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# De-linearizing Linearity: Projective Quantum Axiomatics From Strong Compact Closure

Bob Coecke[1,2]

*Oxford University Computing Laboratory,*
*Wolfson Building, Parks Road,*
*OX1 3QD Oxford, UK.*

**Abstract**

Elaborating on our joint work with Abramsky in [2,3] we further unravel the linear structure of Hilbert spaces into several constituents. Some prove to be very crucial for particular features of quantum theory while others obstruct the passage to a formalism which is not saturated with physically insignificant global phases.
First we show that the bulk of the required linear structure is purely multiplicative, and arises from the strongly compact closed tensor which, besides providing a variety of notions such as scalars, trace, unitarity, self-adjointness and bipartite projectors [2,3], also provides *Hilbert-Schmidt norm*, *Hilbert-Schmidt inner-product*, and in particular, the *preparation-state agreement axiom* which enables the passage from a formalism of the vector space kind to a rather projective one, as it was intended in the (in)famous Birkhoff & von Neumann paper [7].
Next we consider additive types which distribute over the tensor, from which measurements can be build, and the correctness proofs of the protocols discussed in [2] carry over to the resulting weaker setting. A full probabilistic calculus is obtained when the trace is moreover *linear* and satisfies the *diagonal axiom*, which brings us to a second main result, characterization of the necessary and sufficient additive structure of a both qualitatively and quantitatively effective categorical quantum formalism without redundant global phases. Along the way we show that if in a category a (additive) monoidal tensor distributes over a strongly compact closed tensor, then this category is always enriched in commutative monoids.

*Keywords:* Strong compact closure, quantum mechanics, global phases, projective geometry, categorical trace, quantum logic.

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[2] Email: coecke@comlab.ox.ac.uk

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# 1 Introduction

The formalism of the most successful physical theory of the previous century has many redundant and operationally insignificant ingredients e.g. the redundancy of *global phases*. Its creator himself, John von Neumann [23], was very aware of this fact [20]. The key insight of the (in)famous 1936 Birkhoff-von Neumann paper entitled “The Logic of Quantum Mechanics” [7] is that when eliminating redundant global scalars one passes from a vector space to a *projective space*. Such a projective space has a non-distributive lattice of subspaces and hence the deducted natural level of abstraction was a lattice-theoretic one, but after seven decades there is still no satisfactory abstract counterpart to the role which the tensor product plays in von Neumann’s Hilbert space formalism. Also, the world of lattices is insufficiently comprehensive to give any explicit account on probabilities, which are traditionally left implicit by relying on Gleason’s theorem [14] e.g. Piron’s book [19]. As discussed in [2], another shortcoming of von Neumann’s formalism is the total lack of *types* reflecting kinds e.g. $f : \mathcal{H} \to \mathcal{H}$ can be reversible dynamics, a measurement, either destructive or non-destructive, or a mixed state.

Typing and finding an appropriate abstraction of the quantum formalism was (re-)addressed by Abramsky and myself in [2] where we recast the formalism of quantum mechanics in purely category-theoretic terms. We considered *strongly compact closed categories* with *biproducts* and we showed that all the Hilbert space machinery necessary for quantum mechanics arises in that setting, but now equipped with appropriate types and high-level tools for reasoning about entanglement — following the tradition of *linear logic* we will refer to the strongly compact closed structure as the *multiplicative* part of the structure and to the *biproducts* as the *additive* part of the structure. The abstract counterpart to the Hilbert space tensor product is now a structural primitive from which, surprisingly, most of the required ingredients for a quantum formalism can be derived [2,3]. Hence we postulated the axioms of (finitary dimensional) quantum mechanics in terms of strong compact closure, and biproducts, and it turned out that many non-trivial results obtained within von Neumann’s formalism such as quantum teleportation, logic-gate teleportation and entanglement swapping become almost trivial in the abstract setting. Moreover, the abstract setting is far more *expressive* and is *explicitly operational* (in the compositional sense), and of course, admits a lot more *axiomatic freedom*, and, last but definitely not least, turns out to still be a *quantitative* setting. But unfortunately the biproduct structure comes together with redundant global phases and also with semi-additive enrichment,[3] in layman's terms, a vector space like calculus which excludes anything of the projective kind which is non-trivial. Biproducts as in [2][4] and in particular the pairing operation of the product structure also cause a collapse of the classical information flow onto the superposition structure, due to which the physically and syntactically different entities 'classical bit equipped with probability weights' and 'qubit' become categorically isomorphic.

The goal of this paper is to address these problems of the additive part of the structure by reconsidering von Neumann's initial concern which led to quantum logic, but this time not with Birkhoff but with category theory as a close friend. While in [7], starting from a single Hilbert *space*, one first eliminates global scalars and then aims at finding the appropriate abstraction, i.e. diagramatically,

$$
\begin{tikzcd}
\textbf{Hilbert space } \mathcal{H} \arrow[d, "\text{kill redundant global scalars}"'] \arrow[dr, "\text{Birkhoff-von Neumann [7]}"] & \\
\textbf{lattice of subspaces } \mathbb{L}(\mathcal{H}) \arrow[r, "\text{go abstract}"'] & \textbf{abstract lattices}
\end{tikzcd}
$$

we will start from the whole *category* of finite dimensional Hilbert spaces and linear maps **FdHilb**, with strongly compact closed categories with 'some additive structure' as its appropriate abstraction, and then study the abstract counterpart to 'elimination of redundant global scalars', i.e. diagramatically,

$$
\begin{tikzcd}
\textbf{FdHilb} \arrow[r, "\text{go abstract}"] \arrow[dr, "\text{our approach}"'] & \textbf{`vectorial' strong compact closure} \arrow[d, "\text{kill redundant global scalars}"] \\
& \textbf{`projective' strong compact closure}
\end{tikzcd}
$$

Since the bulk of the required linear structure is already present in the strong compact closed structure, there is no need for commitment to the highly demanding biproduct structure, and we expose the *necessary* and *sufficient* additive structure required for an effective categorical quantum formalism which

[3] Following Selinger [21,22] semi-additive enrichment does admit a probabilistic interpretation when considering density matrices, but these only arise from an irreversible construction on the initial biproduct category cf. [22] and Definition 3.1 in this paper.

[4] By this we mean biproducts as the type for superposition e.g. defining a qubit as $Q \simeq \mathrm{I} \oplus \mathrm{I}$. In [21] biproducts do not play this role, they encode classical control. In [22] there are two levels of biproducts, one which generates superposition but of which the pairing operation gets erased in more or less the same manner as we do it in the construction in Def. 3.1 of this paper — see also the paragraph 'parallel work' below.

includes a probability calculus, but excludes global phases. Abstract counterparts to 'eliminating global phases' and 'absence of global phases' are introduced in Sections 2 and 3. In Sections 4, 7 we study the qualitative and the quantitative structural requirements on the additive component of a categorical quantum formalism, respectively referred to as an *ortho-structure* and an *ortho-Bornian structure*. In Section 5 we re-address the categorical semantics of [2] and deal with the above mentioned problem of the collapse of the classical information flow onto the superposition structure in two possible ways. An important physically significant feature of dumping biproducts is that the dominant role of the scalars vanishes — cf. in the case of biproducts all finitary morphisms arise as matrices in the semiring of scalars. The resulting sole significance of a scalar is that of a probability weight e.g. there is no connection anymore with the relative phases responsible for interference phenomena. We discuss this issue briefly in Section 8.

**Proofs, details, discussion and some more results.**

These can be found in [9] which is an extended version of this paper. Additional sections includes a construction which adds abstract global phases and hence provides a (partial) converse to Def. 3.1; this yields an abstract equivalence which resembles the fundamental theorem of projective geometry relating projective spaces and vector spaces.

**Other work.**

The aim of this paper and the conception of the utterance 'quantum logic' is different from the work by Abramsky and Duncan in [4] and by Abramsky in [1]. Their aim is to find a geometric model and syntax for automated reasoning within our categorical formalism of [2,3] in the spirit of the *proof-net* calculus for linear logic, anticipating on the fact that many quantum protocols such as quantum teleportation have an underlying diagrammatic interpretation in terms of the *quantum information-flow*, introduced in [8] and abstractly axiomatized by Abramsky and myself

**Parallel work.**

Selinger's latest [22] and this paper — which were simultaneously and independently written — have a non-empty intersection. Our *WProj*-construction for strongly compact closed categories coincides with Selinger's canonical embedding of a strongly compact closed category $\mathbf{C}$ in its category of *completely positive maps* $\mathbf{CPM}(\mathbf{C})$. Also in [22], Selinger proposes a graphical language for strong compact closure for which he proved completeness for equational reasoning — we have been using a similar language in a more informal manner

[2,3] and continue(d) to do so in this paper. We also mention the independent work by Baez [5] which relates to the developments in [2,3] and by Kauffman [16] which relates to those in [8].

