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The Principle and Models of Dynamic Programming

CHUNG-LIE WANG*

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada

Submitted by E. Stanley Lee

1. INTRODUCTION

The successful application of Operations Research to military problems during World War II (e.g., see Moder and Elmaghraby [38, p. 1]) has led over the past forty years to the development of a variety of sophisticated mathematical programming techniques to treat management and industrial problems (e.g., see Moder and Elmaghraby [38, 39], Ralston and Reilley [41]). In particular, while no universal approach exists to nonlinear mathematical programming problems involving arbitrary objective functions of multiple variables, convex, quadratic, and geometric programming methods are now well developed for use with convex (or concave) functions (e.g., see Beveridge and Schechter [14], Duffin *et al.* [19], Luenberger [35], Rockafellar [43]). These developments have been accelerated by the evolution of computer technology and advances in numerical analysis and approximation.

One of the most useful nonlinear mathematical programming methods appears to be dynamic programming (abbreviated DP). In the last thirty years, DP has developed many useful structures and has broadened its scope to include finite and infinite models and discrete and continuous models, as well as deterministic and stochastic models (e.g., see Bellman [8], and Bellman and Lee [13]). In fact, the DP concept has been used to solve a variety of optimization problems analytically and numerically: For the general problems see Aris [2], Bellman [6–8], Bellman and Dreyfus [11], and Beveridge and Schechter [14]; for the particular problems, see Angel and Bellman [1], Bellman [9, 10], Bellman and Kalaba [12], Bertsekas [15], Dano [16], Dreyfus [17], Dreyfus and Law [18], Jacob-

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son and Mayne [31], Kaufmann and Cruon [32], Larson [33], Lee [34], Mitten [37], Nemhauser [40], Tou [44], and White [58].

The DP approach has been long regarded as most suitable to optimization problems which are sequential and recursive in scheme (e.g., see Denardo [38], Wald [45]). A major objective of this paper is to demonstrate, by examples, that DP can also be used to treat optimization problems with nonsequential and/or nonrecursive schemes.

DP is conceptionally simple: Its foundation is the principle of optimality, which is spelled out by Bellman [8, p.83] (see also Aris [2], Dreyfus [17], Denardo [38], Nemhauser [40]) as follows.

Principle of Optimality. An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

As observed by Bellman [8, p. 83], a proof of the principle of optimality by contradiction is immediate. (See also Aris [2, pp. 27, 133], Dreyfus [14, pp. 8, 14], Nemhauser [40, p. 33].) The key phrase of the principle is "state resulting from the first decision." In other words, the first decision is, for instance, to choose a constraint (or constraints) (e.g., see Wang [48]) or a transformation, to adapt Lagrange multipliers (e.g., see Dreyfus and Law [18], Nemhauser [40]), to modify the objective function, or whatever else; the state resulting from the first decision must vary to ensure the remaining decisions constitute an optimal policy. (For certain particular situations, see, e.g., Aris [2], Beveridge and Schechter [4], White [58].)

The principle of optimality yields all kinds of workable functional equations as models for problems arising in many fields. As stated in Bellman and Lee [13, p. 1], the basic form of the functional equation of DP is

$$f(p) = \operatorname*{opt}_{q} [H(p, q, f(T(p, q)))]$$

where p and q represent the state and decision vectors, respectively, T represents the transformation of the process, and $f(p)$ represents the optimal return function with initial state p. (Here opt denotes max or min.)

Furthermore, DP problems just as the other mathematical programming problems may or may not possess constraints. However, in some cases, we may introduce one or more transition constraints in order to facilitate solving the problems (e.g., see Wang [51, 55]).

DEFINITION. A transition constraint of a mathematical programming problem is an additional constraint consistent with the original constraint (or constraints), designed to facilitate solving the problem.

In this paper, we only consider the problem of finite and infinite discrete models. In subsequent sections, we reestablish most of the known problems cited from [2, 3, 14, 40] using only two simple inequalities: the arithmetic and geometric (abbreviated AG) inequality

$$A \geqslant G \tag{1}$$

and the Jensen inequality

$$f(A) \leqslant [af(x) + bf(y)]/(a+b) \tag{2}$$

with equalities holding if and only if $x = y$; where $(a+b)A = ax + by$, $G^{a+b} = x^a y^b$, f is a convex function, $a, b, x, y > 0$. (Note: for the "only if" equality case of (2), f is also required to be strictly convex [42].) By so doing, we shall be pursuing a second objective of the paper: to study the use of simple inequalities to streamline the DP approach in solving optimization problems.

