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Measurable Linear Transformations on Abstract Wiener Spaces

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Measurable linear transformations from an abstract Wiener space to a Hilbert space are characterized. It is shown that the measure on any infinite dimensional abstract Wiener space can be transformed to that on any other by a measurable linear transformation.

1. INTRODUCTION

All topological vector spaces (TVS) which occur in this paper will be assumed to be real, Hausdorff, and locally convex, unless otherwise noted.

Let H be a separable Hilbert space and let γ be a cylinder measure on H. There is a natural mapping from H to $L^0(\Omega, \mathscr{F}, P)$ where $(\Omega, \mathscr{F}, P)$ is a probability space and $L^0(\Omega, \mathscr{F}, P)$ denotes the vector space of measurable functions on $(\Omega, \mathscr{F}, P)$ (see [1, Exposé no. 1]). The mapping is defined by $x \rightarrow \langle \cdot, x\rangle^{\sim}$, $x \in H$, where $\langle\ ,\ \rangle$ is the inner product on H and $\langle \cdot, x\rangle^{\sim}$ is the random variable corresponding to $\langle \cdot, x\rangle$ on H. If this mapping is continuous when $L^0(\Omega, \mathscr{F}, P)$ is given the topology of convergence in measure (usually not locally convex), then γ satisfies the *scalar concentration condition.* Henceforth we shall assume that all cylinder measures discussed satisfy this condition.

A *Lusin space* is a topological space for which there exists a stronger topology under which the space is a complete separable metric space (see [16] for facts about Lusin spaces). A Borel probability measure on a Lusin space is regular, i.e., it is a Radon measure. A finite product of Lusin spaces is Lusin. A subset of a Lusin space is a Borel subset if and only if it is a Lusin space in the relative topology. A continuous injection from a Lusin space to a Hausdorff space carries Borel subsets to Borel subsets. For a TVS which is Lusin, the weak and strong Borel algebras are the same.

All Borel, or Radon, measures discussed here will be probability measures, i.e., positive with total measure 1. The Borel algebra of a TVS, E, will be that generated by the $\sigma(E, E')$ open sets, unless otherwise noted. A *Borel mapping*

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between topological spaces is a mapping defined on a Borel subset of a topological space and for which the inverse image of any Borel subset is a Borel subset.

Let E, F be TVS and let μ be a Borel measure on E.

DEFINITION 1.1. A linear transformation T defined on a Borel subspace $D_T \subset E$ with values in F is *μ-measurable* if it is a Borel mapping and $\mu(D_T) = 1$.

Definition 1.1 is equivalent to that of weakly measurable essentially linear transformations given in [6, 10, 17]. We shall use the notation $T: E \to F$ even if $D_T \neq E$. Such a mapping induces a Borel measure $T(\mu)$ on F.

A linear transformation $T: E \to F$ is closed if its graph is closed in the product topology.

LEMMA 1.2. *Let E, F be* TVS *which are Lusin spaces. Let $T: E \to F$ be a closed linear transformation defined on a subspace $D_T \subset E$. Then T is a Borel mapping and D_T is a Borel subset of E. If E is a Fréchet space, then $T(E)$ is Borel in F.*

Proof. $E \times F$ is Lusin, and since the graph is closed, it is also Lusin. It follows by [16, Lemma 13, p. 106], that T is Borel and D_T is a Borel subset of E.

Ker T is closed in E [15, p. 156] and the induced injection $T_1: E_1 \to F$, where $E_1 = E/\ker T$, is closed. If E is Fréchet, then so is E_1, and therefore E_1 is a Lusin space. Since T_1 is injective with closed graph, $T_1(D_{T_1}) = T(E)$ is Borel by [16, Lemma 14, p. 107]. ∎

2. MEASURABLE LINEAR TRANSFORMATIONS

Let H be a separable Hilbert space and let γ be the standard Gaussian cylinder measure on H. That is, γ is the cylinder measure for which $\langle \cdot, x\rangle^{\sim}$, $x \in H$, is Gaussian with mean 0 and $\mathrm{Cov}(\langle \cdot, x\rangle^{\sim}, \langle \cdot, y\rangle^{\sim}) = \langle x, y\rangle$. Let B be a separable Banach space and let $S: H \to B$ be a 1–1 continuous linear transformation with dense image. If $S(\gamma)$ can be completed to a Radon measure $\overline{S(\gamma)}$ on B, then the triple (H, B, S) is called an *abstract Wiener space* (AWS) (see [8]). If B is a Hilbert space then in order that (H, B, S) be an AWS it is necessary and sufficient that S be a Hilbert–Schmidt transformation [16, Theorem 2, p. 215 and Theorem 1, p. 341].

The following generalizes [3, Lemma 2].