**Subsequent work.**

In [12,11] we take a very different approach than the one proposed in the second part of this paper. Rather than relying on additive types for describing quantum measurements we abstract over classical data and define quantum measurements purely multiplicatively, by considering self-adjoint Eilenberg-Moore coalgebras for comonads induced by a special kind of internal comonoid. Via Selinger's construction we were then able to build a *decoherence* morphism which takes into account the informatic irreversibility, exactly what we which to accommodate in this paper by relaxing the additive structure. Moreover, this approach extends to POVMs via an abstract variant of Naimark's theorem. It remains to be seen how that approach relates to the results presented here, but it's fair to say that the results in [11,12] seem at the moment more compelling than those presented in the second part of this paper. In recent work [10] we also provide an axiomatic characterization of Selinger's construction, which turns out to be strongly intertwined with the preparation-state agreement axiom that we introduce in the first part of this paper.

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## 2 Some observations on strong compact closure

Recall that a *strongly compact closed category* (SCCC) [3] is a symmetric monoidal category (SMC) [18], hence with unit I, natural isomorphisms $\lambda_A : A \simeq \mathrm{I}\otimes A$ and $\rho_A : A \simeq A\otimes\mathrm{I}$, associativity $\alpha_{A,B,C} : A\otimes(B\otimes C) \simeq (A\otimes B)\otimes C$ and symmetry $\sigma_{A,B} : A \otimes B \simeq B \otimes A$, and, with

- an involution $A \mapsto A^*$ on objects called *dual*,
- a contravariant identity-on-objects monoidal involution $f \mapsto f^\dagger$ on morphisms called *adjoint*, and,
- for each object a distinct morphism $\eta_A : \mathrm{I} \to A^* \otimes A$ called *unit*,

which satisfy

$$\lambda_A^\dagger \circ (\eta_{A^*}^\dagger \otimes 1_A) \circ (1_A \otimes \eta_A) \circ \rho_A = 1_A \tag{1}$$

and the coherence condition $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$, and all natural isomorphisms $\chi$ of the symmetric monoidal structure should satisfy $\chi^{-1} = \chi^\dagger$, that is, they are *unitary*. Every SCCC is also a *compact closed category* (CCC) [5] [17] and we recall that a CCC is a $*$-autonomous category [6] with a self-dual tensor i.e. with natural isomorphisms $u_{A,B} : (A \otimes B)^* \simeq A^* \otimes B^*$ and $u_{\mathrm{I}} : \mathrm{I}^* \simeq \mathrm{I}$. For an SCCC we will assume that $u_{\mathrm{I}}$ is also unitary and that $u_{A,B}$ is strict. As shown in [3] the adjoint of an SCCC decomposes as

$$f^\dagger = (f^*)_* = (f_*)^*$$

where both $(-)^*$ and $(-)_*$ are involutive, respectively contravariant and covariant, and have $A \mapsto A^*$ as action on objects. We will be using two distinct unfoldings of the *name* $\ulcorner f \urcorner : \mathrm{I} \to A^* \otimes B$ of a morphism $f : A \to B$, either the usual definition, or, the *absorption lemma* in [2] (Lemma 3.7), respectively,

$$(1_{A^*} \otimes f) \circ \eta_A \overset{\textbf{Defn}}{=:} \ulcorner f \urcorner \overset{\textbf{Lemma}}{:=} (f^* \otimes 1_B) \circ \eta_B\,. \tag{2}$$

Still following [3] each morphism also defines a *bipartite projector*

$$\mathrm{P}_f := \ulcorner f \urcorner \circ (\ulcorner f \urcorner)^\dagger : A^* \otimes B \to A^* \otimes B\,.$$

In any SMC $\mathbf{C}$ there exists a commutative monoid of *scalars*, namely $\mathbf{C}(\mathrm{I}, \mathrm{I})$ the endomorphism monoid of the tensor unit [17]. As in [2,3] we define *scalar multiplication* by setting

$$s \bullet f := \lambda_B^{-1} \circ (s \otimes f) \circ \lambda_A : A \to B\,.$$

**Lemma 2.1** *Let $f$ and $g$ be a morphisms and $s, t$ scalars in an SMC, then*

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g) \quad \text{and} \quad (s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)\,.$$

Each complex number can be written as $r \cdot e^{i\theta}$ with $r \in \mathbb{R}$ and $\theta \in [0, 2\pi[$ to which we respectively refer as the *amplitude* and the *phase*. Quantum theory dictates that the states of quantum systems are represented as one-dimensional subspaces of a Hilbert space, that is, (non-zero) vectors in a Hilbert space *up to a* (non-zero) *scalar multiple*. Hence when specifying operations on quantum systems we need only to express to which vector a vector is mapped *up to a* (non-zero) *scalar multiple*. Hence **FdHilb** is saturated with *global scalars* which are superfluous for quantum theory. If we eliminate these, then, since

[5] It should be clear to the reader that in the context of this paper a compact closed category cannot be confused with a cartesian closed category.

states are also encoded as morphisms, we also eliminate the redundancy in their description. We would moreover like to eliminate these global scalars using a procedure which applies to any SCCC. But in fact we only want to eliminate *global phases*, since, as shown in [2], *global amplitudes* allow us to encode probability weights, and are crucially intertwined with the abstract inner-product via the abstract *Born rule*. In the case of **FdHilb**, if $f = e^{i\theta} \cdot g$ with $\theta \in [0, 2\pi[$ for $f, g : \mathcal{H}_1 \to \mathcal{H}_2$ then

$$f \otimes f^\dagger = e^{i\theta}\cdot g \otimes (e^{i\theta}\cdot g)^\dagger = e^{i\theta}\cdot g \otimes e^{-i\theta}\cdot g^\dagger = g \otimes g^\dagger\,.$$

The following lemma indicates that the passage $f \mapsto f \otimes f^\dagger$ causes also abstract global phases to vanish in some sort of similar manner.

**Proposition 2.2** *For $f$ and $g$ morphisms and $s, t$ scalars in an SCCC,*

$$s \bullet f = t \bullet g\ ,\ s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \quad \Longrightarrow \quad f \otimes f^\dagger = g \otimes g^\dagger.$$

Observing that $1_{\mathrm{I}}^{-1} = 1_{\mathrm{I}}$ it actually suffices to assume existence of a scalar $x$ such that $s \circ s^\dagger = t \circ t^\dagger = x^{-1}$. But the real surprise is the fact that there exists a converse to Proposition 2.2. It is moreover a stronger result in the sense that it extends beyond cases where there exists an inverse to $s \circ s^\dagger = t \circ t^\dagger$. This shows that abstract removal of global phases is truly genuine and not merely generalization by analogy.

**Proposition 2.3** *For $f$ and $g$ morphisms in an SCCC with scalars $S$,*

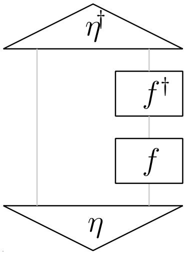
$$f \otimes f^\dagger = g \otimes g^\dagger \quad \Longrightarrow \quad \exists s, t \in S\ :\ s \bullet f = t \bullet g\ ,\ s \circ s^\dagger = t \circ t^\dagger.$$

*In particular can we set*[6]

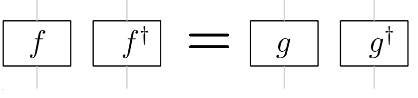
(3) $$s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner \qquad \text{and} \qquad t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner.$$

We prove Proposition 2.3 using pictures. We represent units by triangles and their adjoints by the same triangle but depicted upside down where we take a from bottom to top reading convention. Other morphisms are depicted by square boxes. E.g. the scalar $s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$ is depicted as

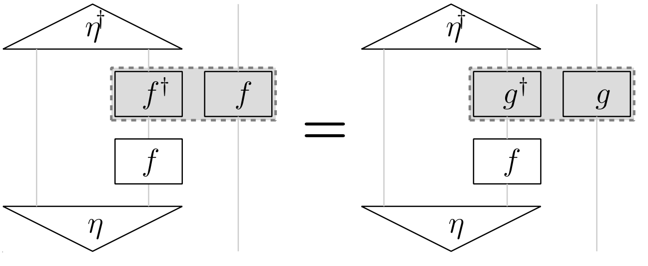
[6] By symmetry we could also set $s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner g \urcorner$ and $t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner g \urcorner$.