We adopt the notations and symbols from the cited sources with little or no modification for the purpose of comparing our results with theirs directly.

2. Several Simple Examples

In this section, we choose several simple examples to demonstrate not only the versatility and novelty of DP but also the much broader scope of the DP approach in handling optimization problems. We show also how to introduce a suitable transition constraint to an optimization problem.

2.1. Consider the problem

$$\phi_2(a) = \min(4x_1^2 + 4x_2^3) \tag{3}$$

subject to

$$2x_1 + 3x_2 = a, \qquad x_1, x_2 \geqslant 0, a > 0. \tag{4}$$

Routinely, with $\phi_1(a) = a^2$, we have

$$\phi_2(a) = \min_{x_2} [(a - 3x_2)^2 + 5x_2^3]. \tag{5}$$

Setting

$$x_2 = u - \tfrac{3}{5},$$

a manipulation yields

$$(a-4x_2)^2+5x_2^3=f(u)+g(a), \tag{6}$$

where

$$f(u)=5u^3-\tfrac{3}{5}(9+10a)\,u \tag{7}$$

and

$$g(a)=a^2+\frac{18}{5}a+\frac{54}{25}. \tag{8}$$

Substituting (6) into (5), we have

$$\phi_2(a)=\min_u f(u)+g(a). \tag{9}$$

Rewriting (7), an application of the AG inequality (1) yields

$$f(u)\geqslant -\frac{5}{\sqrt{2}}\left[\frac{1}{3}(2u^2)+\frac{2}{3}\left\{\frac{3}{25}(9+10a)-u^2\right\}\right]^{3/2}$$

and

$$\min_u f(u)=-\frac{2}{25}(9+10a)^{3/2}. \tag{10}$$

The minimum is attained at

$$2u^2=\frac{3}{25}(9+10a)-u^2$$

or

$$u=\tfrac{1}{5}(9+10a)^{1/2}.$$

From (9), (10), follows

$$\phi_2(a)=g(a)-\frac{2}{25}(9+10a)^{3/2}.$$

The minimum $\phi_2(a)$ is attained at

$$x_1=\frac{1}{2}a+\frac{9}{10}-\frac{3}{10}(9+10a)^{1/2},$$

$$x_2=\frac{1}{5}(9+10a)^{1/2}-\frac{3}{5}.$$

2.2. Consider the problem

$$\phi_2 = \min_{x_1, x_2} [ax_1 + b(x_1x_2)^{-1} + cx_2 + d], \qquad a, b, c, d > 0. \tag{11}$$

We introduce a transition constraint for the problem (11) in two ways.

2.2.1. Consider the problem (11) subject to

$$x_1x_2 = t, \qquad t > 0. \tag{12}$$

Combining

$$\phi_2(t) = \min_{x_2} [\phi_1(t, x_2) + cx_2 + d]$$

with

$$\phi_1(t, x_2) = \min_{x_1 = t} [ax_1 + b(x_1x_2)^{-1}] = at + b(tx_2)^{-1},$$

the minimum

$$\phi_2(t) = 2(act)^{1/2} + bt^{-1} + d$$

is attained at

$$atx_2^{-1} = cx_2 \qquad \text{with (12)}$$

or

$$x_2 = (at/c)^{1/2}, \qquad x_1 = (ct/a)^{1/2}. \tag{13}$$

Furthermore, the minimum

$$\min_t \phi_2(t) = 3(abc)^{1/3} + d \tag{14}$$

is attained at

$$(act)^{1/2} = bt^{-1}$$

or

$$t = (b^2/ac)^{1/3}. \tag{15}$$

In turn, by (13) and (15), the minimum ϕ_2 is attained at

$$x_1 = (bc/a^2)^{1/3}, \qquad x_2 = (ab/c^2)^{1/3}. \tag{16}$$

2.2.2. Consider the problem (11) subject to

$$ax_1 + cx_2 = s, \qquad s > 0. \tag{17}$$

Combining

$$\phi_2(s) = \min_{x_2} [\phi_1(s, x_2) + cx_2 + d]$$

with

$$\phi_1(s, x_2) = \min_{ax_1 = s} [ax_1 + b(x_1 x_2)^{\;1}] = s + ab(sx_2)^{\;1},$$

the minimum

$$\phi_2(s) = s + 4abcs^{\;2} + d$$

is attained at

$$s - cx_2 = cx_2 \qquad \text{with (16)}$$

or

$$x_1 = s/2a, \qquad x_2 = s/2c. \tag{18}$$

Furthermore, the minimum ϕ_2 given in (14) is attained at

$$s/2 = abc(s/2)^{\;2}$$

or

$$s/2 = (abc)^{1/3}. \tag{19}$$

In turn, by (18) and (19) the minimum ϕ_2 is attained again at x_1 and x_2 given in (16).

