. A regular N-gon for odd n has $D(V) = n\lfloor n/2 \rfloor = \binom{n}{2}$. We also find $D(V) = \binom{n}{2}$ for even $n \geqslant 6$ based on rotated superposition of two regular $n/2$-gons of slightly different sizes. The following appears to be new.

Conjecture 1(b). $D_n = \binom{n}{2}$ for all $n \geqslant 2$.

We have verified it for $n \leqslant 7$. Since Conjecture 1(b) implies 1(a), our new conjecture is stronger.

When comparative sizes of the d_k are suppressed, the most detailed information about multiplicities is given by

$$S_{n,m} = \{r(X)\colon |X| = n, r(X) \text{ has } m \text{ terms}\},$$

$$T_{n,m} = \{r(V)\colon |V| = n, r(V) \text{ has } m \text{ terms}\}.$$

Very little is known about $S_n = \bigcup_m S_{n,m}$ and $T_n = \bigcup_m T_{n,m}$. Section 2 completely specifies S and T for $n = 5$ and $m \in \{2, 3\}$. Section 3 focuses on max r_1, Section 4 on max $\sum r_k^2$, Section 5 on large r_k for all $d_k < \delta$, and Section 6 on multiple common distances from every point.

2. Multiplicities for $n = 5$

Clearly, $S_{3,1} \neq \emptyset$ and $S_{n,1} = \emptyset$ for $n \geqslant 4$. We leave cases of S_4 and T_4 as exercises.

Theorem 2. $S_{5,2} = T_{5,2} = \{(5,5)\}$. *$S_{5,3}$ consists of all feasible (r_1, r_2, r_3) except* $(8,1,1)$. $T_{5,3} = S_{5,3} \setminus \{(7,2,1)\}$.

Proof. Let $n = 5$ throughout. The regular pentagon has $r(X) = (5,5)$. Suppose $r(X) = (r_1, r_2)$, $r_1 > r_2$, $r_1 + r_2 = 10$. It is easily checked that either the r_1 set or the r_2 set requires an equilateral triangle. Fig. 1 shows the only ways to add a fourth point x to the three of an equilateral triangle so that $m = 2$. Addition of a fifth point forces a third interpoint distance. Therefore, $S_{5,2} = T_{5,2} = \{(5,5)\}$.

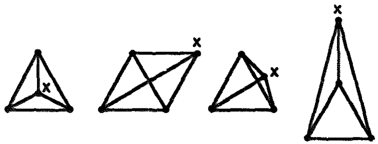


Fig. 1.

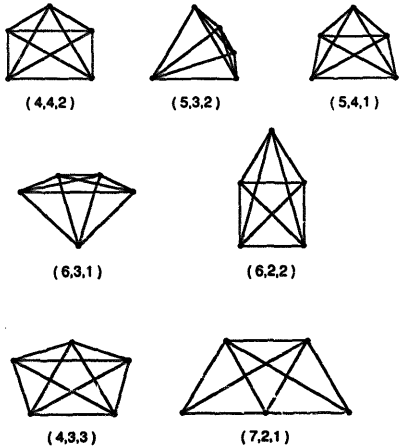


Fig. 2.

Fig. 2 verifies memberships in $T_{5,3}$ and $S_{5,3}$. If (8, 1, 1) were in S_5, then one point is equidistant from the other four, which among themselves have at most three copies of that same distance, contradicting $r_1 = 8$. (See the lower right of Fig. 2 for $r_1 = 7$.)

To show that $(7, 2, 1) \notin T_5$, suppose otherwise. If one point for $r_3 = 1$ is deleted, the resulting 4-gon has either (5, 1) or (4, 2) for its (r'_1, r'_2). The only case of (5, 1) is the second diagram in Fig. 1, and the addition of a fifth point to obtain $r_1 = 7$ forces the nonconvex diagram on the lower right of Fig. 2. If (4, 2) obtains, we have either an equilateral triangle in the r'_1 set, which by convexity forces the third diagram in Fig. 1, or the r'_1 set forms a square. In both cases it is impossible to add a fifth point and get $r_1 = 7$ while preserving convexity. □

What is done in Theorem 2 might be feasible for slightly larger n. For general n, one might look for sizable families of multiplicity vectors, all of whose members are included in or excluded from S_n or T_n.

3. Maximum multiplicity

Let $f(n) = \max r_1$ over all V with $|V| = n$, and let $F(n) = \max r_1$ over all X with $|X| = n$. Erdős [5] conjectured that

$$F(n) \leqslant n^{1+c/\log\log n} \quad \text{for some } c > 0.$$

This bound is attained by $X = \{(i,j): 0 \leqslant i,j < \sqrt{n}\}$, so in fact $F(n)$ is at least as large as the bound for some $c > 0$ and large n. The same X shows that $\min m \leqslant cn/\sqrt{\log n}$ for some $c > 0$, which for large n is much smaller than min m for V in Theorem 1. The best upper bound on $F(n)$ is presently $O(n^{4/3})$. See [3, 17] for more on the upper bound, and [8] for further references on min m and $F(n)$. Another Erdős conjecture related to that for $F(n)$ and the notions of uniformity in Section 6 is that some point in X has fewer than $n^{c/\log\log n}$ points in X equidistant from it.

Erdős and Moser [9] made the following conjecture for convex polygons.

Conjecture 2(a). $f(n) < cn$ for some $c > 0$.

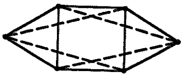
Established bounds are given by the following result.

Theorem 3. $2n - 7 \leqslant f(n) \leqslant 6n(2\log_2 n - 1)$.