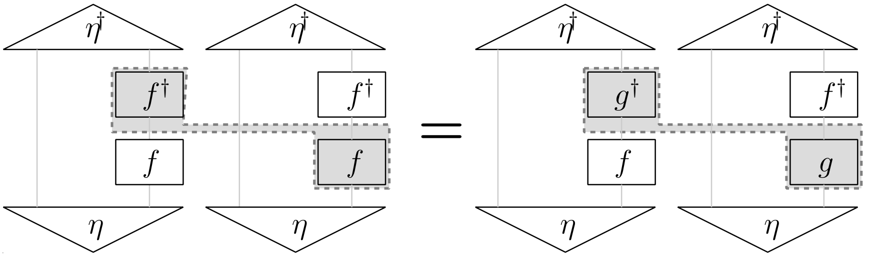

Bifunctoriality means that we can move these boxes upward and downward, and naturality provides additional modes of movement e.g. scalars admit arbitrary movements — one could say that they are not localized in time nor in space but, in Kripke's terms, they *provide a weight for a whole world*. Given that $f \otimes f^\dagger = g \otimes g^\dagger$, that is, in a picture,




it follows from the picture below that $s \bullet f = t \bullet g$,




while the picture below shows that $s \circ s^\dagger = t \circ t^\dagger$,




This completes the proof of Proposition 2.3.

In an SCCC one can also show that

$$\psi \circ \psi^\dagger = \rho_A^\dagger \circ (\psi \otimes \psi^\dagger) \circ \lambda_A \tag{4}$$

where we note that in the case of **FdHilb** the linear operator $\psi \circ \psi^\dagger : A \to A$ is the density matrix representing the pure state $\psi : \mathrm{I} \to A$ i.e. the state usually represented by the vector $\psi(1) \in \mathcal{H}$. In other words (see [2]), since $\psi$ and $\psi^\dagger$ are respectively to be conceived as a ket $|\psi\rangle$ and a bra $\langle\psi|$, their composite $\psi \circ \psi^\dagger$ corresponds to the ket-bra $|\psi\rangle\langle\psi|$. Consider now von Neumann's formalism in **FdHilb**. When passing from vectors $\psi$ to density matrices $\psi \circ \psi^\dagger$ we cancel out global phases. The global amplitudes are squared and hence provide true probability weights. This *trick* however does not extend to morphisms. Indeed, for $U : A \to B$ unitary we have $U \circ U^\dagger = 1_B$ so we lose all its content. But eq.(4) tells us that for states we obtain the same effect (that is eliminating global phases) by passing to $\psi \otimes \psi^\dagger$ instead of $\psi \circ \psi^\dagger$, and this method does extend in abstract generality. Propositions 2.2 and 2.3 then tell us that the desired effect also extends in abstract generality for arbitrary morphisms.

Assignments (3) show that to any morphism, and also to any pair of morphisms we can attribute a special scalar. Recall that the Hilbert-Schmidt norm of a bounded linear map $f : \mathcal{H}_1 \to \mathcal{H}_2$, if it exists, is $\sqrt{\sum_i \langle f(e_i) \mid f(e_i)\rangle}$ [13]. Such a map which admits a Hilbert-Schmidt norm is an Hilbert-Schmidt map. When $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ all Hilbert-Schmidt maps $\mathcal{S}(\mathcal{H})$ constitute a Banach algebra with $\sum_i \langle f(e_i) \mid g(e_i)\rangle$ as an inner-product [13]. Hence $\mathcal{S}(\mathcal{H})$ is itself a Hilbert space. We still have such a Hilbert space structure if $\mathcal{H}_1 \neq \mathcal{H}_2$ (we only lose the compositional structure).

**Definition 2.4** For each morphism $f$ in an SCCC $\mathbf{C}$ we define its *squared Hilbert-Schmidt norm* as $||f|| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner \in \mathbf{C}(\mathrm{I}, \mathrm{I})$.

In **FdHilb** we have $||f||(1) = \sum_i \langle f(e_i) \mid f(e_i)\rangle_{\tilde{\mathcal{H}}}$ for $f : \mathcal{H} \to \tilde{\mathcal{H}}$. For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$ the Hilbert space of all Hilbert-Schmidt maps $f : \mathcal{H}_1 \to \mathcal{H}_2$, we have $\mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2) \simeq \mathcal{H}_1 \otimes \mathcal{H}_2$, so it should not be a surprise that exactly this norm naturally arises in our setting.

**Definition 2.5** For morphisms $f, g : A \to B$ in an SCCC $\mathbf{C}$ we define the *Hilbert-Schmidt inner-product* as $\langle f \mid g\rangle := (\ulcorner f \urcorner)^\dagger \circ \ulcorner g \urcorner \in \mathbf{C}(\mathrm{I}, \mathrm{I})$.

Recall from [2,3] that the *inner-product of states* $\psi, \phi : \mathrm{I} \to A$ in an SCCC is given by $\psi^\dagger \circ \phi \in \mathbf{C}(\mathrm{I}, \mathrm{I})$. The Hilbert-Schmidt inner-product provides a genuine generalization of this inner-product for states.

**Proposition 2.6** *For morphisms $\psi, \phi : \mathrm{I} \to A$ in an SCCC we have*

$$(\ulcorner \psi \urcorner)^\dagger \circ \ulcorner \phi \urcorner = \psi^\dagger \circ \phi \,.$$

A nice application of Proposition 2.6 is the derivation of the version of the Born rule which uses the trace and density matrices. Recall that a projector in the spectral decomposition attributed to a measurement decomposes as

$\mathrm{P} = \pi^\dagger \circ \pi$ and that

$$\mathrm{Prob}(\psi, \mathrm{P}) := \psi^\dagger \circ \mathrm{P} \circ \psi = (\pi \circ \psi)^\dagger \circ (\pi \circ \psi)$$

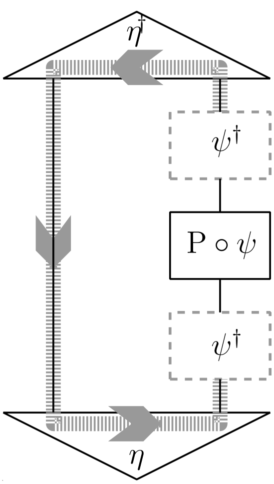
is the corresponding *abstract probability* of P for measuring a system in state $\psi : \mathrm{I} \to A$. In the density matrix version of quantum mechanics [23] the probability rule is $\mathrm{Prob}(\rho, \mathrm{P}) := \mathrm{Tr}(\mathrm{P} \circ \rho)$ where $\rho = \psi \circ \psi^\dagger$ is the density matrix corresponding to the state $\psi$ and Tr assigns to a matrix $(f_{ij})_{ij}$ the trace $\sum_i f_{ii}$. Now recall from [3] that any strongly compact closed category admits a categorical *partial trace* in the sense of [15] — this follows straightforwardly from the corresponding result for compact closed categories [17] — for which the corresponding (*full*) *trace* of $f : A \to A$ is

$$\mathrm{Tr}(f) := \eta_A^\dagger \circ (1_{A^*} \otimes f) \circ \eta_A \in \mathbf{C}(\mathrm{I}, \mathrm{I}) .$$

Hence $\langle f \mid g \rangle = \mathrm{Tr}(f^\dagger \circ g)$. In **FdHilb** this categorical trace coincides with the linear algebraic one. Passing to the Hilbert-Schmidt inner-product through Proposition 2.6, applying eq.(2), bifunctoriality and again eq.(2),

$$\begin{aligned}
\mathrm{Prob}(\psi, \mathrm{P}) &= \psi^\dagger \circ (\mathrm{P} \circ \psi) \\
&= \eta_\mathrm{I}^\dagger \circ (1_{\mathrm{I}^*} \otimes \psi^\dagger) \circ (1_{\mathrm{I}^*} \otimes (\mathrm{P} \circ \psi)) \circ \eta_\mathrm{I} \\
&= \eta_A^\dagger \circ ((\psi^\dagger)^* \otimes 1_A) \circ (1_{\mathrm{I}^*} \otimes (\mathrm{P} \circ \psi)) \circ \eta_\mathrm{I} \\
&= \eta_A^\dagger \circ (1_{A^*} \otimes (\mathrm{P} \circ \psi)) \circ ((\psi^\dagger)^* \otimes 1_\mathrm{I}) \circ \eta_\mathrm{I} \\
&= \eta_A^\dagger \circ (1_{A^*} \otimes (\mathrm{P} \circ \psi)) \circ (1_{A^*} \otimes \psi^\dagger) \circ \eta_A \;=\; \mathrm{Tr}(\mathrm{P} \circ \rho) .
\end{aligned}$$

But in a picture all this boils down to merely moving $\psi^\dagger$ around a loop:



## 3 The preparation-state agreement axiom

Following Propositions 2.2 and 2.3 the following construction aims at eliminating global phases i.e. it tries to turns a category with *vector space flavored* objects into one with *projective space 'with weights' flavored* objects.