3. A Generalized AG Inequality

Consider the problem

$$\phi_n(a) = \min \sum a_j x_j^{b_j} \tag{20}$$

subject to

$$\prod x_j^{c_j} = a, \qquad a > 0, a_j, b_j, c_j > 0, x_j \geqslant 0, 1 \leqslant j \leqslant n. \tag{21}$$

Here and in what follows $\sum$ and $\prod$ are used to designate $\sum_{j=1}^{n}$ and $\prod_{j=1}^{n}$ whenever confusion is unlikely to occur. We introduce also

$$s_K = \sum_{j=1}^{K} c_j/b_j, \qquad t_K = \prod_{j=1}^{K} (a_j b_j/c_j)^{c_j/b_j}, \qquad 1 \leqslant k \leqslant n.$$

Since

$$\phi_1(a) = \min_{x_1^{c_1} = a} a_1 x_1^{b_1} = s_1(t_1 a)^{1/s_1}, \tag{22}$$

$$\phi_2(a) = \min_{x_2} (a x_2^{-c_2})^{1/s_1} + a_2 x_2^{b_2}] \tag{23}$$

$$= \min_{x_2} \left[s_1(t_1 a x_2^{-c_2})^{1/s_1} + \frac{c_2}{b_2}\left(\frac{a_2 b_2}{c_2} x^{b_2}\right)\right]$$

$$\geqslant \min_{x_2} s_2 \left[(t_1 a x_2^{-c_2})^{1/s_2} \left(\frac{a_2 b_2}{c_2} x_2^{b_2}\right)^{c_2/b_2 s_2}\right].$$

The minimum

$$\phi_2(a) = s_2(t_2 a)^{1/s_2}$$

is attained at

$$(t_1 a x_2^{-c_2})^{1/s_1} = \frac{a_2 b_2}{c_2} x_2^{b_2}$$

which is equivalent to

$$\frac{a_1 b_1}{c_1} x_1^{b_1} = \frac{a_2 b_2}{c_2} x_2^{b_2} = (t_2 a)^{1/s_2}.$$

In general we can readily derive ϕ_{K+1} from ϕ_K for any $K \geqslant 1$ by exactly the same procedure as above ((22) and (23)) to derive ϕ_2 from ϕ_1. So, we obtain inductively that the minimum

$$\phi_n(a) = s_n(t_n a)^{1/s_n} \tag{24}$$

is attained at

$$\frac{a_1 b_1}{c_1} x_1^{b_1} = \cdots = \frac{a_n b_n}{c_n} x_n^{b_n} = (t_n a)^{1/s_n}. \tag{25}$$

It is now clear that a generalized AG inequality

$$\sum a_j x_j^{b_j} \geqslant \left(\sum c_j/b_j\right)\left[\prod (a_j b_j/c_j)^{c_j/b_j} \prod x_j^{c_j}\right]^{1/\sum c_j/b_j} \tag{26}$$

follows from (20), (21), and (24). Equality holds in (26) if and only if the x_j satisfy (25).

Remark. Compare the AG inequality (26) with the results given in Beckenbach and Bellman [4, p. 6], Iwamoto [27, 28], and Wang [47, 48].

4. Inequality of Weighted Means

The monotonicity of weighted means $M(t)$ provides a very useful inequality for any real numbers $s < t$,

$$M(s) \leqslant M(t) \tag{27}$$

where

$$M(t) = \left(\sum a_j x_j^t \Big/ \sum a_j \right)^{1/t}, \qquad a_j, x_j > 0,\ 1 \leqslant j \leqslant n.$$

A proof of (27) can be found in [4, 23, 35].

Here we generalize the inequality (27) through the DP approach (see also Wang [50]). In doing so, we consider the problem

$$\text{Opt} \sum a_j x_j^t \tag{28}$$

subject to

$$\sum b_j x_j^s = a, \qquad a > 0,\ a_j, b_j > 0,\ x_j \geqslant 0. \tag{29}$$

There are two cases for the problem (28)–(29) to be considered: Opt = min for $s < 0 < t$ or $0 < s < t$; Opt = max for $s < t < 0$.

