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On secant varieties of compact Hermitian symmetric spaces[☆]

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ABSTRACT

We show that the secant varieties of rank three compact Hermitian symmetric spaces in their minimal homogeneous embeddings are normal, with rational singularities. We show that their ideals are generated in degree three—with one exception, the secant variety of the 21-dimensional spinor variety in $\mathbb{P}^{63}$ where we show that the ideal is generated in degree four. We also discuss the coordinate rings of secant varieties of compact Hermitian symmetric spaces.

1. Introduction

Let K be an algebraically closed field of characteristic zero, let $V = K^{N+1}$, let $X \subset \mathbb{P}V = \mathbb{P}^N$ be a projective variety, and let $\sigma(X) \subset \mathbb{P}V$ denote its *secant variety*, the Zariski closure of the set of points on the secant lines to X. Recently there has been interest in the ideals of secant varieties of homogeneous varieties [1–8], and this paper contributes to their study.

If the ideal of a variety X is generated in degree two, the minimal possible degree of generators for the ideal of $\sigma(X)$ is three ([5] Cor. 3.2), although in general one does not expect generators in degree three (e.g. this almost always fails for complete intersections of quadrics). On the other hand, when X is homogeneous, i.e., V is an irreducible G-module where G is a semi-simple algebraic group and X is the orbit of a highest weight line (so in particular, the ideal of X is generated in degree two), in all previously known examples (mostly just the rank two compact Hermitian symmetric spaces), the ideal of $\sigma(X)$ is generated in degree three.

In this paper we determine the generators of the ideals of the secant varieties of rank three compact Hermitian symmetric spaces in their minimal homogeneous embeddings, which we abbreviate CHSS. There is one surprise, the ideal of the secant variety of the D_7 spinor variety is not generated in degree three, which answers a question posed in [5], Section 3. Recently, L. Manivel has made significant progress towards determining the generators of the ideals of secant varieties of spinor varieties in general, see [9].

While determining the generators of the ideals of secant varieties of higher rank CHSS seems out of reach at the moment, we show that for all CHSS other than spinor varieties, there are indeed generators in degree three.

Regarding singularities of secant varieties, we show:

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Theorem 1.1. *Let $X \subset \mathbb{P}V$ be a rank three CHSS in its minimal homogeneous embedding. Then $\sigma(X)$ is normal, with rational singularities.*

Let $G/P_j \subset \mathbb{P}V_{\omega_j}$ denote the embedded rational homogeneous variety where P_j is the maximal parabolic associated to the simple root α_j, using the ordering of the roots as in [10]. Among the rank three CHSS are the Legendrian varieties, $E_7/P_7, D_6/P_6, A_5/P_3 = G(3, K^6), C_3/P_3 = G_{Lag}(3, K^6)$, and $Seg(\mathbb{P}^1 \times Q)$ (where the last is the Segre product of a $\mathbb{P}^1$ with a quadric hypersurface), which have the property that their secant varieties are the ambient $\mathbb{P}V$, so there is no need to study their ideals and singularities.

Let $S_\pi W$ denote the irreducible $GL(W)$-module associated to the partition π. The generators of the ideals in the remaining cases are as follows:

Theorem 1.2. *Let $\dim W \geq 7$. The ideal of the secant variety of the Grassmannian of 3-planes in its Plucker embedding, $\sigma(G(3, W^*)) \subset \mathbb{P}\Lambda^3 W^*$ is generated in degree three by the $SL(W)$-module of highest weight $2\omega_1 + \omega_7$ occurring in $S^3(\Lambda^3 W)$, i.e., $S_{3,1^6}W = S_{3111111}W \subset S^3(\Lambda^3 W)$.*

Warning: In an attempt to minimize the presence of $*$'s, when we study the ideal of a variety, we often write $X \subset \mathbb{P}V^*$ (as in Theorem 1.2), but when dealing with the variety directly we write $X \subset \mathbb{P}V$ (as in Theorem 1.1).

Theorem 1.2 is proved in Section 5.

Theorem 1.3. *Let $X = \mathbb{S}_7 \subset \mathbb{P}^{63}$ be the D_7-spinor variety. Then the ideal of $\sigma(X)$ is generated by the irreducible D_7-module with highest weight ω_4 in degree four.*

Theorem 1.3 is discussed in Section 6.

For a vector space A, let $K_S(A) \subset A^{\otimes 3}$ denote the kernel of the symmetrization map $S^2 A \otimes A \to S^3 A$, it is a $GL(A)$-module isomorphic to $S_{21}A$. We let $\pi_S : (A \otimes B)^{\otimes 3} \to (A \otimes B)^{\otimes 3}$ denote the symmetrization map whose image is $S^3(A \otimes B)$.

Theorem 1.4. *Let $Y \subset \mathbb{P}W^*$ be a rank two CHSS in its minimal homogeneous embedding other than a quadric hypersurface, so that $X := Seg(\mathbb{P}A^* \times Y) \subset \mathbb{P}(A^* \otimes W^*)$ is a reducible rank three CHSS in its minimal homogeneous embedding (other than $Seg(\mathbb{P}^1 \times Q)$). Let $I_3(\sigma(Y)) \subset S^3 W$ and $S_1(Y) \subset K_S(W)$ respectively denote the modules generating the ideal of the secant variety of Y and the space of linear syzygies for the ideal of Y. Then the ideal of $\sigma(X)$ is generated in degree three by*

$$\Lambda^3 A \otimes \Lambda^3 W, \quad S^3 A \otimes I_3(\sigma(Y)), \quad \text{and} \quad \pi_S(K_S(A) \otimes S_1(Y)).$$

Explicitly, the modules are

Y	$I_3(\sigma(Y))$	$S_1(Y)$
$G(2,6)$	K	$V^{A_n}_{\omega_1+\omega_5}$
$G(2,n+1),\ n \geq 6$	$V^{A_n}_{\omega_6}$	$V^{A_n}_{\omega_1+\omega_5}$
$Seg(\mathbb{P}B^* \times \mathbb{P}C^*)$	$\Lambda^2 B \otimes \Lambda^2 C$	$(S_{21}B \otimes \Lambda^2 C) \oplus (\Lambda^3 B \otimes S_{21}C)$
$\mathbb{S}_5$	0	$V^{\mathfrak{d}_5}_{\omega_4}$
$\mathbb{OP}^2$	K	$V^{\mathfrak{e}_6}_{\omega_2}$

The degree three statement in Theorem 1.4 is a consequence of the more general result:

Proposition 1.5. *Let $Y \subset \mathbb{P}A^*$ and $Z \subset \mathbb{P}B^*$ be varieties and let $Seg(Y \times Z) \subset \mathbb{P}(A^* \otimes B^*)$ denote their Segre product. Let $I_3(\sigma(Y)) \subset S^3 W$ and $S_1(Y) \subset K_S(W)$ respectively denote the modules generating the ideal of the secant variety of Y and the space of linear syzygies for the ideal of Y and similarly for Z. Then*

$$I_3(\sigma(Seg(Y \times Z))) = \Lambda^3 A \otimes \Lambda^3 B \oplus \pi_S(S_1(Y) \otimes K_S(B) \oplus K_S(A) \otimes S_1(Z)) \oplus I_3(\sigma(Y)) \otimes S^3 B \oplus S^3 A \otimes I_3(\sigma(Z)).$$

Theorem 1.4 and Proposition 1.5 are proven in Section 7.

For higher rank irreducible CHSS we have the following result, which is proved in Section 3:

Proposition 1.6. *The ideal of $\sigma(G(k, K^m)) \subset \mathbb{P}V^*_{\omega_k} = \mathbb{P}(\Lambda^k K^m)$ with $m \geq 2k$ contains the irreducible A_{m-1}-modules $V_{2\omega_{k-2}+\omega_{k+4}} \oplus V_{\omega_{k-4}+2\omega_{k+2}} \in S^3 V_{\omega_k}$ among its generators.*

In [9] it is shown that $D_7/P_7, D_8/P_8$ in their minimal homogeneous embeddings are the only spinor varieties whose secant varieties have empty ideal in degree three, thus these are the only CHSS not having cubics in the ideal of its secant variety. (Segre products and Veronese re-embeddings of any varieties contain cubics in the ideals of their secant varieties as there are cubics in the ideals of the Segre products and Veronese embeddings of projective spaces.)