**Definition 3.1** For each SCCC $\mathbf{C}$ we define a category $\mathit{WProj}(\mathbf{C})$.

- The objects of $\mathit{WProj}(\mathbf{C})$ are those of $\mathbf{C}$.
- The Hom-sets of $\mathit{WProj}(\mathbf{C})$ are

  (5) $$\mathit{WProj}(\mathbf{C})(A,B) := \{f \otimes f^\dagger \mid f \in \mathbf{C}(A,B)\}$$

  with $1_A \otimes 1_A \in \mathit{WProj}(\mathbf{C})(A,A)$ being the identities.
- Composition in $\mathit{WProj}(\mathbf{C})$, for $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{C}$, is given by

$$(f \otimes f^\dagger)\,\bar{\circ}\,(g \otimes g^\dagger) := (f \circ g) \otimes (f \circ g)^\dagger.$$

**Proposition 3.2** *Let $\mathbf{C}$ be an SCCC. Then $\mathit{WProj}(\mathbf{C})$ is also an SCCC.*

**Proposition 3.3** *If $f$ and $g$ are morphisms in an SCCC then*

$$f \otimes f^\dagger = g \otimes g^\dagger \iff f \otimes f_* = g \otimes g_* \iff \mathrm{P}_f = \mathrm{P}_g\,.$$

While one easily verifies that $A \mapsto A$, $f \otimes f^\dagger \mapsto f$ yields

$$(\mathit{WProj} \circ \mathit{WProj})(\mathbf{FdHilb}) \simeq \mathit{WProj}(\mathbf{FdHilb})\,,$$

idempotence of $\mathit{WProj}$ fails to be true for arbitrary SCCC. Hence we are mainly interested in *invariance* (up to isomorphism) under $\mathit{WProj}$.

**Theorem 3.4** *$\mathit{WProj}(\mathbf{C}) \simeq \mathbf{C}$ (canonically) for an SCCC $\mathbf{C}$ if and only if*

(6) $$f \otimes f^\dagger = g \otimes g^\dagger \implies f = g\,.$$

Condition (6) expresses that an SCCC is a *fixed point* of the *WProj*-construction — as our main example $\mathit{WProj}(\mathbf{FdHilb})$ is one — and hence it guarantees absence of redundant global phases. Roughly speaking one can think of these fixed points as being the result of consecutively applying *WProj* until all global redundancies are erased. But condition (6) also admits a lucid interpretation in its own right which is moreover a truly compelling physical motivation to adopt condition (6) as an *axiom* for any categorical model of abstract quantum mechanics.

**Corollary 3.5** *$\mathit{WProj}(\mathbf{C}) \simeq \mathbf{C}$ (canonically) for an SCCC $\mathbf{C}$ if and only if*

(7) $$\mathrm{P}_f = \mathrm{P}_g \implies \ulcorner f \urcorner = \ulcorner g \urcorner.$$

Condition (7) states that if two preparations $\mathrm{P}_f$ and $\mathrm{P}_g$ of bipartite states coincide then we have (of course) that the bipartite states $\ulcorner f \urcorner$ and $\ulcorner g \urcorner$ which they produce coincide. And without loss of generality this fact extends to arbitrary states — recall here form [2] that $\psi \circ \psi^\dagger : A \to A$ is the projector which prepares the state $\psi : \mathrm{I} \to A$.

**Corollary 3.6** *$WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) for an SCCC $\mathbf{C}$ if and only if*

$$\forall\, \psi, \phi : \mathrm{I} \to A\,, \quad \psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi\,. \tag{8}$$

**Definition 3.7** An SCCC satisfies the *preparation-state agreement axiom* iff the equivalent conditions (6), (7) and (8) are satisfied.

## 4 Ortho-structure

Besides being an SCCC **FdHilb** also has *biproducts* i.e. it is *semi-additive*. For an SCCC with biproducts the endomorphism monoid of the tensor unit is always an *involutive abelian semiring*, and the full subcategory of objects of type $\mathrm{I} \oplus \ldots \oplus \mathrm{I}$ is isomorphic to the category of matrices in that involutive abelian semiring, and conversely, each *matrix calculus* over an involutive abelian semiring provides an example of an SCCC with biproducts [2].

**Theorem 4.1** *There exist no SCCC with biproducts which both satisfies the preparation-state agreement axiom and for which the endomorphism monoid of the tensor unit is a ring with non-trivial negatives (i.e. $-1 = 1$).*

So if a category with as morphisms matrices over a commutative involutive semiring $R$ satisfies the preparation-state agreement axiom then $R$ cannot have non-trivial negatives, with the fatal consequence that *interference phenomena* relying on cancellation of negatives cannot be modeled. Note that our key example $WProj(\mathbf{FdHilb})$ is not isomorphic to the matrix calculus over its scalar monoid $\mathbb{R}^+$. Next set $[f] := \{g \mid f \otimes f^\dagger = g \otimes g^\dagger\}$.

**Theorem 4.2** *The product structure and the symmetric monoidal $- \oplus -$ structure of* **FdHilb** *do <u>not</u> carry over to $WProj(\mathbf{FdHilb})$. In particular, in an SCCC with biproducts $f' \in [f]$ and $g' \in [g]$ do <u>not</u> imply $f' \oplus g' \in [f \oplus g]$ and hence the operation $[f]\,\bar{\oplus}\,[g] := [f \oplus g]$ is <u>ill</u>-defined.*

**Proof:** While $1_{\mathbb{C}} \in [1_{\mathbb{C}}]$ and $(e^{i\theta} \circ -) \in [1_{\mathbb{C}}]$ we have $\langle 1_{\mathbb{C}}, (e^{i\theta} \circ -)\rangle \notin [\langle 1_{\mathbb{C}}, 1_{\mathbb{C}}\rangle]$ and $1_{\mathbb{C}} \oplus (e^{i\theta} \circ -) \notin [1_{\mathbb{C}} \oplus 1_{\mathbb{C}}]$. □

There is a physical argument why we do not want a product structure. A pairing operation $\langle -, - \rangle : \mathbf{C}(\mathrm{I}, \mathrm{I}) \times \mathbf{C}(\mathrm{I}, \mathrm{I}) \to \mathbf{C}(\mathrm{I}, \mathrm{I} \oplus \mathrm{I})$ would allow to deduce the initial state $\psi : \mathrm{I} \to \mathrm{I} \oplus \mathrm{I}$ of a *qubit* from the probabilities

$$\mathrm{Prob}(\psi\,,\, p^\dagger_{\underline{\mathrm{I}},\mathrm{I}} \circ p_{\underline{\mathrm{I}},\mathrm{I}}) \in \mathbf{C}^+(\mathrm{I},\mathrm{I}) \qquad \text{and} \qquad \mathrm{Prob}(\psi\,,\, p^\dagger_{\mathrm{I},\underline{\mathrm{I}}} \circ p_{\mathrm{I},\underline{\mathrm{I}}}) \in \mathbf{C}^+(\mathrm{I},\mathrm{I})$$

of it being subjected to the dichotomic measurement with projectors

$$p^\dagger_{\underline{\mathrm{I}},\mathrm{I}} \circ p_{\underline{\mathrm{I}},\mathrm{I}} : \mathrm{I} \oplus \mathrm{I} \to \mathrm{I} \oplus \mathrm{I} \qquad \text{and} \qquad p^\dagger_{\mathrm{I},\underline{\mathrm{I}}} \circ p_{\mathrm{I},\underline{\mathrm{I}}} : \mathrm{I} \oplus \mathrm{I} \to \mathrm{I} \oplus \mathrm{I}$$

since $\psi = \langle \mathrm{Prob}(\psi\,,\, p^\dagger_{\underline{\mathrm{I}},\mathrm{I}} \circ p_{\underline{\mathrm{I}},\mathrm{I}})\,,\, \mathrm{Prob}(\psi\,,\, p^\dagger_{\mathrm{I},\underline{\mathrm{I}}} \circ p_{\mathrm{I},\underline{\mathrm{I}}})\rangle$. But this contradicts the *empirical evidence* that a qubit state comprises *relative phase data* — which is responsible for interference phenomena — which gets erased by a measurement. So the intrinsic *informatic irreversibility* [7] of quantum measurements clashes with the very nature of the concept of a categorical product.

Part of the symmetric monoidal structure can actually be retained. The problem exposed in Theorem 4.2 can be overcome if both $[f]$ and $[g]$ contain a particular distinguished morphism, say respectively $f$ and $g$ themselves. Then we can define their monoidal sum by setting $[f] \,\bar{\oplus}\, [g] := [f \oplus g]$. There are important equivalence classes which have such a distinguished element:

- *Identities* $1_A : A \to B$ in $[1_A]$ as a part of the categorical structure.
- *Natural isos* $\lambda_A, \rho_A, \sigma_{A,B}, \alpha_{A,B,C}, u_\mathrm{I}$ as part of the SCCC structure.
- *Positive scalars* $s \circ s^\dagger : \mathrm{I} \to \mathrm{I}$ whenever they are unique in $[s \circ s^\dagger]$.