First, consider the problem

$$\phi_n(a) = \min \sum a_j x_j^t \tag{30}$$

subject to (29).

In this case, the convexity of the function $x^{t/s}$ for $0 < s < t$ or $s < 0 < t$ is used. We also introduce

$$\lambda_k = \sum_{j=1}^{k} \mu_j \qquad \text{with} \quad \mu_j = (b_j^{1/s} a_j^{-1/t})^{1/(1/s - 1/t)},\ 1 \leqslant k \leqslant n.$$

Since

$$\phi_1(a) = \min_{b_1 x_1^s = a} a_1 x_1^t = \lambda_1 (a/\lambda_1)^{t/s},$$

$$\phi_2(a) = \min_{x_2} [\phi_1(a - b_2 x_2^s) + a_2 x_2^t]$$

$$= \min_{x_2} \left[\lambda_1 \left(\frac{a - b_2 x_2}{\lambda_1} \right)^{t/s} + \mu_2 \left(\frac{b_2 x_2^s}{\mu_2} \right)^{t/s} \right]$$

$$\geqslant \min_{x_2} \lambda_2 \left[\frac{\lambda_1}{\lambda_2} \left(\frac{a - b_2 x_2^s}{\lambda_1} \right) + \frac{\mu_2}{\lambda_2} \left(\frac{b x_2^s}{\mu_2} \right) \right]^{t/s}.$$

The minimum

$$\phi_2(a) = \lambda_2 (a/\lambda_2)^{t/s}$$

is attained at

$$\frac{a - b_2 x_2^s}{\mu_1} = \frac{b_2 x_2^s}{\mu_2}$$

or

$$\frac{b x_1^s}{\mu_1} = \frac{b_2 x_2^s}{\mu_2} = \frac{a}{\lambda_2}.$$

Using the same argument as mentioned in the previous section, we obtain inductively that the minimum

$$\phi_n(a) = \lambda_n (a/\lambda_n)^{t/s} \tag{31}$$

is attained at

$$\frac{b_1 x_1^s}{\mu_1} = \cdots = \frac{b_n x_n^s}{\mu_n} = \frac{a}{\lambda_n}. \tag{32}$$

For the case $s < t < 0$, the above result can be duplicated by using the concavity of the function $x^{t/s}$ and opt = max.

It is now clear that the inequality (27) follows from (30), (29), and (31) with $a_j = b_j$ ($1 \leqslant j \leqslant u$). Equality holds in (27) if and only if $x_1 = \cdots = x_n$ (by (32)).

Remark. Equations (30), (29), and (31) produce an inequality which generalizes not only the inequality (27) of weighted means but also the Hölder inequality (e.g., see Wang [56, 57] or below).

5. Problems Concerning Cross-current Extraction

Regarding the n-stage cross-current extraction process, we refer to Beveridge and Schechter [14, pp. 222, 254, 664, 682] for details. For our purpose, we consider only the problem

$$\max \sum y_j \tag{33}$$

where

$$y_j = p_{j+1} - p_j - Bq_j = y_k(q_k)$$

and

$$p_{j+1} = p_j(1 + m'q_j), \qquad 1 \leqslant j \leqslant n. \tag{34}$$

Here B and m' are constants while p_j and q_j are variables as indicated in [14, p. 254].

Using the principle of optimality, we obtain the recurrence relation for $k > 1$

$$\begin{aligned} M_k(p_{k+1}) &= \max \sum_{j=1}^{k} y_j = \max\,[M_{k-1}(p_k) + y_k] \\ &= \max_{q_k}\,[M_{k-1}(p_{k+1}(1 + m'q_k)^{-1}) + y_k(q_k)]. \end{aligned}$$

In particular,

$$\begin{aligned} M_1(p_2) &= \max_{q_1}\,[p_2 - p_2(1 + m'q_1)^{-1} - Bq_1] \\ &= \max_{q_1}\left[p_2 - p_2(1 + m'q_1)^{-1} - \frac{B}{m'}(1 + m'q_1) + \frac{B}{m'}\right]. \end{aligned}$$