In Section 8 we discuss the coordinate ring of $\sigma(X)$ for arbitrary rational homogeneous varieties. A key point is that when $X \subset \mathbb{P}V$ is homogeneous, $\sigma(X)$ is the closure of the orbit of the sum of a highest weight vector and a lowest weight vector.

We obtain our results using the methods of [11], as described in Theorem 2.1, along with some new results about induced representations. In brief, in each case we obtain a desingularization of $\sigma(X)$, by exploiting that fact that each X has a Legendrian "smaller cousin", and apply Weyman's method to this desingularization.

The case of $Seg(\mathbb{P}A^* \times Q) \subset \mathbb{P}(A^* \otimes W^*)$ is immediate as $\sigma(Seg(\mathbb{P}A^* \times Q)) = \sigma(Seg(\mathbb{P}A^* \times \mathbb{P}W^*))$.

Notation. For a variety $Z \subset \mathbb{P}V$, we let $\hat{Z} \subset V$ denote the corresponding cone. We adopt the following conventions: K is an algebraically closed field of characteristic zero, G is a complex semi-simple algebraic group, P a parabolic subgroup, $X = G/P \subset \mathbb{P}V$ denotes a rational homogeneous variety in its minimal homogeneous embedding. We use German letters to denote Lie algebras associated to algebraic groups. We use the ordering of roots as in [10]. The fundamental weights and the simple roots of $\mathfrak{g}$ are respectively denoted ω_i and α_i. P_k denotes the maximal parabolic of G obtained by deleting the root spaces corresponding to negative roots having a nonzero coefficient on the simple root α_k. More generally, for $J = (j_1, \dots, j_s)$, P_J denotes the parabolic obtained by deleting the negative root spaces having a nonzero coefficient on any of the simple roots $\alpha_{j_1}, \dots, \alpha_{j_s}$. $\Lambda_{\mathfrak{g}}$, Λ_G respectively denote the weight lattices of $\mathfrak{g}$, G, and $\Lambda_{\mathfrak{g}}^+ \subset \Lambda_{\mathfrak{g}}$, $\Lambda_G^+ \subset \Lambda_G$ the dominant weights. We let $L \subset P$ be a (reductive) Levi factor and $\mathfrak{f} = [\mathfrak{l}, \mathfrak{l}]$ a semi-simple Levi factor. We write $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{n}$ is nilpotent. $V_\lambda^{\mathfrak{g}}$ denotes the irreducible $\mathfrak{g}$-module with highest weight λ and we often supress $\mathfrak{g}$ in the notation. Unless otherwise noted, G will be simply connected so $\Lambda_G^+ = \Lambda_{\mathfrak{g}}^+$ and we will freely switch from rational G-modules to $\mathfrak{g}$-modules.

When dealing with $\mathfrak{a}_n$-modules we sometimes use partitions to index highest weights, with the dictionary $\pi = (p_1, \dots, p_{n+1})$ corresponds to the weight $(p_1 - p_2)\omega_1 + (p_2 - p_3)\omega_2 + \dots + (p_n - p_{n+1})\omega_n$. We write $S_\pi K^n$ for the associated module. Sometimes we abbreviate a partition $(i_1, \dots, i_1, i_2, \dots, i_2, \dots, i_k, \dots, i_k) = ((i_1)^{a_1}, (i_2)^{a_2}, \dots, (i_k)^{a_k})$ where i_s occurs a_s times.

2. Method of proof

2.1. The basic theorem of [11]

Theorem 2.1 ([11]). *Let $Y \subset \mathbb{P}V$ be a variety and suppose there is a projective variety $\mathcal{B}$ and a vector bundle $q : E \to \mathcal{B}$ that is a subbundle of a trivial bundle $\underline{V} \to \mathcal{B}$ with $\underline{V}_z \simeq V$ for $z \in \mathcal{B}$ such that $\mathbb{P}E \to Y$ is a desingularization of Y. Write $\eta = E^*$ and $\xi = (\underline{V}/E)^*$.*

If the sheaf cohomology groups $H^i(\mathcal{B}, S^d\eta)$ are all zero for $i > 0$ and $d > 0$ and if the linear maps $H^0(\mathcal{B}, S^d\eta) \otimes V^* \to H^0(\mathcal{B}, S^{d+1}\eta)$ are surjective for all $d \geq 0$, then

(1) *$\hat{Y}$ is normal, with rational singularities.*
(2) *The coordinate ring $K[\hat{Y}]$ satisfies $K[\hat{Y}]_d \simeq H^0(\mathcal{B}, S^d\eta)$.*
(3) *The vector space of minimal generators of the ideal of Y in degree d is isomorphic to $H^{d-1}(\mathcal{B}, \Lambda^d\xi)$ which is also the homology of the middle term of the complex*

$$\cdots \to \Lambda^2 V \otimes H^0(\mathcal{B}, S^{d-2}\eta) \longrightarrow V \otimes H^0(\mathcal{B}, S^{d-1}\eta) \longrightarrow H^0(\mathcal{B}, S^d\eta) \to 0. \tag{1}$$

(4) *More generally, $\oplus_j H^j(\mathcal{B}, \Lambda^{i+j}\xi)$ tensored with $Sym(V)$ with degree shifted by $-(i+j)$ is isomorphic to the i-th term in the minimal free resolution of Y.*
(5) *If moreover Y is a G-variety and the desingularization is G-equivariant, then the identifications above are as G-modules.*

2.2. The basic theorem applies in our case

In this paper the desingularizations will all be by a homogeneous bundle $\mathbb{P}E$ such that the corresponding bundle η is irreducible. In this case we have:

Proposition 2.2. *Notations as above, if Y is a G variety, $\mathcal{B} = G/P$ and η is induced from an irreducible P-module, then the sheaf cohomology groups $H^i(\mathcal{B}, S^d\eta)$ are all zero for $i > 0$ and the linear maps $H^0(\mathcal{B}, S^d\eta) \otimes V^* \to H^0(\mathcal{B}, S^{d+1}\eta)$ are surjective for all $d \geq 0$. In particular all the conclusions of 2.1 apply.*

Proof. An irreducible homogeneous bundle can have nonzero cohomology in at most one degree, but a quotient bundle of a trivial bundle has nonzero sections, thus $H^0(B, \eta)$ is a nonzero irreducible module and all other $H^j(B, \eta)$ are zero. Let $\mathfrak{f} \subset \mathfrak{p} \subset \mathfrak{g}$ be a semi-simple Levi factor, so the weight lattice of $\mathfrak{f}$ is a sublattice of the weight lattice of $\mathfrak{g}$, let $\mathfrak{t}^c$ denote the complement of $\mathfrak{t}_{\mathfrak{f}}$ (the torus of $\mathfrak{f}$) in $\mathfrak{t}_{\mathfrak{g}}$ and let $\mathfrak{l} = \mathfrak{f} + \mathfrak{t}^c$ denote the Levi factor of $\mathfrak{p}$. η is induced from an irreducible $\mathfrak{g}_0$-module U which is a weight space for $\mathfrak{t}^c$ having non-negative weight, say $(w_1, \dots, w_p)$. The bundle $\eta^{\otimes d}$, corresponds to a module which is $U^{\otimes d}$ as an $\mathfrak{f}$-module and is a weight space with weight $(dw_1, \dots, dw_p)$ for the action of $\mathfrak{t}^c$. Thus $S^d\eta$ is completely reducible and each component of $S^d\eta$ is very ample and in particular acyclic.