Such distinguished morphisms are the only ones for which we need monoidal sums, so we are going to let them play a distinguished role within the 'minimally required' additive structure which we will introduce. Indeed, much of what seems to be additive at first sight turns out to be multiplicative e.g. while the usual Hilbert-Schmidt norm involves an explicit *summation* of inner-products parameterized over a basis, abstractly it only involves units and adjoints which are both part of the multiplicative SCCC-structure.

**Definition 4.3** An *ortho-SCCC* is an SCCC $(\mathbf{C}, \otimes, \mathrm{I}, \lambda, \rho, \sigma, \alpha, (-)^*, (-)^\dagger, \eta)$ that comes with an *ortho-structure* i.e. a second monoidal structure $(-\oplus-)$ which is total on objects but can be only partial on morphisms, more specifically, the symmetric monoidal category $(\mathbf{C}^\sharp, \oplus, 0, l, r, s, a)$ is a subcategory of $\mathbf{C}$ of which the objects coincide with those of $\mathbf{C}$, and, which is such that $(-\otimes-) : \mathbf{C}^\sharp \times \mathbf{C}^\sharp \to \mathbf{C}^\sharp$ is a strong symmetric monoidal bifunctor of which the witnessing natural isomorphisms are unitary and with $\lambda, \rho, \alpha, \sigma, u_\mathrm{I}$ symmetric monoidal natural in all variables, and which is also such that the partial bifunctor $(-\oplus-)$ commutes with $(-)^*$ and $(-)^\dagger$ i.e. $0^* = 0$, $(A \oplus B)^* = A^* \oplus B^*$ for all objects and $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$ for all $\mathbf{C}^\sharp$-morphisms.

[7] Not to be confused with the irreversibility of a projector as a linear map.

The strong symmetric monoidal bifunctor provides (by definition) *distributivity* natural isomorphisms

$$\mathsf{DIST}_{0,l} : A \otimes 0 \simeq 0 \qquad \mathsf{DIST}_l : A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$$

$$\mathsf{DIST}_{0,r} : 0 \otimes A \simeq 0 \qquad \mathsf{DIST}_r : (B \oplus C) \otimes A \simeq (B \otimes A) \oplus (C \otimes A)\,.$$

By asking that $(- \otimes -)$ is a strongly symmetric monoidal bifunctor with $\lambda, \rho, \sigma, \alpha, u_{\mathrm{I}}$ all monoidal natural isomorphisms we make sure that these distributivity isomorphisms behave well with respect to the natural isomorphisms of the symmetric monoidal structure on $\mathbf{C}^\sharp$.

**Proposition 4.4** *For each pair of objects $A, B$ in an ortho-SCCC there exists a distinguished morphism $0_{A,B} : A \to B$ which, for all $f : B \to C$ satisfies*

$$f \circ 0_{A,B} = 0_{B,C} \circ f = 0_{A,C}$$

*and which is explicitly defined in*

$$\begin{tikzcd} A \arrow[r, "\simeq"] \arrow[d, "0_{A,B}"'] & \mathrm{I} \otimes A \arrow[r, "\eta_0 \otimes 1_A"] & (0^* \otimes 0) \otimes A \arrow[r, "\simeq"] & 0 \arrow[d, "1_0"] \\ B & \mathrm{I} \otimes B \arrow[l, "\simeq"] & (0^* \otimes 0) \otimes B \arrow[l, "\eta_0^\dagger \otimes 1_B"] & 0 \arrow[l, "\simeq"] \end{tikzcd}$$

Hence, without assuming the universal property of a zero object we do obtain a special family of morphisms which behave similarly. We will set $0_A := 0_{A,0}$, hence $0_{A,B} := 0_B^\dagger \circ 0_A : A \to B$. We define *pseudo-projections* and the *pseudo-injections* respectively as

$$p_{\underline{A},B} := r_A^\dagger \circ (1_A \oplus 0_B) : A \oplus B \to A$$

$$p_{A,\underline{B}} := l_A^\dagger \circ (0_A \oplus 1_B) : A \oplus B \to B$$
$$q_{\underline{A},B} := (1_A \oplus 0_B^\dagger) \circ r_A : A \to A \oplus B$$
$$q_{A,\underline{B}} := (0_A^\dagger \oplus 1_B) \circ l_A : B \to A \oplus B$$

and the *pseudo-components* of a morphism $f : \bigoplus_i A_i \to \bigoplus_j B_j$ are

$$f_{ij} := p_j \circ f \circ q_i : A_i \to B_j\,.$$

Of course in general these do <u>not</u> admit any kind of matrix calculus.

**Proposition 4.5** *In an ortho-SCCC we have*

$$p_{\underline{A},B} \circ q_{\underline{A},B} = 1_A \qquad\qquad p_{\underline{A},B} \circ q_{A,\underline{B}} = 0_{A,B}$$

$$q^\dagger_{\underline{A},B} = p_{\underline{A},B} = p_{B,\underline{A}} \circ s_{A,B} \qquad\qquad p_{\underline{B},D} \circ (f \oplus g) = f \circ p_{\underline{A},C}$$

$$1_A \oplus p_{\underline{B},C} = p_{\underline{A\oplus B},C} \circ a_{A,B,C} \qquad\qquad p_{\underline{A},B} \circ p_{\underline{A\oplus B},C} = p_{\underline{A},B\oplus C} \circ a^\dagger_{A,B,C}\,.$$

**Proposition 4.6** *The components $\pi_i := p_i \circ U : A \to A_i$ of a unitary morphism $U : A \to \bigoplus_i A_i$ are 'co-normalized' i.e. $\pi_i \circ \pi_i^\dagger = 1_{A_i}$ and 'co-orthogonal' for $i \neq j$ i.e. $\pi_j \circ \pi_i^\dagger = 0_{A_i,A_j}$. Analogously, the components $\psi_i := U \circ q_i : A_i \to A$ of unitary morphism $U : \bigoplus_i A_i \to A$ are 'normalized' i.e. $\psi_i^\dagger \circ \psi = 1_{A_i}$ and 'orthogonal' for $i \neq j$ i.e. $\psi_j^\dagger \circ \psi_i = 0_{A_i,A_j}$. Unitary maps preserve normality, co-normality, orthogonality and co-orthogonality.*

Note that the partial monoidal sum on morphisms did not come with an operational significance since its only aim was to provide pseudo-projections and pseudo-injections with appropriate properties. But they do much more than this, they also provide sums.

**Theorem 4.7** *An ortho-SCCC is (partially) enriched in commutative monoids i.e. admits a notion of sum of morphisms, where the sum of $f, g : A \to B$ for which $f \oplus g$ exists is given by*

$$\begin{tikzcd}
B & \mathrm{I}\otimes B \arrow[l, "\simeq"'] & (2^*\otimes 2)\otimes B \arrow[l, "\eta_2^\dagger\otimes 1_B"'] & 2^*\otimes(B\oplus B) \arrow[l, "\simeq"'] \\
A \arrow[u, "f+g"] \arrow[r, "\simeq"'] & \mathrm{I}\otimes A \arrow[r, "\eta_2\otimes 1_A"'] & (2^*\otimes 2)\otimes A \arrow[r, "\simeq"'] & 2^*\otimes(A\oplus A) \arrow[u, "1_{2^*}\otimes(f\oplus g)"']
\end{tikzcd}$$

*and with the additive units as in Proposition 4.4.*

We can put this in a slogan:

$$\boxed{\text{SCCC} + \oplus + \text{distributivity} \quad\Longrightarrow\quad \textbf{CMon}\text{-enrichment}}$$

## 5 Categorical semantics for protocols

An ortho-SCCC provides enough structure for the description and correctness proofs of the protocols considered in [2]. Two approaches are possible.