The maximum

$$M_1(p_2) = p_2 - 2\left(\frac{B}{m'}\right)^{1/2} p_2^{1/2} + \frac{B}{m'}$$

is attained at

$$p_2(1 + m'q_1)^{-1} = \frac{B}{m'}(1 + m'q_1)$$

or

$$q_1 = \frac{1}{m'}\left[\left(\frac{m'p_2}{B}\right)^{1/2} - 1\right] \qquad \text{with} \quad p_1^2 = \frac{B}{m'}\,p_2.$$

For $k=2$, we have

$$M_2(p_3)=\max_{q_2}\,[M_1(p_3(1+m'q_2)^{-1})+y_2(q_2)]$$
$$=\max_{q_2}\left[p_3-\left\{2\left(\frac{B}{m'}\right)^{1/2}\left(\frac{p_3}{1+m'q_2}\right)^{1/2}+\frac{B}{m'}(1+m'q_2)\right\}+2\frac{B}{m'}\right].$$

The maximum

$$M_2(p_3)=p_3-3\left(\frac{B}{m'}\right)^{2/3}p_3^{1/3}+2\frac{B}{m'}$$

is attained at

$$\left(\frac{B}{m'}\right)^{1/2}\left(\frac{p_3}{1+m'q_2}\right)^{1/2}=\frac{B}{m'}(1+m'q_2)$$

or

$$q_2=\frac{1}{m'}\left[\left(\frac{m'p_3}{B}\right)^{1/3}-1\right]\qquad\text{with}\quad p_2^3=\frac{B}{m'}p_3^2.$$

Using the same argument as above, we obtain inductively that the maximum

$$M_k(p_{k+1})=p_{k+1}-(k+1)\left(\frac{B}{m'}\right)^{k/(k+1)}p_{k+1}^{1/(k+1)}+k\frac{B}{m'}$$

is attained at

$$q_k=\frac{1}{m'}\left[\left(\frac{m'p_{k+1}}{B}\right)^{1/(k+1)}-1\right]\qquad\text{with}\quad p_k^{k+1}=\frac{B}{m'}p_{k+1}^k.$$

6. Problems Concerning Chemical Reaction

Problems of chemical reaction in a sequence of stirred tanks have been intensively studied by Aris [2]. For our purpose, we cite only the following problem from Aris [2, p. 20]:

$$\min\sum\theta_j \tag{35}$$

subject to

$$\theta_j\geqslant 0,\qquad c_1=\gamma \tag{36}$$

where

$$c_j = \frac{c_{j+1} + \theta_j k_1}{1 + \theta_j(k_1 + k_2)}, \qquad 1 \leqslant j \leqslant n. \tag{37}$$

Here k_1, k_2, γ are parameters (or constants) while c_j and θ_j are variables as indicated in [2, p. 20].

Using the principle of optimality, we obtain the recurrence relation for $k > 1$

$$\begin{aligned} f_k(c_{k+1}) &= \min \sum_{j=1}^{k} \theta_j \\ &= \min_{c_k} [f_{k-1}(c_k) + \theta_k]. \end{aligned} \tag{38}$$

Rewriting (37) as

$$\theta_j = \frac{1}{k_1 + k_2}\left(\frac{c_e - c_{j+1}}{c_e - c_j} - 1\right), \qquad 1 \leqslant j \leqslant n,$$

where $(k_1 + k_2)\, c_e = k_1$, we have

$$f_1(c_2) = \frac{1}{k_1 + k_2}\left(\frac{c_e - c_2}{c_e - \gamma} - 1\right). \tag{39}$$

From (38) and (39), follows

$$\begin{aligned} f_2(c_3) &= \min_{c_2}\left[f_1(c_2) + \frac{1}{k_1 + k_2}\left(\frac{c_e - c_3}{c_e - c_2} - 1\right)\right] \\ &= \frac{1}{k_1 + k_2} \min_{c_2}\left[\frac{c_e - c_2}{c_e - \gamma} + \frac{c_e - c_3}{c_e - c_2} - 2\right]. \end{aligned}$$

The minimum

$$f_2(c_3) = \frac{2}{k_1 + k_2}\left[\left(\frac{c_e - c_3}{c_e - \gamma}\right)^{1/2} - 1\right]$$

is attained at

$$\frac{c_e - c_2}{c_e - \gamma} = \frac{c_e - c_3}{c_e - c_2}$$

or

$$\theta_1 = \theta_2 = \tfrac{1}{2} f_2(c_2).$$

Finally, we conclude by the same argument as used above that the minimum

$$f_n(c_{n+1}) = \frac{n}{k_1 + k_2}\left[\left(\frac{c_e - c_{n+1}}{c_e - \gamma}\right)^{1/n} - 1\right]$$

is attained at

$$\theta_1 = \cdots = \theta_n = \frac{1}{k_1 + k_2}\left[\left(\frac{c_e - c_{n+1}}{c_e - \gamma}\right)^{1/n} - 1\right].$$

Remark. Concerning various modifications of the problem, consult Aris [2] for complete details. It should also be noted that there are misprints on page 26 of [2].