To prove the second assertion, consider the maps $V^* \otimes H^0(\mathcal{B}, S^{r-1}\eta) \to H^0(\mathcal{B}, S^r\eta)$. Note that $H^0(\mathcal{B}, S^j\eta) \subset S^jV^*$. The proof of Proposition 2.2 will be completed by Lemma 2.3 below applied to U and each irreducible component of $H^0(\mathcal{B}, S^r\eta)$. $\square$

Let $M^{\mathfrak{g}}_{\mathfrak{g}_0}$ denote the sub-category of the category of $\mathfrak{g}_0$-modules generated under direct sum by the irreducible $\mathfrak{g}_0$-modules with highest weight in $\Lambda^+_{\mathfrak{g}} \subset \Lambda^+_{\mathfrak{g}_0}$ and note that it is closed under tensor product. Let $M_{\mathfrak{g}}$ denote the category of $\mathfrak{g}$-modules. Define an additive functor $\mathcal{F} : M^{\mathfrak{g}}_{\mathfrak{g}_0} \to M_{\mathfrak{g}}$ which takes an irreducible $\mathfrak{g}_0$-module with highest weight λ to the corresponding irreducible $\mathfrak{g}$-module with highest weight λ.

Lemma 2.3. *Let $\mathfrak{l} \subset \mathfrak{g}$ and $\mathcal{F}$ be as above. Let U, W be irreducible $\mathfrak{l}$-modules Then*

$$\mathcal{F}(U \otimes W) \subseteq \mathcal{F}(U) \otimes \mathcal{F}(W).$$

Proof. Let $N \subset P$ denote the unipotent radical of P. Any L-module W may be considered as a P-module where N acts trivially. Saying $V = \mathcal{F}(W)$ means that V is the G-module parabolically induced from W and W is the set of N-invariants of V. The N-invariants of $\mathcal{F}(U) \otimes \mathcal{F}(W)$ contain $U \otimes W$. □

3. Proof of Proposition 1.6

Proof. We recall some facts from [5], Section 3. For any variety $X \subset \mathbb{P}W^*$ whose ideal is generated in degree two, $I_3(\sigma(X)) = S^3W \cap (I_2(X) \otimes W)$ with the intersection being taken inside $S^2W \otimes W$.

Let G be semi-simple, let $W = V_\lambda$ and let $X = G/P \subset \mathbb{P}W^*$ be the orbit of a highest weight line. Assume that an irreducible G-module V_μ appears in S^3V_λ and does not appear in $V_{2\lambda} \otimes V_\lambda$, where $V_{2\lambda}$ is the unique submodule of S^2V_λ isomorphic to $V_{2\lambda}$. Then V_μ must be in $I_3(\sigma(X))$ as $S^2V_\lambda = I_2(X) \oplus V_{2\lambda}$.

Let $G = SL(n, K)$ and let $W = \Lambda^k K^m = V_{\omega_k}$. It follows from the Pieri formulas that the modules $V_{2\omega_{k-2}+\omega_{k+4}}$ and $V_{\omega_{k-4}+2\omega_{k+2}}$ do not occur in $V_{2\omega_k} \otimes V_{\omega_k}$. To see that they occur in S^3W, identify V_{ω_k} with $\Lambda^k(K^n)$ where K^n has a basis $\{e_1, \ldots, e_n\}$. First observe that $\Lambda^6(K^6) \subset S^3(\Lambda^2(K^6))$, in fact if $(f_1, \ldots, f_6)$ is a basis of K^6, then the inclusion takes it to the Pfaffian

$$f_1 \wedge \cdots \wedge f_6 \mapsto \sum_\sigma \mathrm{sgn}(\sigma)(f_{\sigma(1)} \wedge f_{\sigma(2)}) \circ (f_{\sigma(3)} \wedge f_{\sigma(4)}) \circ (f_{\sigma(5)} \wedge f_{\sigma(6)}) \tag{2}$$

where we sum over all permutations $\sigma \in \mathfrak{S}_6$ satisfying

$$\sigma(1) < \sigma(2), \qquad \sigma(3) < \sigma(4), \qquad \sigma(5) < \sigma(6), \qquad \sigma(1) < \sigma(3) < \sigma(5).$$

Now consider $K^6 \subset K^n$ as the span of $\{e_{k-1}, \ldots, e_{k+4}\}$, we produce a highest weight vector of $V_{2\omega_{k-2}+\omega_{k+4}}$ by wedging each term $f_i \wedge f_j = e_{i+k-2} \wedge e_{i+k-2}$ in the summation (2) with $e_1 \wedge \cdots \wedge e_{k-2}$. We leave it to the reader to check that the resulting vector has the desired properties. The module $V_{\omega_{k-4}+2\omega_{k+2}}$ occurs in S^3W as well by symmetry (or one can define an analogous map). □

4. Desingularizations for secant varieties of rank 3 CHSS

In our situation the desingularizations are based on the observation that in each case X is swept out by the union of Legendrian varieties X_{small} and $\sigma(X)$ is the union of the $\sigma(X_{small})$'s which are linear spaces.

Here is a table of X, X_{small}, $\mathcal{B}$ and the desingularizing bundle E over $\mathcal{B}$:

X	X_{small}	$\mathcal{B}$ = base	E = bundle
$G(3, U)$	$G(3, 6)$	$G(6, U)$	$\Lambda^3\mathcal{R}_U$
$\mathbb{S}_7$	$\mathbb{S}_6$	Q^{12}	$Spin(p^\perp/\hat{p})$
$\mathbb{P}A \times Y$	$\mathbb{P}^1 \times Q$	$G(2, A) \times Y_Q$	$\mathcal{R}_A \otimes S$

Here, for the last case, if $Y = G/P \subset \mathbb{P}W$, $Y_Q = G/P'$ is the variety obtained via Tits' *shadows* that parametrizes a space of quadric sections of Y (see [12], Section 2). One takes the marked Dynkin diagram for $Y \subset \mathbb{P}W$ and looks for the largest subdiagram whose resulting marked diagram is a quadric hypersurface. The marked diagram of Y_Q is obtained by marking all nodes bounding the nodes of the subdiagram corresponding to the quadric hypersurface. The irreducible homogeneous bundle $S \to Y_Q$ is obtained from the irreducible $\mathfrak{p}$-module of highest weight equal to the highest weight of W (where we have included the weight lattice of the Levi factor of $\mathfrak{p}$ into the weight lattice of $\mathfrak{g}$). As such, it is an irreducible sub-bundle of the trivial bundle with fiber W and satisfies the hypotheses of Proposition 2.2.

Example 1. Here are the relevant spaces for $X = \mathbb{P}A \times E_6/P_1$ (see Fig. 1):

Here is a table for the cases $X = \mathbb{P}A \times Y$:

Y	Y_Q	$\dim Q$
$\mathbb{P}B \times \mathbb{P}C$	$G(2, B) \times G(2, C)$	2
$G(2, W)$	$G(4, W)$	4
$\mathbb{S}_5$	Q^8	6
E_6/P_1	E_6/P_6	8

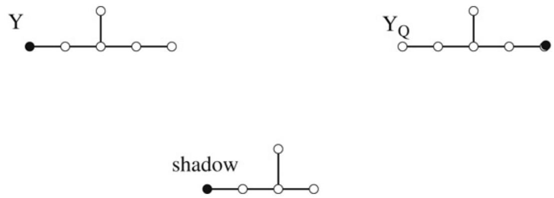

Fig. 1. Shadow of $Y_Q = E_6/P_6$ on $Y = E_6/P_1$ is a $Q^8 = D_5/P_1$.

When $\dim Q = 2^k$, we have a uniform model, $W = \mathcal{J}_n(\mathbb{B})$, where $\mathcal{J}_n(\mathbb{B})$ is the Jordan algebra of $n \times n$ $\mathbb{B}$-Hermitian symmetric matrices. In this model $\sigma(Y)$ is the set of rank at most 2 elements with ideal generated by the 3×3 minors. (In the case $\mathbb{B} = \mathbb{O}$ the octonions, we have $n = 3$ and care must be taken when defining the determinant.) In these cases the fiber of S is isomorphic to $\mathcal{J}_2(\mathbb{B})$.

When $Y = \mathbb{S}_5$ we have $W \simeq \mathcal{S}_5 \simeq \Lambda^{even}K^5$ and the fiber of S is isomorphic to $\Lambda^{even}K^4$.