### 5.1. Distinct types for superposition and weighted branching.

When starting from an ortho-SCCC it suffices to add classical branching *freely* as a product structure i.e. *sum types* $(A_1, \ldots, A_n)$, *pairing* $\langle f_1, \ldots, f_n\rangle : C \to (A_1, \ldots, A_n)$ and *projections* $\tilde{p}_i : (A_1, \ldots, A_n) \to A_i$. This *branching structure* enables classical statistics and measurement outcome dependent manipulation of data i.e. *classical information flow*, while the ortho-structure provides the interface between the quantum state space and the classical world. We adapt some examples from [2] to the context of an ortho-SCCC with freely added products. Each unitary morphism $U : A \to \bigoplus_{i=1}^{i=n} A_i$ defines a *non-destructive measurement* $\langle \mathrm{P}_i\rangle_{i=1}^{i=n} : A \to (A)_{i=1}^{i=n}$ where $\mathrm{P}_i := \pi_i^\dagger \circ \pi_i$ with $\pi_i := p_i \circ U$. While in [2] classical communication is encoded as distributivity isomorphisms here we have

$$\mathrm{CC}_{A\leftarrow(B,C)} := \langle 1_A \otimes \tilde{p}_1, 1_A \otimes \tilde{p}_2\rangle : A \otimes (B, C) \to (A \otimes B, A \otimes C)$$

which admits no inverse, reflecting the fact that in absence of the ability to erase information, distributing information is irreversible. Also, while there is a canonical map $\langle p_{\underline{\mathrm{I}},\mathrm{I}}, p_{\mathrm{I},\underline{\mathrm{I}}}\rangle : \mathrm{I}\oplus\mathrm{I} \to (\mathrm{I}\,, \mathrm{I})$, namely the destructive measurement associated to the unitary morphism $1_{\mathrm{I}\oplus\mathrm{I}}$, this map has no inverse, and hence there exists no isomorphism between a *qubit* $Q \simeq \mathrm{I}\oplus\mathrm{I}$ and a *weighted bit* $(\mathrm{I}, \mathrm{I})$. More concretely, we define a *destructive teleportation measurement* by means of a unitary morphism $T : Q \otimes Q \to \mathrm{I} \oplus \mathrm{I} \oplus \mathrm{I} \oplus \mathrm{I}$ which is such that there exist unitary maps $\beta_1, \beta_2, \beta_3, \beta_4 : Q \to Q$ with $\ulcorner\beta_i\urcorner = T^\dagger \circ q_i$. The destructive teleportation measurement itself is

$$\langle p_i \circ T\rangle_{i=1}^{i=4} = \langle \ulcorner\beta_i\urcorner^\dagger\rangle_{i=1}^{i=4} : Q \otimes Q \to (\mathrm{I}, \mathrm{I}, \mathrm{I}, \mathrm{I})\,.$$

**Theorem 5.1** *The theorems stated in* [2] *on correctness of the example protocols for an SCCC with biproducts carry over to any ortho-SCCC with freely added products when using the above definitions.*

Hence it indeed suffices for the ortho-structure to be limited to assuring coherent coexistence of the pseudo-projections with the SCCC structure since for all the other qualitative uses of the biproduct structure in [2] we can as well use the freely added product structure which does not genuinely interact with the SCCC structure. Conclusively, we decomposed the additives in a *fundamental structural component*, namely the ortho-structure, and, a *classical branching structure*, which can be freely added as a product structure. This classical branching structure can of course be of a more sophisticated nature than the one we used here, for example one might want to capture classical mixing, but the bottom line is that it can be introduced on top of the ortho-SCCC structure and hence is not an intrinsic ingredient.

### 5.2. [2]-style semantics.

One keeps a minimal number of non-isomorphic types by distinguishing between explicit and non-explicit sums. For example, when $Q \simeq \mathrm{I} \oplus \mathrm{I}$ then $Q$ represents a qubit i.e. the *superposition* of I and I, while $\mathrm{I}\oplus\mathrm{I}$ represents a pair of *probabilistic weights* attributed to two *branches* of scalar type e.g. the respective probabilities of a destructive non-trivial qubit measurement. Since this semantics is discussed in detail in [2] we only point at the required modification when starting from an ortho-SCCC rather than from an SCCC with biproducts. The key observation is that given a unitary morphism $U : A \to \bigoplus_{i=1}^{i=n} A_i$ with $q_i : A_i \to A$ pseudo-injections we can define the corresponding non-destructive measurement as

$$\left(\bigoplus_{i=1}^{i=n} U^\dagger\right) \circ \left(\bigoplus_{i=1}^{i=n} q_i\right) \circ U : A \to \bigoplus_{i=1}^{i=n} A$$

which in the case of biproducts coincides with $\langle \pi_i^\dagger \circ \pi_i \rangle_{i=1}^{i=n} : A \to \bigoplus_{i=1}^{i=n} A$. Note that $q_i \in \mathbf{C}^\sharp$ and that it is also reasonable to assume meaningfulness of $\bigoplus_{i=1}^{i=n} U^\dagger$ i.e. $n$ copies of the same morphism. However, branch dependent operations $\bigoplus_{i=1}^{i=n} f_i$ do require a sufficiently large additive monoidal structure. The state of the $j$th branch is obtained by applying

$$p_j \circ - : \mathbf{C}\left(A, \bigoplus_{i=1}^{i=n} A_i\right) \to \mathbf{C}(A, A_j)\,,$$

and the absence of a pairing operation as in *WProj*(**FdHilb**) prevents the collapse of the classical information flow onto the superposition structure.

## 6 What is a Born-rule?

Given a model which intends to describe quantum mechanics, including *states* $\mathcal{S}$, *measurements* $\mathcal{M}$, *probability weights* $\mathbb{W}$, a *unit* $1 \in \mathbb{W}$ and an addition $- + - : \mathbb{W} \times \mathbb{W} \to \mathbb{W}$, a *Born rule* assigns to each tuple consisting of a state $\psi \in \mathcal{S}$, a measurement $M \in \mathcal{M}$ which applies to $\psi$ and an outcome $i \in \mathrm{spec}(M)$ a probability weight $s(\psi, M, i) \in \mathbb{W}$ such that

$$\sum_{i \in \mathrm{spec}(M)} s(\psi, M, i) = 1\,.$$

If our model is both *compositional* and if states *carry a probability weight* which can be extracted by means of a *valuation* $|-|_\xi : \mathcal{S} \to \mathbb{W}$ we obtain

$$\sum_i |M_i \circ \psi|_\xi = |\psi|_\xi$$

where $M_i \circ -$ stands for the *action* of $M$ on the state $\psi$ whenever the outcome $i$ occurs in that measurement. E.g. in **FdHilb** we have $|\psi|_{\mathbf{FdHilb}} := \langle \psi \mid \psi \rangle$ for $\psi \in \mathcal{H}$ and $M_i := \mathrm{P}_i : \mathcal{H} \to \mathcal{H}$ for the non-destructive measurement represented by the self-adjoint operator $M = \sum_i a_i \cdot \mathrm{P}_i : \mathcal{H} \to \mathcal{H}$, so since $|\mathrm{P}_i \circ \psi|_{\mathbf{FdHilb}} = \langle \mathrm{P}_i \circ \psi \mid \mathrm{P}_i \circ \psi \rangle = \langle \psi \mid \mathrm{P}_i \circ \psi \rangle$ by self-adjointness of $\mathrm{P}_i$ and since $\sum_i \mathrm{P}_i = 1_\mathcal{H}$ the usual Born rule slightly generalized to the case that $|\psi|_{\mathbf{FdHilb}} \neq 1$ arises i.e.

$$\sum_i |\mathrm{P}_i \circ \psi|_{\mathbf{FdHilb}} = |\psi|_{\mathbf{FdHilb}} \, .$$

Using $\mathrm{P}_i = \pi_i^\dagger \circ \pi_i$ with $\pi_i : \mathcal{H} \to \mathcal{G}_i$ and hence $\mathcal{H} \simeq \bigoplus_i \mathcal{G}_i$ we have $|\mathrm{P}_i \circ \psi|_{\mathbf{FdHilb}} = |\pi_i \circ \psi|_{\mathbf{FdHilb}}$, and for $U : \mathcal{H} \to \bigoplus_i \mathcal{G}_i$ the unique unitary map satisfying $p_i \circ U = \pi_i$, where $p_j : \bigoplus_i \mathcal{G}_i \to \mathcal{G}_j$ are the canonical projections, when setting $\phi := U \circ \psi : \mathbb{C} \to \bigoplus_i \mathcal{G}_i$ we obtain $|\psi|_{\mathbf{FdHilb}} = |\phi|_{\mathbf{FdHilb}}$. When also introducing the *components* of $\phi$ as $\phi_i := p_i \circ \phi = p_i \circ U \circ \psi = \pi_i \circ \psi : \mathbb{C} \to \mathcal{G}_i$ all the above results in

$$\sum_i |\phi_i|_{\mathbf{FdHilb}} = |\phi|_{\mathbf{FdHilb}} \, .$$

When replacing the *squared vector norm* $|-|_{\mathbf{FdHilb}} = \langle - \mid - \rangle : (\mathcal{H} \to \mathbb{C}) \to \mathbb{C}$ which only applies to morphisms of the type $\mathbb{C} \to \mathcal{H}$ by the *squared Hilbert-Schmidt norm* $|-|_{\mathbf{FdHilb}} = \sum_i \langle -(e_i) \mid -(e_i) \rangle : (\mathcal{H}_1 \to \mathcal{H}_2) \to \mathbb{C}$ we obtain by an analogous calculation that

$$\sum_i |f_i|_{\mathbf{FdHilb}} = |f|_{\mathbf{FdHilb}} \tag{9}$$

where now $f : \mathcal{H} \to \bigoplus_i \mathcal{G}_i$ and $f_i := p_i \circ f : \mathcal{H} \to \mathcal{G}_i$ with $\mathcal{H}$ arbitrary. For obvious reasons eq.(9) is our favorite incarnation of the orthodox Born rule.