7. Optimal Allocation on Sampling

Optimal allocation of sample sizes in multivariate stratified random sampling has been carefully studied in Arthanari and Dodge [3]. For our purpose, we cite only the following problem from [3, p. 30]:

$$\phi_n(a) = \min \sum C_j/x_j \tag{40}$$

subject to

$$\sum a_j x_j = a, \qquad a > 0, a_j, x_j > 0, \ 1 \leqslant j \leqslant n. \tag{41}$$

Using the principle of optimality, we obtain the recurrence relation for $k > 1$

$$\phi_k(a) = \min_{x_k}\left[\phi_{k-1}(a - a_k x_k) + \frac{C_k a_k}{a_k x_k}\right]. \tag{42}$$

In particular,

$$\phi_1(a) = \min_{a_1 x_1 = a} C_1/x_1 = w_1/a, \tag{43}$$

where

$$w_k = [w_{k-1}^{1/2} + (C_k a_k)^{1/2}]^2, \qquad k = 1, 2, \ldots; \ w_0 = 0.$$

From (42) and (43), follows

$$\phi_2(a) = \min_{x_2} \left[\frac{w_1}{a - a_2 x_2} + \frac{C_2 a_2}{a_2 x_2} \right]$$
$$\geqslant \min_{x_2} w_2 \left[w_1^{1/2} \frac{a - a_2 x_2}{w_1^{1/2}} + (C_2 a_2)^{1/2} \frac{a_2 x_2}{(C_2 a_2)^{1/2}} \right]^{-1}.$$

The minimum

$$\phi_2(a) = w_2/a$$

is attained at

$$\frac{w_1^{1/2}}{a - a_2 x_2} = \frac{(C_2 a_2)^{1/2}}{a_2 x_2}$$

or

$$\frac{a_1 x_1}{(C_1 a_1)^{1/2}} = \frac{a_2 x_2}{(C_2 a_2)^{1/2}} = \frac{a}{w_2^{1/2}}.$$

Finally, we conclude by the same argument as used above that the minimum

$$\phi_n(a) = w_n/a$$

is attained at

$$x_j = a(C_j a_j)^{1/2} \Big/ a_j \sum (C_j a_j)^{1/2}, \qquad 1 \leqslant j \leqslant n.$$

8. Inequalities and DP

Inequalities such as the AG inequality, the Hölder inequality, and the Minkowski inequality, etc., can be established by the functional equation approach of DP (e.g., see Beckenbach and Bellman [4, p. 6], Iwamoto [24–30], and Wang [47–53]). In the above, we demonstrated that DP problems can be inductively established by simple inequalities (1) or (2). In fact, as studied in Wang [54–55], inequalities can be used to establish DP problems directly.

8.1. For the problem (20)–(21), we apply the AG inequality (see Wang [54, p. 157] for complete details).

8.2. For the problem (28)–(29), we apply a generalized Hölder inequality (e.g., see Wang [57, p. 554]):

$$\left(\sum b_j x_j^s\right)^{1/s} \leqslant \lambda_n^{(1/s)-(1/t)} \left(\sum a_j x_j^t\right)^{1/t} \tag{44}$$

for $0 < s < t$ or $s < 0 < t$, where λ_n is given above. The sign of inequality is reversed in [44] for $s < t < 0$. In both cases equality holds if and only if the x_j satisfy (32). Using (44) we establish the problem (28)–(29) exactly as indicated by (31) and (32).