Lemma 4.1. *Let $\tilde{\sigma}$ denote the total space of E. The image of $q : \tilde{\sigma} \to V$ is $\hat{\sigma}(X)$ and $q : \tilde{\sigma} \to \hat{\sigma}(X)$ is a resolution of singularities of $\hat{\sigma}(X)$.*

Proof. In each case, the fiber $E_x \subset V_x = V$ over $x \in \mathcal{B}$ is $\hat{\sigma}(X_{small,x}) = E_x \subset V$ and $X_{small,x} \subset X$ by construction, so the image of q is contained in $\hat{\sigma}(X)$. On the other hand, they both have the same dimension and $\hat{\sigma}(X)$ is reduced and irreducible. Thus we need only show that the map is generically one to one. It is clear that q restricted to each fiber is generically one to one, so it is sufficient to show that there is a unique fiber over a general point. For $\sigma(G(3, W))$, a general point determines a unique 6 plane in W. For $\sigma(Seg(\mathbb{P}A \times Y))$, a general point clearly determines a unique 2-plane A' in A. When $Y \neq \mathbb{S}_5$, two elements of Y in general position lie in a unique $\mathcal{J}_2(\mathbb{B})$ and the unique fiber is $A' \otimes \mathcal{J}_2(\mathbb{B})$. (The intersection $Y \cap \mathbb{P}\mathcal{J}_2(\mathbb{B})$ is the set of rank one elements in $\mathcal{J}_2(\mathbb{B})$, i.e., a quadric of dimension $\dim \mathbb{B}$, see, e.g., [13], chapter VI or [14].) For the case $Y = \mathbb{S}_5$, fix an isotropic line $L \subset K^{10}$, the set $\{F \in \mathbb{S}_5 \mid L \subset F\}$ is the shadow of L in $\mathbb{S}_5$ and the span of this shadow is the image of the fiber over $[L] \in Q^8$ in K^{16}. Two general points $F, F' \in \mathbb{S}_5$, considered as $\mathbb{P}^4$'s in Q^8 will intersect in a point, i.e., an isotropic line in K^{10}, which determines the unique fiber above a general point in their linear span.

Finally for the case $X = \mathbb{S}_7$, the same argument for $\mathbb{S}_5$ applies, as it is still true two general $F, F' \in \mathbb{S}_7$ will intersect in an isotropic line. □

Lemma 4.1 combined with Theorem 2.1 and Proposition 2.2 prove Theorem 1.1.

We now proceed with a case by case study.

5. Case of $X = G(3, W^*)$

Proof of Theorem 1.2. Let

$$R_p(\Lambda^k W^*) = \{T \in \Lambda^k W^* \mid \exists K^p \subset W^* \text{ such that } T \in \Lambda^k K^p\}$$

$R_p(\Lambda^k W^*)$ is called a *rank variety* (or *subspace variety*). Such varieties are discussed in detail in ([11], Section 7). Their ideals are easy to describe, namely $I_d(R_p(\Lambda^k W^*))$ consists of all modules corresponding to copies of $S_\pi W$ occurring in $S^d(\Lambda^k W)$ where $\ell(\pi) > p$. However it is in general difficult to determine generators of the ideal. $R_p(\Lambda^k W^*)$ is desingularized by $\Lambda^k \mathcal{S} \to G(p, W^*)$, and the corresponding bundle $\xi = (\Lambda^k \underline{W^*}/\Lambda^k \mathcal{S})^*$ in general is not irreducible. When $p = \dim W - 1$ however ξ is irreducible, which will be the key to our proof.

Proposition 5.1. $\hat{\sigma}(G(3, W^*)) = R_6(\Lambda^3 W^*)$.

Proof. A general point of $\hat{\sigma}(G(3, W^*))$ is of the form $v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6$, so $\hat{\sigma}(G(3, W^*)) \subseteq R_6(\Lambda^3 W^*)$ because the latter is compact and the former is connected. But both varieties are of the same dimension $6(\dim W) - 17$ and are reduced and irreducible so they must be equal. □

Write $V = \Lambda^3 W$. We have

$$S^3 V = S_{33}W \oplus S_{322^1 1^2}W \oplus S_{2^3 1^3}W \oplus S_{31^6}W.$$

The last module corresponds to a partition of length seven and thus by the remark above, it is among the generators of $I(R_6(\Lambda^3 W^*))$ (because the ideal in degree two of any secant variety is empty), and the rest are not as their partitions have length at most six.

To show there are no generators in degree greater than three, we need to prove exactness in the middle step of (1) which in this case is:

$$(S_{1^6}W \oplus S_{2211}W) \otimes S^{r-2}(S_{111}W)\,|_{|\pi|\leq 6} \to S_{111}W \otimes S^{r-1}(S_{111}W)\,|_{|\pi|\leq 6} \to S^r(S_{111}W)\,|_{|\pi|\leq 6}$$

for $r > 3$.

The largest partition that can show up in the middle has length nine, so once we have solved the problem for $G(3, K^9)$ we are done.

Thus one could proceed by calculating $H^d(G(6, K^9), \Lambda^{d+1}\xi)$ with the aid of a computer to conclude (although the passage from the cohomology of $\Lambda^{d+1}gr(\xi)$ to $\Lambda^{d+1}\xi$ might require some effort). We will proceed differently, resolving the cases of $\dim W = 7, 8, 9$ iteratively using rank varieties with $p = \dim W - 1$.

For $\dim W = 7$, the method in [11], Section 7.3 shows that the ideal of $R_6(\Lambda^3 W)$ is generated by $S_{31^6}W$ and we are done. For the next two cases we proceed indirectly, calculating the ideal of $R_7(\Lambda^3 K^8)$ (resp. $R_8(\Lambda^3 K^9)$), and show these are in the ideal generated by $S_{31^6}W$ to complete the proof.

Proposition 5.2. *The ideal of the rank variety $R_6(\Lambda^3 K^{7*})$ is generated in degree three by $S_{3,1^6}K^7$ included in $S^3(\Lambda^3 K^7)$ as described in the recipe in the proof.*

The ideal of the rank variety $R_7(\Lambda^3 K^{8*})$ is generated in degree four by $S_{3,2^2,1^5}K^8$ included in $S^4(\Lambda^3 K^8)$ as described in the recipe in the proof.

The ideal of the rank variety $R_8(\Lambda^3 K^{9*})$ is generated in degrees four and five by $S_{4,1^8}K^9$ and $S_{3^2,2^2,1^5}K^9$ respectively included in $S^4(\Lambda^3 K^9)$ and $S^5(\Lambda^3 K^9)$ as described in the recipe in the proof.

Proof. Thanks to the irreducibility of ξ and its exterior powers, determination of modules generating the ideal is a straightforward application of the methods of [11] and is left to the reader. It remains to show that the above modules are all in the ideal generated by $S_{31^6}W$. To do this we give explicit descriptions of the modules as spaces of polynomials.

We will encode the representations occurring in the d-th symmetric powers of $\Lambda^3 W$ by Young tableaux D with $3d$ boxes, filled with the numbers $1, \dots, d$ with each number occurring three times. These tableaux are also assumed to be weakly increasing in rows and strictly increasing in columns. We associate to such tableau D the map $\rho(D)$

$$\rho(D) : \Lambda^{d'_1} W \otimes \cdots \otimes \Lambda^{d'_r} W \to S_d(\Lambda^3 W)$$

where $\pi' = (d'_1, \dots, d'_r)$ is the conjugate partition to $\pi = \pi(D)$, the partition associated to the Young diagram of D.

The map $\rho(D)$ is defined as the composition of the following maps:

(a) assuming there are $e_{i,s}$ boxes filled with s in the i-th row, apply the embedding

$$\Lambda^{d'_i} W \to \Lambda^{e_{i,1}} W \otimes \cdots \otimes \Lambda^{e_{i,d}} W$$

for each row of D,

(b) Noting that for each s, $e_{1,s} + \cdots + e_{r,s} = 3$, wedge the factors coming from different rows corresponding to the same number s in D, i.e., after rearranging the factors define the projection to $(\Lambda^3 W)^{\otimes d}$ by sending, for each s,

$$\Lambda^{e_{1,s}} W \otimes \cdots \otimes \Lambda^{e_{r,s}} W \to \Lambda^3 W$$

and tensoring the results.

(c) Project $(\Lambda^3 W)^{\otimes d} \to S^d(\Lambda^3 W)$ by symmetrizing.

