Sufficient conditions for the convergent splittings of non-Hermitian positive definite matrices

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Abstract

We present sufficient conditions for the convergent splitting of a non-Hermitian positive definite matrix. These results are applicable to identify the convergence of iterative methods for solving large sparse system of linear equations. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Iterative methods based on matrix splittings play an important role for solving large sparse systems of linear equations (see [1,2]), and convergence property of a matrix splitting determines the numerical behaviour of the corresponding iterative method. There are many studies for the convergence property of a matrix splitting for important matrix classes such as M-matrix, H-matrix and Hermitian positive definite matrix. However, a little is known about a general non-Hermitian positive definite matrix.
matrix. To our knowledge, the only result for the latter case is Theorem 5.41 in [1]. In this paper, we will present some sufficient conditions which guarantee that a splitting of a non-Hermitian positive definite matrix is a convergent splitting. These results are applicable to construct convergent matrix splitting methods for solving the large sparse non-Hermitian positive definite linear systems.

In the following, we will first introduce some essential notations and concepts.

We use $\mathbb{C}^{n \times n}$ to denote the $n \times n$ complex matrix set, and $\mathbb{C}^n$ the $n$-dimensional complex vector set. In particular, we use $\mathbb{R}^{n \times n}$ to denote the $n \times n$ real matrix set, and $\mathbb{R}^n$ the $n$-dimensional real vector set. For an $x \in \mathbb{C}^n$, $x^*$ is used to represent the conjugate transpose of the vector $x$. For $A \in \mathbb{C}^{n \times n}$, $H(A)$ and $S(A)$ are used to denote the Hermitian and skew-Hermitian parts of the matrix $A$, respectively, i.e., $H(A) = \frac{1}{2}(A^* + A)$ and $S(A) = \frac{1}{2}(A^* - A)$. Moreover, we use $r(A)$, $\rho(A)$ and $\lambda(A)$ to denote the numerical radius, the spectral radius and the spectrum of the matrix $A$, respectively. That is to say, $\lambda(A) = \{ \lambda \in \mathbb{C}^1 \mid \text{there exists a nonzero vector } x \in \mathbb{C}^n, x^*x = 1 \}$, and $\rho(A) = \max \{ |\lambda| \mid \lambda \in \lambda(A) \}$. From [1] we know that $\|A\|_2 \geq r(A) \geq \rho(A)$ holds, where $\|A\|_2$ is the spectral norm of the matrix $A \in \mathbb{C}^{n \times n}$, which is defined by $\|A\|_2 = \max(\sqrt{\sigma} \mid \sigma \in \lambda(A^*A))$.

We call $A \in \mathbb{C}^{n \times n}$ a positive definite matrix if its Hermitian part $H(A)$ is positive definite. Evidently, $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if $A = H(A)$. A $= M - N$ is called a splitting of the matrix $A \in \mathbb{C}^{n \times n}$ if $M \in \mathbb{C}^{n \times n}$ is nonsingular. This splitting is called a convergent splitting if $\rho(M^{-1}N) < 1$.

2. Main results

Let $A \in \mathbb{C}^{n \times n}$ be a positive definite matrix, and $A = M - N$ be a splitting. Define $B = M + N$. Then we have

$H(M) = \frac{1}{2}(H(B) + H(A))$, \quad $H(N) = \frac{1}{2}(H(B) - H(A))$,

$S(M) = \frac{1}{2}(S(B) + S(A))$, \quad $S(N) = \frac{1}{2}(S(B) - S(A))$.

If both $A$ and $B$ are positive definite matrices, then both $H(A)$ and $H(B)$ are Hermitian positive definite matrices, and hence, $H(M)$ is a Hermitian positive definite matrix, too. This implies that $M \in \mathbb{C}^{n \times n}$ is a positive definite matrix.

**Theorem 2.1.** Let $A, B \in \mathbb{C}^{n \times n}$ be positive definite matrices. If

$$x^*H(A)x \cdot x^*H(B)x > |x^*S(A)x| \cdot |x^*S(B)x|$$

holds for all $x \in \mathbb{C}^n$, then $\rho(M^{-1}N) < 1$.