PROPOSITION 2.1. *Let (H, B, S) be an* AWS *and let $\mu = \overline{S(\gamma)}$ be the induced Radon measure on B. Let H_1 be a separable Hilbert space and let $T: B \to H_1$ be a μ-measurable linear transformation. Then $T \cdot S$ is Hilbert–Schmidt.*

Proof. $\mu(D_T) = 1$ so $S(H) \subset D_T$ (see [13]), and therefore $T \cdot S$ is defined everywhere on H. $T \cdot S$ is a Borel mapping, so by a theorem of Douady [16, Theorem 1, p. 157] it is continuous. $T \cdot S(\gamma)$ can be completed to the Radon measure $T(\mu)$ on H_1, so $(H, H_1, T \cdot S)$ is an AWS and $T \cdot S$ is Hilbert–Schmidt. ▌

LEMMA 2.2. *Let γ be the standard Gaussian cylinder measure on H. Let $S: H \to B$ be a continuous linear transformation into a separable Banach space B such that $S(\gamma)$ can be completed to a Radon measure. Then for any sequence $\{P_n\}$ of finite dimensional orthogonal projections on H such that $P_n \to I$, the identity operator, and $\epsilon > 0$ there exists N such that if $m, n > N$ then*

$$\gamma\{\| SP_n x - SP_m x \|_B > \epsilon\} < \epsilon.$$

Proof. See [2; 7, Corollary 5.2]. ▌

A cylinder measure γ for which Lemma 2.2 remains valid will be said to satisfy *condition* 1.

PROPOSITION 2.3. *Let H be a separable Hilbert space and let γ be a cylinder measure on H which satisfies condition* 1. *Let B be a separable Banach space and let $S: H \to B$ be a continuous linear transformation such that $S(\gamma)$ can be completed to a Radon measure μ. Let B_1 be a separable Banach space and let $T: B \to B_1$ be a closed linear transformation with domain $D_T \supset S(H)$. Then $T \cdot S(\gamma)$ can be completed to a Radon measure on B_1 if and only if T is μ-measurable.*

Proof. Since T is closed, it is a Borel mapping by Lemma 1.2. Since $T \cdot S$ is a Borel mapping defined everywhere on H, it is continuous by Douady's theorem. If T is μ-measurable then $T \cdot S(\gamma)$ can be completed to $T(\mu)$. To prove the converse it suffices to show that $\mu(D_T) = 1$.

It was proved in [4] that the transformation $S: H \to B$ can be factored $S = A \cdot W$ where $W: H \to H$ is a positive-definite Hilbert–Schmidt transformation and $A: H \to B$ is a closed, 1–1, $\overline{W(\gamma)}$-measurable linear transformation. ($\overline{W(\gamma)}$ is a Radon measure since W is Hilbert–Schmidt.) Moreover $A^{-1}: B \to H$ is bounded and $D_{A^{-1}} = B$. Let us now suppose that S has been so factored and that $\nu = \overline{W(\gamma)} = A^{-1}(\mu)$.

Let $\{e_k\}_{k \in \mathbf{N}}$ be an orthonormal basis of H consisting of eigenvectors of W. Let P_n be the orthogonal projection onto the subspace generated by $\{e_1, \ldots, e_n\}$. Then $P_n(H) \subset W(H)$ so $P_n(H) \subset D_A$. Let $Q_n = A \cdot P_n \cdot A^{-1}$. Q_n is a Borel mapping defined everywhere on B, so by Douady's theorem it is continuous. Moreover for $m \leqslant n$, $Q_m \cdot Q_n = Q_n \cdot Q_m = Q_m$. $Q_n(B) \subset A(W(H)) = S(H)$ so $Q_n(B) \subset D_T$.

Now, for $\epsilon > 0$,

$$\begin{aligned}\mu\{\| Q_n x - Q_m x \|_B > \epsilon\} &= \nu\{\| AP_n x - AP_m x \|_B > \epsilon\} \\ &= \gamma\{\| SP_n x - SP_m x \|_B > \epsilon\} \\ &< \epsilon\end{aligned}$$

for m and n sufficiently large by condition 1.

Therefore Q_n converges in μ-measure so some subsequence, also denoted by Q_n, converges μ-almost everywhere. Let $x \in B$ and suppose that $Q_n x$ converges. Since $Q_n x = AP_n A^{-1}x$ and $P_n A^{-1}x \to A^{-1}x$ and since A is closed, if $AP_n A^{-1}x$ converges, it must converge to $AA^{-1}x = x$. Therefore $Q_n \to I_B$, the identity operator on B, μ-almost everywhere.

Again let $\epsilon > 0$. Then

$$\begin{aligned}&\mu\{\| TQ_n x - TQ_m x \|_{B_1} > \epsilon\} \\ &= \nu\{\| TAP_n x - TAP_m x \|_{B_1} > \epsilon\} \\ &= \gamma\{\| TSP_n x - TSP_m x \|_{B_1} > \epsilon\} \\ &< \epsilon\end{aligned}$$

for m and n large enough by condition 1.