We call *D the numbering scheme* associated to the map $\rho(D)$. It will give rise to an appropriate copy of $S_{\pi(D)}W \subset S^d(\Lambda^3 W)$. The four Schur functors mentioned in the statement of the lemma correspond to four numbering schemes.

$$D_1 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}.$$

$$D_2 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3&4\\ \hline \multicolumn{1}{|c|}{2}&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{4} \\ \cline{1-3} \multicolumn{1}{|c|}{3} \\ \cline{1-1}\end{array}.$$

$$D_3 = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3&4&4\\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1}\end{array}.$$

$$D_4 = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3&4&5\\ \hline \multicolumn{1}{|c|}{2}&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{5} \\ \cline{1-4} \multicolumn{1}{|c|}{3}&\multicolumn{1}{c|}{5} \\ \cline{1-2}\end{array}.$$

It is clear that the images of the corresponding maps are in the ideals of corresponding rank varieties because of the length of the first row of each numbering scheme.

Decomposing the domain and range of $\rho(D_i)$ into irreducible representations, in all four cases $S_{\pi(D_i)}W$ is the only Schur functor occurring in both the domain and range of $\rho(D_i)$, and it occurs there with multiplicity one. Thus it only remains to see that the maps $\rho(D_j)$ are nonzero, which is the purpose of the following lemma. $\square$

Lemma 5.3. *The numbering schemes D_1, D_2, D_3, D_4 all yield nonzero modules.*

Proof. We prove that the image of the map

$$\rho(D_2) : \Lambda^8 W \otimes \Lambda^3 W \otimes W \to S^4(\Lambda^3 W)$$

corresponding to the numbering scheme

$$D_2 = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

is nonzero in $S_4(\Lambda^3 W)$. The other cases are similar.

Consider the contribution to the monomial $(e_1 \wedge e_2 \wedge e_3)(e_1 \wedge e_2 \wedge e_3)(e_1 \wedge e_4 \wedge e_5)(e_6 \wedge e_7 \wedge e_8)$ in the image of highest weight vector $\rho(D_2)(e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7 \wedge e_8) \otimes (e_1 \wedge e_2 \wedge e_3) \otimes (e_1)$. All occurrences of this monomial can be divided to 24 classes (corresponding to permutations of $\{1, 2, 3, 4\}$) according to the order in which the factors appear in $(\Lambda^3 W)^{\otimes d}$ after applying parts (a) and (b) of the definition of $\rho(D_2)$. In fact only two classes out of 24 are non-empty. The factor $e_6 \wedge e_7 \wedge e_8$ has to come from the first factor, and one of the factors $e_1 \wedge e_2 \wedge e_3$ has to come from the fourth factor. The factor $e_1 \wedge e_4 \wedge e_5$ can come from the third factor (and this gives contribution 3 to the coefficient) or from the second factor (and this gives contribution 1 to the coefficient). Thus the coefficient is nonzero and therefore $\rho(D_2) \neq 0$. □

To finish the proof of Theorem 1.2 we need to show that the ideal generated by the first module contains the other modules. But this is clear by the definition of maps $\rho(D)$ and by the last part of the proof of Proposition 5.2 as the other numbering schemes all contain the first. □

6. Case of $X = \mathbb{S}_7$

A desingularization of $\hat{\sigma}(\mathbb{S}_7) \subset V_{\omega_7}^{D_7}$ is given by the sub-bundle of $q : E \to Q^{12} = D_7/P_1$ whose fiber is isomorphic to $\hat{\sigma}(\mathbb{S}_6) = V_{\omega_6}^{D_6}$. Thus we can apply Theorem 2.1.

Note that $V_{\omega_7}^{D_7}$ decomposes to $V_{\omega_6}^{D_6} \oplus V_{\omega_5}^{D_6}$ as a D_6-module, and this splitting gives rise to the bundles ξ and η over Q^{12}. Thus they are both irreducible and dual to one another. In this case it is straightforward to calculate $H^j(\Lambda^{j+1}\xi)$ if one knows the decomposition of $\Lambda^{j+1} V_{\omega_6}^{D_6}$. In fact we calculated the entire minimal free resolution which is available at http://www.math.neu.edu/~weyman/mathindex.html for the interested reader. In particular the only generator of the ideal is the module V_{ω_4} as stated in the theorem.

7. Case of $\sigma(Seg(\mathbb{P}A^* \times Y))$

Proof of Proposition 1.5. All spaces discussed in this section are to be considered as linear subspaces of $(A \otimes B)^{\otimes 3} = A^{\otimes 3} \otimes B^{\otimes 3}$ and all evaluations are as multi-linear forms. In particular, the symmetrization map

$$\begin{aligned} \pi_S : (A \otimes B)^{\otimes 3} &\to (A \otimes B)^{\otimes 3} \\ a_1 \otimes a_2 \otimes a_3 \otimes b_1 \otimes b_2 \otimes b_3 &\mapsto \frac{1}{6} \sum_{\sigma \in \mathfrak{S}_3} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)} \otimes b_{\sigma(1)} \otimes b_{\sigma(2)} \otimes b_{\sigma(3)} \end{aligned}$$

realizes $S^3(A \otimes B) \subset (A \otimes B)^{\otimes 3}$ as $\pi_S((A \otimes B)^{\otimes 3})$. Similarly we regard $S^2 A \otimes A \subset A^{\otimes 3}$ as the image of the symmetrization map $x \otimes y \otimes z \mapsto \frac{1}{2}(x \otimes y \otimes z + y \otimes x \otimes z)$ and likewise for $S^2 B \otimes B \subset B^{\otimes 3}$.

As mentioned in Section 3, for any variety $X \subset \mathbb{P}V$, $I_3(\sigma(X)) = (I_2(X) \otimes V^*) \cap S^3 V^*$. The $GL(A) \times GL(B)$ decomposition of $S^3(A \otimes B) \subset (A \otimes B)^{\otimes 3}$ is:

$$S^3(A \otimes B) = \Lambda^3 A \otimes \Lambda^3 B \oplus \pi_S(K_S(A) \otimes K_S(B)) \oplus S^3 A \otimes S^3 B,$$

where $K_S(A)$ is the kernel of the map $S^2 A \otimes A \to S^3 A$, which is a $GL(A)$-module isomorphic to $S_{21} A$. In particular, if $R \in K_S(A)$, we have $R(u, v, w) = R(v, u, w)$ for all $u, v, w \in A^*$ and

$$R(u, v, u) = -\frac{1}{2} R(u, u, v) \quad \forall u, v \in A^* \tag{3}$$

because $R(u, v, w) + R(u, w, v) + R(v, w, u) + R(v, w, u) + R(w, u, v) + R(w, v, u) = 0$, and setting $w = u$ gives $2R(u, v, u) + 2R(u, u, v) + 2R(v, u, u) = 4R(u, v, u) + 2R(u, u, v) = 0$.

The factor $\Lambda^3 A \otimes \Lambda^3 B$ is in $I_3(\sigma(Seg(Y \times Z)))$ because it is $I_3(\sigma(Seg(\mathbb{P}A^* \times \mathbb{P}B^*)))$.

By polarization and symmetry, a polynomial $P \in S^3 V^* \subset V^{*\otimes 3} = \{\text{trilinear maps } V \times V \times V \to \mathbb{C}\}$ is in the ideal of $\sigma(X)$ if and only if $P(x_1, x_1, x_2) = 0$ for all $x_1, x_2 \in \hat{X}$. In what follows we consider P as a trilinear form on V.

By definition $S_1(Y) = \{T \in I_2(Y) \otimes A \mid \pi_{S,A}(T) = 0\}$, where

$$\pi_{S,A}(a_1 \otimes a_2 \otimes a_3) = \sum_{\sigma \in \mathfrak{S}_3} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)}.$$

Elements of $S_1(Y) \subset K_S(A)$ have the property that as trilinear forms they vanish on any triple of the form (v, v, w) with $[v] \in Y$ and w arbitrary.