8.3. For the problem (33)–(34), by using (34) recursively we have

$$p_1 = p_{n+1} \prod (1 + m'q_j)^{-1}. \tag{45}$$

Regrouping the summation in (33), using (45), and applying the AG inequality, we obtain

$$\begin{aligned}\sum y_j &= p_{n+1} - p_{n+1} \prod (1 + m'q_j)^{-1} - \sum \frac{B}{m'} (1 + m'q_j) + n \frac{B}{m'} \\ &\leqslant p_{n+1} - (n+1) \left(\frac{B}{m'}\right)^{n/(n+1)} p_{n+1}^{1/(n+1)} + n \frac{B}{m'} = M_n(p_{n+1}).\end{aligned}$$

The maximum

$$\max \sum y_j = M_n(p_{n+1})$$

is attained at

$$\frac{B}{m'} (1 + m'q_1) = \cdots = \frac{B}{m'} (1 + m'q_r) = p_{n+1} \prod (1 + m'q_j)^{-1}$$

or

$$q_1 = \cdots = q_n = \frac{1}{m'} \left[\left(\frac{m'p_{n+1}}{B} \right)^{1/(n+1)} - 1 \right]$$

with

$$p_j^{j+1} = \frac{B}{m'} p_{j+1}^j, \qquad 1 \leqslant j \leqslant n.$$

8.4. For the problem (35)–(36), we apply the AG inequality to the summation

$$\begin{aligned}\sum \theta_n &= \frac{1}{k_1+k_2}\sum\left(\frac{c_e - c_{i+1}}{c_e - c_i} - 1\right)\\ &= \frac{n}{k_1+k_2}\left[\frac{1}{n}\sum\left(\frac{c_e - c_{i+1}}{c_e - c_i}\right) - 1\right]\\ &\geqslant \frac{n}{k_1+k_2}\left[\left(\frac{c_e - c_{n+1}}{c_e - \gamma}\right)^{1/n} - 1\right] = f_n(c_{n+1}).\end{aligned}$$

The minimum

$$\min \sum \theta_i = f_n(c_{n+1})$$

is attained at $\theta_1 = \cdots = \theta_n$.

8.5. For the problem (40)–(41), rewriting the summation in (40) as follows,

$$\sum C_i/x_i = w_n^{1/2} \sum \frac{(C_i a_i)^{1/2}}{w_n^{1/2}} \cdot \frac{C_i}{(C_i a_i)^{1/2} x_i}, \tag{46}$$

where w_n is given above, and applying the arithmetic and harmonic inequality to the right-hand side of (46), we have

$$\sum C_i/x_i \geqslant w_n^{1/2}\left[\sum \frac{(C_i a_i)^{1/2}}{w_n^{1/2}} \cdot \frac{(C_i a_i)^{1/2} x_i}{C_i}\right]^{-1} = \frac{w_n}{a}.$$

The minimum

$$\min \sum C_i/x_i = w_n/a$$

is attained at

$$\frac{C_1}{(C_1 a_1)^{1/2} x_1} = \cdots = \frac{C_n}{(C_n a_n)^{1/2} x_n} = \frac{w_n^{1/2}}{a}.$$

9. Bellman–Nemhauser Model Problems

Although we have directly established in the the previous section the DP problems given in Sections 3–7 by using pertinent inequalities, there are certain DP problems which cannot be solved by using inequalities alone. In this regard, we consider a Bellman–Nemhauser model problem of an infinite number of decisions as follows:

$$\min \sum_{n=1}^{\infty} [C_1 d_n^p + C_2(x_n - d_n^p)], \qquad C_1, C_2 > 0,\ p > 1, \tag{47}$$

subject to

$$x_{n+1} = b(x_n - d_n), \qquad 0 < b < 1, \qquad 0 \leqslant d_n \leqslant x_n, \quad n = 1, 2, \ldots. \tag{48}$$

Using the principle of optimally, we obtain the recurrence relation for $k > 1$

$$f_k(x) = \min_{d_k} [C_1 d_k^p + C_2(x_k - d_k)^p + f_{k-1}(b(x_k - d_k))]. \tag{49}$$

In particular,

$$\begin{aligned} f_1(x_1) &= \min_{d_1} [C_1 d_1^p + C_2(x_1 - d_1)^p] \\ &= \min_{d_1} \left[C_2^{1/(p-1)} \left(\frac{C_1^{1/p} d_1}{C_2^{1/p(p-1)}} \right)^p + C_1^{1/(p-1)} \left(\frac{C_2^{1/p}(x_1 - d_1)}{C_1^{1/p(p-1)}} \right)^p \right] \\ &\geqslant \min_{d_1} C \left[\frac{C_2^{1/(p-1)}}{C} \left(\frac{C_1^{1/p} d_1}{C_2^{1/p(p-1)}} \right) + \frac{C_1^{1/(p-1)}}{C} \left(\frac{C_2^{1/p}(x_1 - d_1)}{C_1^{1/p(p-1)}} \right) \right]^p \\ &= k_1 x_1^p, \end{aligned} \tag{50}$$

where

$$k_1 = C_1 C_1 / C^{p-1} \qquad \text{with} \quad C = C_1^{1/(p-1)} + C_2^{1/(p-1)}.$$