Hence expressing a Born rule requires a *scalar sum* $- + -$ and a scalar-valued *valuation* on morphisms $|-|_\xi$, and when interpreting scalars as probabilistic weights these respectively stand for *adding probabilities* and *extracting* the *probabilistic weight* from the morphisms representing physical processes. The Born rule itself should then express that 'taking components of morphisms', that is, physically speaking, 'branching due to measurements', reflects through the valuation at the level of the scalars in terms of a decomposition over the scalar sum, diagrammatically,

$$\begin{tikzcd} \mathbf{C}(A, B_1 \oplus \ldots \oplus B_n) \arrow[r, "\langle p_1 \circ - \,,\ \ldots\ ,\, p_n \circ - \rangle_{\mathbf{Set}}"] \arrow[d, "|-|_\xi"] & \mathbf{C}(A, B_1) \times \ldots \times \mathbf{C}(A, B_n) \arrow[d, "|-|_\xi \times\ \ldots\ \times |-|_\xi"'] \\ \mathbf{C}(\mathrm{I}, \mathrm{I}) & \mathbf{C}(\mathrm{I}, \mathrm{I}) \times \ldots \times \mathbf{C}(\mathrm{I}, \mathrm{I})\,. \arrow[l, "- + \ldots + -"] \end{tikzcd}$$

Physically this means that the total probability weight is preserved when considering all branches i.e. a *conservation law*, which in particular implies that *relative phases lost in measurements carry no probabilistic weight*.

For $f : A \to B$ and $h : A \to A$ we (re-)set

$$||f|| := \eta_A^\dagger \circ (1_{A^*} \otimes (f^\dagger \circ f)) \circ \eta_A \quad \text{ and } \quad \mathrm{Tr}(h) := \eta_A^\dagger \circ (1_{A^*} \otimes h) \circ \eta_A\,.$$

The *trace* Tr only applies to *endomorphic types* and $||f|| = \mathrm{Tr}(f^\dagger \circ f)$. In an SCCC with biproducts the trace is *linear* i.e. $\mathrm{Tr}(h + h') = \mathrm{Tr}(h) + \mathrm{Tr}(h')$.

**Proposition 6.1** *Each SCCC with biproducts admits a Born rule, namely,* $||f_1|| + \ldots + ||f_n|| = ||f||$ *for any morphism* $f : A \to B_1 \oplus \ldots \oplus B_n$.

We set $||\mathbf{C}||$ for the range of $||-||$ and $|f| := \sqrt{||f||}$ if $||f||$ has a unique square-root. We take a scalar to be *positive* iff it decomposes as $x \circ x^\dagger$ and has at most one square-root in

$$\mathbf{C}^+(\mathrm{I}, \mathrm{I}) := \{s^\dagger \circ s \mid s \in \mathbf{C}(\mathrm{I}, \mathrm{I})\}$$

and an SCCC $\mathbf{C}$ to be *positive valued* iff all the scalars in $||\mathbf{C}||$ are positive.

**Proposition 6.2** *If* $\mathbf{C}$ *is a positive valued SCCC with biproducts then*

$$|f_1|_{\mathit{WProj}(\mathbf{C})} + \ldots + |f_n|_{\mathit{WProj}(\mathbf{C})} = |f|_{\mathit{WProj}(\mathbf{C})}$$

*for any* $f \in \mathit{WProj}(\mathbf{C})(A, B_1 \oplus \ldots \oplus B_n)$ *so* $\mathit{WProj}(\mathbf{C})$ *admits a Born rule.*

When does a non-semi-additive ortho-SCCC admit a Born rule? Does it matter that the valuation involves $||-||$ i.e. relies on the multiplicative structure? Is $||-||$ (cf. Prop. 6.1) or $|-|$ (cf. Prop. 6.2) more canonical than the other? The following lemma shows that if $\mathbf{C}(\mathrm{I}, \mathrm{I}) \subseteq \mathbf{C}^\sharp(\mathrm{I}, \mathrm{I})$, then for any valuation which is a rational power $\nu$ of $||-||$ there is a single structural axiom which stands for existence of a Born-rule, and which only relies on the SCCC-structure and on $- \oplus -$ (hence not explicitly on $\nu$ nor on $- + -$).

**Lemma 6.3** *Let* $\mathbf{C}$ *be an ortho-SCCC and let the maps*

$$|-|_\xi : \bigcup_{A,B} \mathbf{C}(A,B) \to \mathbf{C}(\mathrm{I},\mathrm{I}) \quad \text{and} \quad -+- : \mathbf{C}(\mathrm{I},\mathrm{I}) \times \mathbf{C}(\mathrm{I},\mathrm{I}) \to \mathbf{C}(\mathrm{I},\mathrm{I})\,,$$

*be such that for all morphisms* $f : A \to B_1 \oplus B_2$ *we have*

$$|f|_\xi = |f_1|_\xi + |f_2|_\xi\,. \tag{10}$$

- **i.** *Scalars* $s_1, s_2, s_3$ *satisfy an associative rule* $(s_1 + s_2) + s_3 = s_1 + (s_2 + s_3)$ *provided there exists a morphism* $f : A \to B_1 \oplus B_2 \oplus B_3$ *such that* $s_i = |f_i|_\xi$*. Hence* $f : A \to B_1 \oplus \ldots \oplus B_n$ *satisfies* $|f|_\xi = |f_1|_\xi + \ldots + |f_n|_\xi$*. If for all scalars* $|s \bullet -|_\xi = |s|_\xi \circ |-|_\xi$ *then* $s, s_1, s_2$ *satisfy a distributivity rule* $s \circ (s_1 + s_2) = (s \circ s_1) + (s \circ s_2)$ *provided there exists a morphism* $f : A \to B_1 \oplus B_2$ *and a scalar* $t$ *such that* $s_i = |f_i|_\xi$ *and* $s = |t|_\xi$*. If* $|(-)^\dagger|_\xi = |-|_\xi$ *and* $|0_{A,B}|_\xi = 0_{\mathrm{I},\mathrm{I}}$ *then all* $f, g \in \mathbf{C}^\sharp$ *satisfy*
$$|f \oplus g|_\xi = |f|_\xi + |g|_\xi\,. \tag{11}$$
- **ii.** *Let* $\mathbf{C}(\mathrm{I},\mathrm{I}) \subseteq \mathbf{C}^\sharp$ *and assume that for all* $s \in |\mathbf{C}|_\xi$ *there exists a scalar* $s^\zeta$ *such that* $|s^\zeta|_\xi = s$*. Then for all* $s, t \in |\mathbf{C}|_\xi$ *we have*
$$s + t = |s^\zeta \oplus t^\zeta|_\xi\,, \tag{12}$$
*in the presence of which eq.(10) can now be equivalently rewritten as*
$$|f|_\xi = \Big|\, |f_1|_\xi^\zeta \oplus |f_2|_\xi^\zeta \,\Big|_\xi. \tag{13}$$
- **iii.** *If moreover* $|-|_\xi := ||-||^\nu$ *and if the unique square-roots, the* $\nu$th*-powers and* $\frac{1}{\nu}$th*-powers consequently required in eq.(12) and eq.(13) exist, then eq.(13) rewrites equivalently as*
$$||f|| = \mathrm{Tr}(||f_1|| \oplus ||f_2||)\,. \tag{14}$$

By Lemma 6.3 for $|-|_\xi := ||-||$ and $|-|_\xi := |-|$ we respectively have

$$s + t := \mathrm{Tr}(s \oplus t) \qquad \text{and} \qquad s + t = \sqrt{\mathrm{Tr}(s^2 \oplus t^2)} = |s \oplus t|$$

assuming self-adjointness of $s$ and $t$. More generally, $s + t = (\mathrm{Tr}(s^{\frac{1}{\nu}} \oplus t^{\frac{1}{\nu}}))^\nu$ for $|-|_\xi := ||-||^\nu$ — the $\nu$th-power outside the trace and the $\frac{1}{\nu}$th-power inside the trace do not cancel out. Different choices of $|-|_\xi$ yield different sums and hence different abstract integers and rationals e.g. when setting $2_\mathrm{I} := 1_\mathrm{I} + 1_\mathrm{I}$ we have $2_\mathrm{I} = \mathrm{Tr}(1_{\mathrm{I}\oplus\mathrm{I}})$ for $||-||$ and $2_\mathrm{I} = \sqrt{\mathrm{Tr}(1_{\mathrm{I}\oplus\mathrm{I}})}$ for $|-|$ — recall here that $\mathrm{Tr}(1_{\mathbb{C}\oplus\mathbb{C}}) = \dim(\mathbb{C} \oplus \mathbb{C}) = 2$ in **FdHilb**. The key result of Lemma 6.3 is of course eq.(14) which we call the *ortho-Bornian axiom*. We will now decompose this ortho-Bornian axiom in two tangible components.