Any element of $K_S(A) \otimes K_S(B) \subset (A \otimes B)^{\otimes 3}$ may be written as a sum $\sum R_i \otimes T_i$ with $R_i \in K_S(A)$ and $T_i \in K_S(B)$, thus any element of $\pi_S(K_S(A) \otimes K_S(B))$ is of the form $P = \pi_S(\sum R_i \otimes T_i)$ with $R_i \in K_S(A)$ and $T_i \in K_S(B)$. We compute

$$\begin{aligned}
\frac{1}{2}P(v \otimes z, v \otimes z, w \otimes y) &= \sum R_i(v, v, w)T_i(z, z, y) + \sum R_i(v, w, v)T_i(z, y, z) + \sum R_i(w, v, v)T_i(y, z, z) \\
&= \sum R_i(v, v, w)T_i(z, z, y) + 2\sum R_i(v, w, v)T_i(z, y, z) \\
&= \sum R_i(v, v, w)T_i(z, z, y) + 2\sum \left(-\frac{1}{2}R_i(v, v, w)\left(-\frac{1}{2}T_i(z, z, y)\right)\right) \\
&= \frac{3}{2}\sum R_i(v, v, w)T_i(z, z, y).
\end{aligned}$$

The first equality holds because of the six permutations in $\mathfrak{S}_3$, only three yield different elements, the second because $R_i \in S^2A \otimes A$ and $T_i \in S^2B \otimes B$, and the third by (3).

First we show that $S_1(Y) \otimes K_S(B) \subset I_3(\sigma(Seg(Y \times Z)))$. Write $R_i = \sum_\alpha R_{i,\alpha} \otimes \ell^\alpha$ with $R_{i,\alpha} \in I_2(Y)$, so

$$P(v \otimes z, v \otimes z, w \otimes y) = 3\sum R_{i,\alpha_i}(v, v)\ell^{\alpha_i}(w)T_i(z, z, y)$$

which is zero because $R_{i,\alpha} \in I_2(Y)$ for all i, α. Similarly $\pi_S(K_S(A) \otimes S_1(Z)) \subset I_3(\sigma(Seg(Y \times Z)))$.

Now say $P = \pi_S(\sum R_i \otimes T_i) \in \pi_S(K_S(A) \otimes K_S(B)) \cap I_3(\sigma(Seg(Y \times Z)))$. Without loss of generality we assume that the R_i are linearly independent modulo $S_1(Y)$ and the T_i are linearly independent modulo $S_1(Z)$. Fix $y, z \in \hat{Z}$ to obtain a linear equation

$$\sum_i R_i(v, v, w)c_{i,y,z} = 0 \tag{4}$$

where $c_{i,y,z} = T_i(y, y, z)$ and if y, z are chosen generically all the coefficients are nonzero because we are working mod $S_1(Z)$. Note that the index range for i is at most from 1 to $\min\{\dim K_S(A), \dim K_S(B)\}$. We will show each $R_i(v, v, w)$ must be zero for all $v, w \in \hat{Y}$. Once having done so, since $\hat{Y}$ spans A and the expression is linear in w, $R_i(v, v, w)$ must be zero for all $v \in \hat{Y}$ and $w \in A$, but this in turn implies that each $R_i \in S_1(Y)$.

To obtain the desired vanishing, fix v, w and consider the $R_i(v, v, w) = r_i$ as constants. We have an equation

$$\sum_i r_iT_i(y, y, z) = 0 \quad \forall y, z \in \hat{Z}$$

As remarked above, since Z is linearly non-degenerate, we may choose $\dim B$ elements $z_s \in \hat{Z}$ that give a basis of B^*. Similarly, we may choose $\binom{\dim B+1}{2} - \dim I_2(Z)$ elements $y_t \in \hat{Z}$ such that the vectors y_t^2 span $I_2(Z)^\perp \subset S^2B^*$. Thus the vectors $y_t^2 \otimes z_s$ give a basis of $I_2(Z)^\perp \otimes B^*$. Thus the pairing with elements of $K_S(B)/S_1(Z)$ is perfect, which implies that the matrix given by pairing the $y_t^2 \otimes z_s$ with the T_i has a one-sided inverse, so we have enough independent equations to force all the r_i to vanish.

The argument for the $S^3A \otimes S^3B$ factor is similar, but easier, as there is no need to symmetrize. Write $P = \sum R_i \otimes T_i$ with $R_i \in S^3A$ and $T_i \in S^3B$.

$$P(v \otimes z, v \otimes z, w \otimes y) = \sum R_i(v, v, w)T_i(z, z, y).$$

Now $R_i \in I_3(\sigma(Y))$ iff $R_i(v, v, w) = 0$ for all $v, w \in \hat{Y}$ and one concludes as above. $\square$

Proof of Theorem 1.4. We first observe that without loss of generality, we may assume $\dim A = 2$. This is because, continuing the notation of Section 4, $\sigma(Seg(\mathbb{P}A^* \times Y)) \subset \sigma(Seg(\mathbb{P}A^* \times \mathbb{P}W^*))$ and the ideal of the latter already contains all partitions of length greater than two and is generated by $\Lambda^3A \otimes \Lambda^3W$. Thus the only modules of generators of $\sigma(Seg(\mathbb{P}A^* \times Y))$ that occur when $\dim A > 2$ not already accounted for are the components of $\Lambda^3A \otimes \Lambda^3W$.

The modules $I_2(\sigma(Y))$ are all trivial modules except for $V_{\omega_6}^{A_n}$ for $n \geq 6$ (and $\Lambda^2B \otimes \Lambda^2C$ for $Y = Seg(\mathbb{P}B^* \times \mathbb{P}C^*)$).

The case of a triple Segre product was treated in [5]. We now show that there are no new generators in degrees greater than three in the remaining cases of $Y = G(2, B)$ and $Y = \mathbb{OP}^2$. We give two proofs of the $G(2, B)$ case, the first one gives more information, the second one is uniform with the case $Y = \mathbb{OP}^2$.

First proof of $G(2, B)$ case. We need to prove exactness of the middle step of

$$(S_{11}A \otimes S_{11}(\Lambda^2B)) \otimes S^{r-2}(A \otimes \Lambda^2B)\mid_{|\pi|\leq 4} \to (A \otimes \Lambda^2B) \otimes S^{r-1}(A \otimes \Lambda^2B)\mid_{|\pi|\leq 4} \to S^r(A \otimes \Lambda^2B)\mid_{|\pi|\leq 4}.$$

Here by $|_{|\pi|\leq 4}$, we mean the components of $S_{a,b}(\Lambda^2 B)$ that occur in the decomposition of $S^p(A\otimes\Lambda^2 B)=\oplus_{a+b=p}S_{a,b}A\otimes S_{a,b}(S_{1,1}B)$ that as partitions $S_\pi B$ have length at most four.

Since the partitions $S_\pi B$ occurring in the middle entry can have length at most six, it is sufficient to solve the problem for the case $n\leq 6$. The decomposition of $S_{a,b}(\Lambda^2 B)$ is not known in closed form, however at this point we could rely on a computer to compute $H^{d-1}(\mathbb{P}A^*\times G(4,6),\Lambda^d\xi)$. We instead use an induction argument that is computer free.

Let B have dimension n and consider the rank variety

$$R_{n-1}(A^*\otimes\Lambda^2 B^*):=\{T\in A^*\otimes\Lambda^2 B^*\mid \exists U,\dim U=n-1, T\in A^*\otimes\Lambda^2 U\}.$$

The secant variety $\hat{\sigma}(\mathbb{P}A^*\times G(2,B^*))$ coincides with $R_4(A^*\otimes\Lambda^2 B^*)$. We determine the ideal of R_n via that of R_{n-1} which will render the bundles ξ that we use irreducible.

Over $\mathcal{B}:=A^*\otimes G(n-1,B^*)$ consider the bundle with fiber $A^*\otimes\Lambda^2\mathcal{S}$. It provides a desingularization of $R_{n-1}(A^*\otimes\Lambda^2 B^*)$. Our corresponding bundles are $\eta=A\otimes\Lambda^2\mathcal{S}^*$ and $\xi=A\otimes(\Lambda^2 B/\Lambda^2\mathcal{S})=A\otimes\mathcal{S}^*\otimes\mathcal{Q}^*$. Note that rank $\xi=2(n-1)$.