Therefore $T \cdot Q_n$ converges in μ-measure so a subsequence, also denoted by $T \cdot Q_n$, converges μ-almost everywhere. Thus there exists a set of μ-measure 1 where $T \cdot Q_n$ and Q_n both converge and $Q_n \to I_B$. Since T is closed, if $TQ_n x$ converges and $Q_n x \to x$, then $TQ_n x \to Tx$. It follows that $\mu(D_T) = 1$. ∎

Remark. The fact that $TQ_n x$ converges to Tx in the above proof essentially gives a representation for measurable linear transformations. On an AWS, since convergence in measure is equivalent to almost everywhere convergence, this representation is the same as [6, Satz 2.12] (see also [10, Sect. VII]). This approach is developed in [5], but representation theorems of this form can be proved more efficiently using results of [11; 9, Theorem 4.1]. ∎

Suppose that B_1 in Proposition 2.3 is a Hilbert space and that $T \cdot S$ is Hilbert–Schmidt. Then $T \cdot S(\gamma)$ can be completed to a Radon measure so we have

COROLLARY 2.4. *Let (H, B, S) be an AWS and let $T\colon B \to H_1$ be a closed linear transformation into a separable Hilbert space H_1. Then T is $\overline{S(\gamma)}$-measurable if and only if $T \cdot S$ is Hilbert–Schmidt.*

3. TRANSFORMATION OF GAUSSIAN MEASURES

In this section we show that all Gaussian Radon measures with support on an infinite-dimensional Banach space are, in a sense, equivalent.

THEOREM 3.1. *Let* B_1 *and* B_2 *be infinite-dimensional separable Banach spaces with Gaussian Radon measures* μ_1 *and* μ_2 *such that* $\operatorname{supp} \mu_1 = B_1$ *and* $\operatorname{supp} \mu_2 = B_2$. *Then there exists a* 1–1, μ_1*-measurable linear transformation* $T: B_1 \to B_2$ *such that* T^{-1} *is* μ_2*-measurable,* $\mu_2 = T(\mu_1)$, *and* $T^{-1}(\mu_2) = \mu_1$.

Proof. It can be shown that there exist AWS (H_i, B_i, S_i), $i = 1, 2$, such that $\mu_i = \overline{S_i(\gamma)}$ (see [2, 14]). As in the proof of Proposition 2.3 we can factor $S_i = A_i \cdot W_i$ where W_i is a positive-definite Hilbert–Schmidt transformation, $\nu_i = \overline{W_i(\gamma)}$ is the induced Radon measure on H_i, and $A_i: H_i \to B_i$ is a closed, 1–1, ν_i-measurable transformation with continuous inverse. Since H_i is separable and W_i is positive-definite, we can choose a countable orthonormal basis for H_i consisting of eigenvectors of W_i. Thus there is no loss of generality in assuming that $H_i = l^2$ and that W_i is a diagonal transformation. In this case we have $W_1(e_k) = \lambda_k e_k$, $W_2(e_k) = \eta_k e_k$, $\lambda_k, \eta_k > 0$, where $e_k \in l^2$ is the sequence with 1 in the kth position and 0 elsewhere.

Let $V: l^2 \to l^2$ be defined by $Ve_k = \eta_k \lambda_k^{-1} e_k$. Then V is densely defined on l^2 but not necessarily bounded. V is self-adjoint and therefore closed. It is clear that $V \cdot W_1 = W_2$ which is Hilbert–Schmidt so, by Corollary 2.4, V is ν_1-measurable and $V(\nu_1) = \nu_2$.

Let $T = A_2 \cdot V \cdot A_1^{-1}$. Since T is the composition of Borel mappings it is a Borel mapping so to prove that T is μ_1-measurable it suffices to show that $\mu_1(D_T) = 1$. But $\nu_2(D_{A_2}) = 1$ so $\nu_1(V^{-1}(D_{A_2})) = 1$ and $\mu_1(D_T) = \mu_1(A_1(V^{-1}(D_{A_2}))) = 1$. It is clear that $T(\mu_1) = \mu_2$.

Similar properties of $T^{-1} = A_1 V^{-1} A_2^{-1}$ follow similarly. ∎

Kuelbs [12] proved that any tight Borel probability measure on a strict inductive limit of Fréchet spaces (not necessarily separable) has its support on a separable Banach subspace. Combining Kuelbs' result with Theorem 3.1, we have

COROLLARY 3.2. *Let* E_1 *and* E_2 *be strict inductive limits of Fréchet spaces with* μ_1 *and* μ_2 *tight Gaussian Borel measures on* E_1 *and* E_2, *respectively. Assume that* $\operatorname{supp} \mu_i$ *is infinite-dimensional,* $i = 1, 2$. *Then there exists a* 1–1, μ_1*-measurable linear transformation* $T: E_1 \to E_2$ *such that* $T(\mu_1) = \mu_2$.

Proof. We need only note that in Kuelbs' theorem, the Borel subsets of the Banach space supporting μ_i in E_i are also Borel subsets of E_i. ∎

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