The minimum

$$f_1(x_1) = k_1 x_1^p$$

is attained at

$$\frac{C_1^{1/p}}{C_2^{1/p(p-1)}} = \frac{C_2^{1/p}(x_1 - d_1)}{C_1^{1/p(p-1)}} \tag{51}$$

or

$$d_1 = C_2^{1/(p-1)} x_1 / C.$$

From (49) and (50), follows

$$f_2(x_2) = \min_{d_2} [C_1^p d_2 + C_2(x_2 - d_2)^p + f_1(b(x_2 - d_2))]$$

$$= \min_{d_2} [C_1 d_2^p + (C_2 + k_1 b^p)(x_2 - d_2)^p]$$

$$\geqslant k_2 x_2^p, \tag{52}$$

where

$$k_2 = \frac{C_1(C_2 + k_1 b^p)}{[C_1^{1/(p-1)} + (C_2 + K_1 b^p)^{1/(p-1)}]^{p-1}}.$$

Replacing C_2 by $C_2 + k_1 b^p$ in (50) and (52) and noting (52), it is easy to see that the minimum

$$f_2(x_2) = k_2 x_2^p$$

is attained at

$$d_2 = \frac{(C_2 + k_1 b^p)^{1/(p-1)}}{C_1^{1/(p-1)} + (C_2 + k_1 b^p)^{1/(p-1)}} x_2.$$

In general, using the same argument as given above, we conclude inductively that the minimum

$$f_n(x_n) = k_n x_n^p, \tag{53}$$

where

$$k_n = \frac{C_1(C_2 + k_{n-1} b^p)}{[C_1^{1/(p-1)} + (C_2 + k_{n-1} b^p)^{1/(p-1)}]^{p-1}}, \tag{54}$$

$n = 1, 2, \ldots$ with $k_0 = 0$, is attained at

$$d_n = \frac{(C_2 + k_{n-1} b^p)^{1/(p-1)}}{C_1^{1/(p-1)} + (C_2 + k_{n-1} b^p)^{1/(p-1)}} x_n.$$

Finally, passing to the limits, we obtain from (53) and (54)

$$f(x) = kx^p,$$

where k can be found from

$$k = \frac{C_1(C_2 + kb^p)}{[C_1^{1/(p-1)} + (C_2 + kb^p)^{1/(p-1)}]^{p-1}}$$

by the Newton–Raphson method (e.g., see Beveridge and Schechter [4, p. 56]).

Remark. For $p = 2$, our result coincides with that of Nemhauser [40, p. 214] and includes that of Gue and Thomas [22, p. 176] as a special case. Various numerical results of the case $p = 2$ can also be found in [22, 40]. For $0 < p < 1$, we can readily treat the problem (47)–(48) as a maximum problem and obtain the same result.

10. Concluding Remarks

In the results given above we have demonstrated that the simple inequalities (1) and (2) can be used to establish a wide range of DP problems in a unified and concise manner. For our inequality approach, we have not required the functions adopted in the problems to be differentiable. On the other hand, those DP problems solved in [2, 3, 14, 22, 40] by using the direct calculus method or the lagrange multiplier method require the assumption of the differentiability of functions.

The problem (47)–(48) for $0 < p < 1$ is indeed a special case of the famous model of the DP problem introduced by Bellman [8, pp. 11, 44]:

$$f(x) = \max_{0 \leqslant y \leqslant x} [g(y) + h(x - y) + f(ay + b(x - y))]. \tag{55}$$

The problem (55) has been intensively studied by Bellman [8] in many respects. Very recently, Iwamoto [28] has given two inverse versions of the Bellman allocation process (55). In this connection Bellman and Lee [13] have shown more than one way to approach functional equations (such as (55)) in DP.

In Section 8, we briefly mentioned an approximation technique. For many DP problems, approximation techniques produce many useful numerical solutions (e.g., see Bellman [7, 8], Dreyfus and Law [18], Lee [34], Nemhauser [40]).

Finally, it might be worthwhile to point out that by employing the concept of a transition constraint, a further development of the idea of solving DP problems by pertinent inequalities appears to be promising (e.g., see Wang [51, 54, 55]).

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