We have the decomposition into irreducible homogeneous bundles:

$$\Lambda^d\xi=\bigoplus_{\{\pi=(a,b)\mid 2a+b=d,a+b\leq n-1\}}S_\pi A\otimes S_{\pi'}\mathcal{Q}^*\otimes S_d\mathcal{S}^*$$

where π' denotes the conjugate partition to π, so if $\pi=(a,b)$, then $\pi'=(2^a,1^b)$.

We apply the Bott algorithm (see e.g. [11], Section 4.1.5) to the weight $\mu=\omega_a+\omega_{a+b}-(2a+b)\omega_{n-1}$. The only potential places to have nonzero cohomology are at the steps $(n-1)-(a+b)$, $(n-1)-a$ and $n-1$.

To obtain nonzero $H^{(n-1)-(a+b)}(\mathcal{B},\Lambda^{n-(a+b)}\xi)$, $-(2a+b)+(n-1-(a+b))$ must be non-positive and $-(2a+b)+(n-1-(a+b))+2$ must be non-negative, implying $3a+2b=n-1$. Write $i=2a+b$, so $H^i(\mathcal{B},\Lambda^{i+1}\xi)=S_{n-i,2i-n}A\otimes S_{2^{2i-n},1^{2n-2i}}B$.

To obtain nonzero $H^{(n-1)-a}(\mathcal{B},\Lambda^{n-a}\xi)$, $-(2a+b)+(n-1-a)+2$, must be non-positive and $-(2a+b)+(n-1-a)+2$ must be non-negative, implying $3a+b=n+1$, contradicting $a+b\leq n-1$.

To obtain nonzero $H^{n-1}(\mathcal{B},\Lambda^n\xi)$, $-(2a+b)+(n-1)+1$ would have to be negative and $-(2a+b)+(n-1)+2$ would have to be non-negative, implying $2a+b=n+2$ contradicting $a+b\leq n-1$.

In summary:

Proposition 7.1. *Let B, A be vector spaces respectively of dimensions n, 2 and consider the rank variety*

$$R_{n-1}(A^*\otimes\Lambda^2 B^*):=\{T\in A^*\otimes\Lambda^2 B^*\mid \exists U,\dim U=n-1, T\in A^*\otimes\Lambda^2 U\}.$$

Then the ideal of R_{n-1} is generated in degrees $\lceil\frac{n}{2}\rceil\leq d\leq\lfloor\frac{2n}{3}\rfloor$ by the modules

$$S_{n-d,2d-n}A\otimes S_{2^{2d-n},1^{2n-2d}}B\subset S^d(A\otimes\Lambda^2 B).$$

To prove Theorem 1.4, by the discussion above we need to examine the cases $n=5,6$. When $n=5$ we only have $d=3$ and the module $S_{2,1}A\otimes S_{2,1^4}B$ generates the ideal of $R_4(A^*\otimes\Lambda^2 B^*)$ and thus of $\sigma(\mathbb{P}A^*\times G(2,5))$. When $n=6$ we have $d=3,4$ and the modules $S_3A\otimes S_{1^6}B$ in degree 3 and $S_{2,2}A\otimes S_{2^2,1^4}B$ in degree 4 generate the ideal of $R_5(A^*\otimes\Lambda^2 B^*)$, and the ideal of $\sigma(Seg(\mathbb{P}A\times G(2,6)))$ is generated by these and the representation $S_{2,1}A\otimes S_{2,1^4}B$ that already occurs for dim $A=5$. However, $S_{2,2}A\otimes S_{2^2,1^4}B$ is in the ideal generated by $S_{2,1}A\otimes S_{2,1^4}B$. $\square$

Proof of cases $Y=G(2,K^6)$ and $Y=\mathbb{OP}^2$. Write $\mathcal{B}=G/P_{i_0}$. Following the conventions of [10], $i_0=4,6$ respectively. We write the Levi factor of $\mathfrak{p}$ as $\mathfrak{g}_0=\mathfrak{f}+\langle Z_{i_0}\rangle$, where $\mathfrak{f}$ is semi-simple (respectively $\mathfrak{a}_3+\mathfrak{a}_1$ and $\mathfrak{d}_5$) and $\langle Z_{i_0}\rangle$ is the center of $\mathfrak{g}_0$.

We will obtain the result by computing $H^i(\mathcal{B},\Lambda^{i+1}\xi)$ via $H^i(\mathcal{B},\Lambda^{i+1}gr(\xi))$ and applying a result of Ottaviani and Rubei. Here

$$\begin{aligned}gr(\xi)&=A\otimes(\mathcal{S}^*\otimes\mathcal{Q}^*\oplus\Lambda^2 Q^*)=A\otimes(E_{\omega_1-\omega_4+\omega_5}\oplus E_{-\omega_4})\quad\text{for } Y=G(2,6)\\&=A\otimes(E_{\omega_2-\omega_6}\oplus E_{-\omega_6})\quad\text{for } Y=\mathbb{OP}^2\end{aligned}$$

where E_λ denotes the irreducible bundle corresponding to the $\mathfrak{g}_0$-module of highest weight λ.

We first compute the decomposition of the exterior powers of the $\mathfrak{g}_0$-module giving rise to $gr(\xi)$ as an $\mathfrak{f}$-module and then compute the action of Z_{i_0} to determine the coefficient on ω_{i_0} for each irreducible $\mathfrak{f}$-module appearing.

The $\mathfrak{f}$-module decomposition is straightforward with the aid of LiE [15], keeping in mind that $\dim A=2$:

$$\Lambda^k(A\otimes(U\oplus K))=\oplus_{a+b=k}S_{a,b}A\otimes S_{2^a,1^b}(U\oplus K)\tag{5}$$

$$=\oplus_{a+b=k}S_{a,b}A\otimes(S_{2^a,1^b}U\oplus S_{2^a,1^{b-1}}U\oplus S_{2^{a-1},1^{b+1}}U\oplus S_{2^{a-1},1^b}U).\tag{6}$$

One then uses LiE to decompose these $GL(U)$-modules as $\mathfrak{f}$-modules. Next to determine the weight on the marked node (i.e., the coefficient of ω_{i_0}), one uses the grading element $Z_{i_0}\in\mathfrak{t}$ which has the property that $Z_{i_0}(\alpha_j)=\delta_{i_0,j}$. Thus if $\lambda=\sum_{i\neq i_0}\lambda^i\omega_i$ is an irreducible $\mathfrak{f}$-module appearing in $W_\mu^{\otimes k}$, where the ω_i are fundamental weights of $\mathfrak{g}$, to find the coefficient of ω_{i_0} of the $\mathfrak{g}_0$-module, one calculates

$$\sum\lambda^j(c^{-1})_{i_0,j}=kZ_{i_0}(\mu)$$

where (c^{-1}) denotes the inverse of the Cartan matrix. In both our cases $Z_{i_0}(\mu)=-\frac{1}{3}$.

Now one calculates $H^j(\mathcal{B}, \Lambda^p gr(\xi))$. In practice we first calculated $H^{p-1}(\mathcal{B}, \Lambda^p gr(\xi))$, and only if this was nonzero did we calculate the other $H^j(\mathcal{B}, \Lambda^p gr(\xi))$.