A construction in [4] gives the lower bound. Füredi [13] proves the upper bound and says that a proof refinement allows 6 to be replaced by π. Motivated in part by the lower bound, we venture the following.

Conjecture 2(b). $f(n) < 2n$.

Conjectures 2(a) and (b) are intimately related to Conjectures 6 and 7 of Section 6.



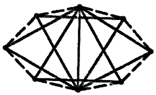


Fig. 3.

4. Maximum $\sum r_k^2$ for convex polygons

For regular convex n-gons we have

$$\sum r_k^2 = n^2(n-1)/2 \quad \text{for odd } n \geqslant 3,$$

$$\sum r_k^2 = n^2(n/2 - 3/4) \quad \text{for even } n.$$

Let $g(n) = \max \sum r_k^2$ over all V with $|V| = n$. Inspection for $n = 3$ and Theorem 2 for $n = 5$ show that $g(n) = n^2(n-1)/2$ for $n \in \{3, 5\}$.

Conjecture 3. For all odd $n \geqslant 3$, $g(n)$ is given by $\sum r_k^2$ for a regular n-gon.

The same thing cannot be true for small even values of n.

Theorem 4. For $n \in \{4, 6, 8\}$, *$g(n)$ exceeds $\sum r_k^2$ for a regular n-gon.*

Proof. The values of $\sum r_k^2$ for a regular n-gon are 20, 81 and 208 for $n = 4, 6, 8$, respectively. The second diagram of Fig. 1 gives $g(4) = 26$. The diagrams of Fig. 3 give $g(6) \geqslant 85$ and $g(8) \geqslant 210$. $\square$

We have not been able to exceed $n^2(n/2 - 3/4)$ for $n = 10$. Perhaps the conclusion of Conjecture 3 holds for all even $n \geqslant 10$. If Conjecture 2(b) is true, then $g(n) < n^3$ for all n. But it is not even known whether $g(n) < cn^3$ for some $c > 0$.

5. Large multiples for sub-diameter distances

A regular n-gon, $n \geqslant 4$, has $r_k = n$ for every $d_k < \delta$. An interesting feature of the second diagram of Fig. 1 is $r_1 > n$ for the one distance smaller than δ. We doubt that this is true for any larger n for all distances smaller than δ. The following was noted in [10].

Conjecture 4. There is no X for $n \geqslant 5$ such that $r_k > n$ for every interpoint distance less than δ.

Theorem 1 verifies the conclusion for $n = 5$. Our next result does likewise for $n = 6$. The conjecture is open for $n \geqslant 7$. The following was conjectured by Endre Makai.

Theorem 5. $(7, 7, 1) \notin S_6$.

Proof. We prove more by showing that $(a, 14 - a, 1) \notin S_6$. Suppose otherwise. Let $X = \{1, 2, \ldots, 6\}$ and assume that the multiplicity of $d(1, 2)$ is 1. Then for each $i \in \{1, 2\}$, $X \backslash \{i\}$ has exactly two distances between vertices. A theorem of Altman [1] says that the only convex 5-gons that determine two distances are regular pentagons. And if five points in the plane are not the vertices of a convex 5-gon, it is easily seen (and well known) that they determine more than two distances. Hence each $X \backslash \{i\}$ describes a regular pentagon. But then points 1 and 2 coincide, and we have a contradiction. □

6. Multiple common distances

We say that X is *k-uniform* if every $x \in X$ has k other points in X equidistant from it, and is *absolutely k-uniform* if there is a distance d_0 such that every x has k other points in X distance d_0 from it. The regular hexagon with a center point is absolutely 3-uniform.

It follows from factorization of Gaussian integers (see for example [16], or [11]) that for every positive integer h there is a $d_0(h)$ such that $a^2 + b^2 = d_0(h)$ has at least h distinct solutions in integers $a \leqslant b$. Given k, a suitably large N for $X = \{(i, j): 0 \leqslant i, j \leqslant N,\ i \text{ and } j \text{ integers}\}$ then assures that X is absolutely k-uniform.

The situation for convex polygons is unclear. Of course every regular polygon ($n \geqslant 3$) is absolutely 2-uniform. Danzer (see [6]) constructed a 9-point V that is 3-uniform but not absolutely 3-uniform. Fishburn and Reeds [12] constructed a 20-point V that is absolutely 3-uniform. Erdős [6] conjectured the following.

Conjecture 5. No V is 4-uniform.

We also suggest the following conjecture.

Conjecture 6. No V is absolutely 4-uniform.

This is very strong. If true, it implies Conjecture 2(a) as follows. Let $d_0 = 1$ for convenience. For some $v \in V$, at most three other points have unit distance from v. Remove such a v and repeat. When only three points remain, at most $3(n - 3)$ unit distances have been deleted, so there were at most $3n - 6$ to start with.

A weak variant of Conjecture 6 goes as follows. We say that a two-part partition $\{A, B\}$ of V is a *cut* if the convex hulls of A and B are disjoint.

Conjecture 7. There is an integer $k \geqslant 4$ such that for every cut $\{A, B\}$ of every V either $|\{b \in B\colon d(a,b) = 1\}| < k$ for some $a \in A$ or $|\{a \in A\colon d(a,b) = 1\}| < k$ for some $b \in B$.

If true for k, an easy proof shows that Conjecture 2(a) holds with $c \leqslant 2(k-1)$. The 20-point construction in [12] shows that the conclusion of Conjecture 7 is false when $k = 3$.

7. Discussion

In reviewing and extending results on multiplicities of interpoint distances in finite subsets of the plane, we hope to encourage new research in this area. We regard our conjectures as difficult but not hopeless.

Acknowledgement

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