**Proof.** Let $\lambda$ be an eigenvalue of the matrix $(M^{-1}N)$ and $x$ the corresponding eigenvector, i.e., $M^{-1}Nx = \lambda x$, or equivalently, $\lambda Mx = Nx$. Then we have $|\lambda| = \ldots$
Here we have noticed that $M \in \mathbb{C}^{n \times n}$ is positive definite and $x \in \mathbb{C}^n$ is nonzero. By direct operations, we get

$$
\begin{cases}
|x^*Nx|^2 = (x^*H(N)x)^2 - (x^*S(N)x)^2, \\
|x^*Mx|^2 = (x^*H(M)x)^2 - (x^*S(M)x)^2.
\end{cases}
$$

Therefore, we have

$$
|x^*Mx|^2 - |x^*Nx|^2 = x^*H(A)x \cdot x^*H(B)x - x^*S(A)x \cdot x^*S(B)x.
$$

By making use of (1) we obtain $|x^*Mx|^2 > |x^*Nx|^2$. This inequality implies that

$$
|\lambda|^2 = \frac{|x^*Nx|^2}{|x^*Mx|^2} < 1.
$$

Therefore, $\rho(M^{-1}N) < 1$, as was to be proved.

Theorem 2.1 affords a sufficient condition for guaranteeing a splitting resulted from a non-Hermitian positive definite matrix being convergent. It straightforwardly results in the following corollaries.

**Corollary 2.1** [1]. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian positive definite matrix, and $A = M - N$ be its splitting. Define $B = M + N$. Then $\rho(M^{-1}N) < 1$ holds if $B \in \mathbb{C}^{n \times n}$ is a positive definite matrix.

**Corollary 2.2.** Let $A \in \mathbb{C}^{n \times n}$ be a positive definite matrix, and $A = M - N$ be its splitting. Define $B = M + N$. Then $\rho(M^{-1}N) < 1$ holds if $B \in \mathbb{C}^{n \times n}$ is a Hermi-

**Corollary 2.3.** Let $A \in \mathbb{C}^{n \times n}$ be a positive definite matrix, and $A = M - N$ be its splitting, with $M = \frac{1}{2}(A^* + A)$ and $N = \frac{1}{2}(A^* - A)$. Then $\rho(M^{-1}N) < 1$ if for all $x \in \mathbb{C}^n$, it holds that $x^*Mx > |x^*Nx|$.

At last, we demonstrate a further property of the Hermitian/skew-Hermitian splitting of a positive definite matrix $A \in \mathbb{C}^{n \times n}$.

**Theorem 2.2.** Let $A \in \mathbb{C}^{n \times n}$ be a positive definite matrix. Then

$$
\|H(A)^{-1}S(A)\|_2 \geq \|H(A)^{-1/2}S(A)H(A)^{-1/2}\|_2 \geq \max_{x \in \mathbb{C}^n} \frac{|x^*S(A)x|}{x^*H(A)x}.
$$

**Proof.** Let $E = H(A)^{-1}S(A)$. Obviously, it holds that

$$
H(A)^{1/2}EH(A)^{-1/2} = H(A)^{-1/2}S(A)H(A)^{-1/2},
$$

that is, the matrix $E$ is similar to the skew-Hermitian matrix $(H(A)^{-1/2}S(A)H(A)^{-1/2})$. Therefore,

$$
\|E\|_2 \geq \rho(E) = \max\{|\lambda| \mid \lambda \in \lambda(E)\}.$$
\[ = \max \{ |\lambda| \mid \lambda \in \lambda(H(A)^{-1/2}S(A)H(A)^{-1/2}) \} \]
\[ = \rho(H(A)^{-1/2}S(A)H(A)^{-1/2}) \]
\[ = \| H(A)^{-1/2}S(A)H(A)^{-1/2} \|_2. \]

In addition, after direct manipulations, we have
\[ \| H(A)^{-1/2}S(A)H(A)^{-1/2} \|_2 \geq \rho(H(A)^{-1/2}S(A)H(A)^{-1/2}) \]
\[ = \max_{y \in \mathbb{C}^n} \frac{|y^*H(A)^{-1/2}S(A)H(A)^{-1/2}y|}{y^*y} \]
\[ = \max_{x \in \mathbb{C}^n} \frac{|x^*S(A)x|}{x^*H(A)x}. \]

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