We got no relevant cohomology except in degrees one and two. In both cases the only modules that appeared in degree two were the cubic generators of the ideal plus $S_{21}A \otimes \mathbb{C}$, which is cancelled by its appearance in H^1, and it is the unique module appearing in H^1. By Proposition 6.7 in [16] cancellation occurs in the spectral sequence for the cohomology of $\Lambda^3\xi$. The program we used for this calculation is publicly available at www.math.tamu.edu/~robles. □

Proof of case $Y = \mathbb{S}_5$. We need to calculate $H^i(\Lambda^{i-1}\xi)$ with $\xi = \mathbb{C}^2 \otimes E$ where E is the vector bundle determined by the P-module with highest weight $\lambda = [-1, 0, 0, 1, 0]$. This calculation is similar to the above, but significantly easier because ξ is irreducible. □

The proof of Theorem 1.4 is now complete. □

8. The coordinate ring of $\sigma(X)$

The following proposition is due to F. Zak ([13], p. 51):

Proposition 8.1. *Let $X = G/P \subset \mathbb{P}V$ be a homogeneously embedded homogeneous variety. Let λ denote the highest weight and μ denote the lowest weight of V, and let v_λ, v_μ be corresponding weight vectors. Then $\sigma(X) = \overline{G.[v_\lambda + v_\mu]}$, where the closure is the Zariski closure.*

Proof.

$$\mathfrak{g}.(v_\lambda + v_\mu) = (\mathfrak{g}_+ + \mathfrak{g}_0 + \mathfrak{g}_-).(v_\lambda + v_\mu) = \mathfrak{g}_+.v_\lambda + \mathfrak{g}_-.v_\mu = \hat{T}_{[v_\lambda + v_\mu]}\sigma(G/P)$$

where the last equality is Terracini's lemma. This proves the result in the case $\mathfrak{g}_+.v_\lambda \cap \mathfrak{g}_-.v_\mu = 0$, i.e. the secant variety is non-degenerate, and the result is easy to verify in the degenerate cases as well. (The degenerate cases are all rank at most 2 CHSS and the adjoint varieties $G/P_{\tilde{\alpha}} \subset \mathbb{P}\mathfrak{g}$ where $\tilde{\alpha}$ is the largest root.) □

We work with the affine variety $\hat{\sigma}(X) \subset V$. We recall the following standard fact:

Proposition 8.2. *Let G be an algebraic group and H a closed subgroup. Then we have the following equality of G-modules:*

$$K[G/H] = \oplus_{\lambda \in \Lambda_G^+} V_\lambda \otimes V_\lambda^{*H}.$$

For a proof see, e.g., [17], Theorem 3, Chapter II, Section 3.

Recall that if $G/H \subset \mathbb{P}V$, then $K[\overline{G/H}]$ maps into $K[G/H]$ by restriction of functions, and $K[\overline{G/H}]$ is equipped with a grading (that depends on the embedding). We do not know of any way to recover the grading from this description in general, in fact the same V_λ may appear in several different degrees. Fortunately, when $G = GL_n$, the degree is recoverable.

Example 2. $X = G(k, W)$, with $k > 2$. Here without loss of generality we may take $\dim W \geq 2k$. Indeed, if $\dim W < 2k$, we may pass to the dual Grassmannian $G(\dim W - k, W^*)$. If $\dim(W) > 2k$, $\hat{\sigma}(G(k, W))$ is contained in the subspace variety $R_{2k}(\Lambda^k W)$ of tensors that can be written using $\leq 2k$ basis vectors. Let $W' \subset W$ be a $2k$ dimensional subspace. Consider the subgroup $H' \subset H$,

$$H' = \{\phi \in SL(W) | \phi|_{W'} = Id_{W'}\}.$$

The quotient $SL(W)/H'$ can be identified with the variety $Hom^{inj}(W', W)$ of injective linear maps from W' to W. Since the complement of $Hom^{inj}(W', W)$ in $Hom(W', W)$ has codimension ≥ 2 every regular function on $Hom^{inj}(W', W)$ extends to $Hom(W', W)$. This means that we have the equalities

$$K[R_{2k}(\Lambda^k W)] = K[SL(W)/H'] = \oplus_{\lambda \in \Lambda^+_{SL(W)}} (S_\lambda W)^* \otimes S_\lambda W^{H'} = \oplus_\lambda (S_\lambda W)^{\oplus \dim S_\lambda W'}.$$

Here in the last equality we may view λ as a partition.

This reduces the calculation of $(S_\lambda W)^H$ to the case $W = W'$. Assuming now that $\dim(W) = 2k$, with basis $e_1, \ldots, e_{2k}$, we may take $v_\lambda + v_\mu = e_1 \wedge \cdots \wedge e_k + e_{k+1} \wedge \cdots \wedge e_{2k}$. Then

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid det(A) = det(B) = 1 \right\}.$$

Considering $G(k, W) = SL(W)/P_k$, it is clear that no fundamental representation other than $W_{\omega_k} = \Lambda^k W$ has an H-fixed vector and in W_{ω_k} there is a 2-dimensional subspace of such spanned by $e_1 \wedge \cdots \wedge e_k$ and $e_{k+1} \wedge \cdots \wedge e_{2k}$. The corresponding two copies of $\Lambda^k W$ generate the ring of invariants in the following sense. We claim:

$$K[SL(W)/H] = \oplus_{r,s \geq 0} S_{(r^k)} W \otimes S_{(s^k)} W.$$

Two generating fundamental representations are in bidegrees $(1, 0)$ and $(0, 1)$. If we work instead with $GL(W)$, since H acts trivially on the determinant, we get

$$K[GL(W)/H] = \oplus_{r,s\geq 0,m\in\mathbb{Z}} S_{(r^k)}W \otimes S_{(s^k)}W \otimes (\Lambda^{\dim W} W)^m.$$

This second description has the advantage that when we consider the embedded space $G/H \subset \mathbb{P}\Lambda^k W$ we can determine the degree these modules could appear in $K[\overline{GL(W)/H}] = K[\sigma(G(k, W))]$.

To see this, write $E = \langle e_1, \dots, e_k\rangle$, $F = \langle e_{k+1}, \dots, e_{2k}\rangle$, we want to see how many instances of the trivial representation of $SL(E) \times SL(F)$ occur in the irreducible $SL(W)$ module $S_\lambda W$. Now, since $W = E \oplus F$

$$S_\lambda(W) = \oplus_\mu S_\mu E \otimes S_{\lambda/\mu} F = \oplus_{\mu,\nu} c^\lambda_{\mu,\nu} S_\mu E \otimes S_\nu F$$

we have

$$\dim\ (S_\lambda W^*)^H = \sum_{r,s\geq 0} c^\lambda_{(r^k),(s^k)}.$$

This gives

$$K[SL(W)/H] = \oplus_{r,s\geq 0} S_{(r^k)}W \otimes S_{(s^k)}W.$$

This means that this ring is the homomorphic image of the symmetric algebra on two copies of $\Lambda^k W$ corresponding to components in bidegrees $(1, 0)$ and $(0, 1)$ which is our claim.

Example 3. $X = Seg(\mathbb{P}A_1 \otimes \cdots \otimes \mathbb{P}A_k) = \Pi GL(A_i)/P$, $\dim A_i = 2$. Here we may take $v_\lambda + v_\mu = e_1 \otimes \cdots \otimes e_k + f_1 \otimes \cdots \otimes f_k$ with e_j, f_j a basis of A_j. Then

$$H = \left\{ \Pi_j \begin{pmatrix} s_j & 0 \\ 0 & t_j \end{pmatrix} \mid s_1 \cdots s_k = t_1 \cdots t_k = 1 \right\}.$$

Now consider the action of H on

$$S_{a_1,b_1}A_1 \otimes \cdots \otimes S_{a_k,b_k}A_k = (detA_1)^{b_1} S^{a_1-b_1}A_1 \otimes \cdots \otimes (detA_k)^{b_k} S^{a_k-b_k}A_k.$$

The weight vectors are

$$e_1^{i_1+b_1} f_1^{a_1-i_1} \otimes \cdots \otimes e_k^{i_k} f_k^{a_k-i_k}, \quad 0 \leq i_j \leq a_j - b_j$$

which is acted on by H by

$$(s_1^{i_1+b_1} \cdots s_k^{i_k+b_k})(t_1^{a_1-b_1-i_1} \cdots t_k^{a_k-b_k-i_k})$$

so in order to be H-invariant, the exponents must satisfy $i_1 + b_1 = i_2 + b_2 = \cdots = i_k + b_k$ and $a_1 - b_1 - i_1 = a_2 - b_2 - i_2 = \cdots = a_k - b_k - i_k$. This means that the vectors $e_j^{i_j+b_j} f_j^{a_j-i_j}$ have to be all of the same weight, for $j = 1, \dots, k$.

Thus means the dimension of the subspace of H-invariant vectors in the module

$$S_{a_1,b_1}A_1 \otimes \cdots \otimes S_{a_k,b_k}A_k,$$

is $\min_j\ a_j - \max_j\ b_j + 1$.

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