# 7 Ortho-Bornian structure

An endomorphism $h : A \to A$ is called *positive* iff it decomposes as $h = f^\dagger \circ f$. If $A = A_1 \oplus \ldots \oplus A_n$ then we call $h_{11} \oplus \ldots \oplus h_{nn} : A \to A$ the *pseudo-diagonal* of $h$ with respect to that decomposition of $A$. The collection of all positive morphisms of an SCCC $\mathbf{C}$ will be denoted by $\mathbf{C}^+$. The ortho-Bornian axiom is now equivalent to validity of

$$\mathrm{Tr}(h) = \mathrm{Tr}(\mathrm{Tr}(h_{11}) \oplus \mathrm{Tr}(h_{22}))$$

for all positive morphisms

$$h := f^\dagger \circ f : A_1 \oplus A_2 \to A_1 \oplus A_2 \in \mathbf{C}^+ .$$

**Proposition 7.1** *In an ortho-SCCC for $h, h' \subseteq \mathbf{C}^\sharp \cap \mathbf{C}^+$ we have*

$$\mathrm{Tr}(h + h') = \mathrm{Tr}(h \oplus h')$$

*with respect to the sum defined in Theorem 4.7. Hence on positive scalars this sum is the one corresponding to the valuation $|| - ||$ (cf. Lemma 6.3) i.e.*

$$s + t := \mathrm{Tr}(s \oplus t) .$$

**Proof:** The proof of the first claim proceeds by graphical calculus. The second claim follows by the fact that on scalars $\mathrm{Tr}(s) = s$. □

**Definition 7.2** Let $\mathbf{C}$ be an ortho-SCCC with $\mathbf{C}^+ \subseteq \mathbf{C}^\sharp$. Its trace satisfies the *diagonal axiom* iff for all $h : A_1 \oplus A_2 \to A_1 \oplus A_2 \in \mathbf{C}^+$ we have

$$\mathrm{Tr}(h) = \mathrm{Tr}(h_{11} + h_{22})$$

and it is *linear* iff for all $h, h' \in \mathbf{C}^+$ we have

$$\mathrm{Tr}(h) + \mathrm{Tr}(h') = \mathrm{Tr}(h + h') .$$

Both the diagonal axiom and linearity are stable under the *WProj*-construction.

**Theorem 7.3** *For an ortho-SCCC* $\mathbf{C}$ *with* $\mathbf{C}^+ \subseteq \mathbf{C}^\sharp$ TFAE:

- *The trace of* $\mathbf{C}$ *satisfies the ortho-Bornian axiom.*
- *The trace of* $\mathbf{C}$ *is linear and satisfies the diagonal axiom.*

The (full) trace $\mathrm{Tr}(-) = \eta_A^\dagger \circ (1_{A^*} \otimes -) \circ \eta_A$ which we have been using so far is a specialization (set $B = C := \mathrm{I}$) of the categorical partial trace

(15) $$\mathrm{Tr}(-) = \lambda_C^\dagger \circ (\eta_A^\dagger \otimes 1_C) \circ (1_{A^*} \otimes -) \circ (\eta_A \otimes 1_B) \circ \lambda_B$$

which exists as primitive data in so-called *traced monoidal categories* introduced in [15], and of which compact closed categories are a special case. As also shown in [3] the required equation for strong compact closure, that is, eq.(1), is equivalent to the *yanking axiom* for the partial trace i.e.

$$\mathrm{Tr}(\sigma_{A,A}) = 1_A\,.$$

This allows us to end with a conclusive definition in which an ortho-Bornian category arises from three assumptions on the canonical categorical trace — the definition below is not a self-contained definition but relies on the rest of the paper in order to be understood.

**Definition 7.4** An *ortho-Bornian* SCCC is a category $\mathbf{C}$ which comes with a special object I of which the endomorphisms are called *scalars*, with *tensors* $A\otimes B$ and $f\otimes g$ of objects and morphisms, with *duals* $A^*$ of objects, with *adjoints* $f^\dagger : B \to A$ of morphisms $f : A \to B$, with a special morphism $\eta_A : \mathrm{I} \to A^*\otimes A$ called *unit* for each object, with *monoidal sums* $A\oplus B$ and $f\oplus g$ of arbitrary objects and of those morphisms which are included in a subcategory $\mathbf{C}^\sharp$, all of these pieces of data being subject to conditions which establish harmonious coexistence (incl. Def. 4.3), furthermore $\mathbf{C}^\sharp$ includes all zero and all positive morphisms, and, the canonical trace $\mathrm{Tr}(-)$ on $\mathbf{C}$ which is build from units and their adjoints as in eq.(15)

**1a.** satisfies the *yanking axiom* as part of the SCCC-structure,

**1b.** satisfies the *diagonal axiom* as part of the ortho-Bornian structure,

**1c.** is *linear* also as part of the ortho-Bornian structure.

This category is moreover *projective with weights* iff it

**2.** satisfies the *preparation-state agreement axiom.*

# 8 Weight and relative phase as distinct entities

Passing from a category such as *WProj*(**FdHilb**) — or any other one obtained by applying the *WProj*-construction to an SCCC with biproducts — to a *genuine* ortho-Bornian SCCC involves separating the entities which play the role of probabilistic weight and of relative phase i.e. the extra chunk of state space one gains by considering superpositions of two underlying state spaces. In *WProj*(**FdHilb**) these two entities respectively are

$$\mathbb{R}^+ := \{\bar{c}\cdot c \mid c\in\mathbb{C}\} \quad \text{and} \quad \Big\{\{c\cdot(c_1,c_2)\mid c\in\mathbb{C}_0\}\,\Big|\,(c_1,c_2)\in(\mathbb{C}\times\mathbb{C})_0\Big\}$$

where $\mathbb{C}_0 := \mathbb{C}\backslash\{0\}$ and $(\mathbb{C}\times\mathbb{C})_0 := \mathbb{C}\times\mathbb{C}\backslash\{(0,0)\}$, hence both are constructed starting from $\mathbb{C}$, the scalar monoid of **FdHilb**. Writing these down when using more conceptual categorical machinery we get

$$\{s^\dagger \circ s \mid s \in \mathbb{S}\} \qquad \text{and} \qquad \Big\{\{s \bullet \langle s_1, s_2\rangle \mid s \in \mathbb{S}\} \,\Big|\, s_1, s_2 \in \mathbb{S}\Big\}$$

where $\mathbb{S} := \mathbf{FdHilb}(\mathbb{C},\mathbb{C})$. The crucial ingredient which enables us to do this is the pairing operation of the biproduct structure which allows to express the morphisms $f \in \mathbf{FdHilb}(\mathbb{C}, \mathbb{C}\oplus\mathbb{C})$ in terms of those in $\mathbf{FdHilb}(\mathbb{C},\mathbb{C})$ as $f := \langle s_1, s_2\rangle$. But when the ortho-structure of a weighted projective ortho-Bornian SCCC is not inherited from a biproduct structure we do not have such a connection. Denoting the scalar monoid as $\mathbb{W} := \mathbf{C}(\mathrm{I},\mathrm{I})$ — where every member is now to be interpreted a probability weight — the new player is the set $\mathbb{X}$ implicitly defined within $\mathbf{C}(\mathrm{I}, \mathrm{I}\oplus\mathrm{I}) = \mathbb{W}\times\mathbb{X}$, that is, the *qubit states* stripped off from any information concerning probabilistic weight. While these two entities do not share a common parent anymore they do interact in an important manner via the *measurement statistics*

$$\mathbf{C}(\mathrm{I}, \mathrm{I}\oplus\mathrm{I}) \times \mathbf{C}(\mathrm{I}, \mathrm{I}\oplus\mathrm{I}) \to \mathbb{W} :: (\psi,\phi) \mapsto \phi^\dagger \circ \psi$$

where $\mathbf{C}(\mathrm{I}, \mathrm{I}\oplus\mathrm{I}) \times \mathbf{C}(\mathrm{I}, \mathrm{I}\oplus\mathrm{I}) \simeq \mathbb{W}^2 \times \mathbb{X}^2$, and in which the crucial component relating probabilistic weight and relative phase is of type $\mathbb{X}^2 \to \mathbb{W}$.

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