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A new operational matrix for solving fractional-order differential equations

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ABSTRACT

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason we need a reliable and efficient technique for the solution of fractional differential equations. This paper deals with the numerical solution of a class of fractional differential equations. The fractional derivatives are described in the Caputo sense. Our main aim is to generalize the Legendre operational matrix to the fractional calculus. In this approach, a truncated Legendre series together with the Legendre operational matrix of fractional derivatives are used for numerical integration of fractional differential equations. The main characteristic behind the approach using this technique is that it reduces such problems to those of solving a system of algebraic equations thus greatly simplifying the problem. The method is applied to solve two types of fractional differential equations, linear and nonlinear. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

1. Introduction

Fractional differential operators have a long history, having been mentioned by Leibniz in a letter to L'Hospital in 1695. A history of the development of fractional differential operators can be found in [1,2]. One of the most recent works on the subject of fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order, is the book of Podlubny [3], which deals principally with fractional differential equations. Today, there are many works on fractional calculus (see for example [4,5]).

For the past three centuries, this subject has been dealt with by the mathematicians, and only in the last few years, this was pulled to several (applied) fields of engineering, science and economics [5]. However, the number of scientific and engineering problems involving fractional calculus is already very large and still growing and perhaps the fractional calculus will be the calculus of the twenty-first century. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. Fractional differentials and integrals provide more accurate models of systems under consideration. Many authors have demonstrated applications of fractional calculus in the nonlinear oscillation of earthquakes [6], fluid-dynamic traffic model [7], to model frequency dependent damping behavior of many viscoelastic materials [8,9], continuum and statistical mechanics [10], colored noise [11], solid mechanics [12], economics [13], bioengineering [14–16], anomalous transport [17], and dynamics of interfaces between nanoparticles and substrates [18].

The analytic results on the existence and uniqueness of solutions to the fractional differential equations have been investigated by many authors (see, for example [3,4]). During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and

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dynamic systems containing fractional derivatives, such as Adomian's decomposition method [19–23], He's variational iteration method [24–26], homotopy perturbation method [27,28], homotopy analysis method [29], collocation method [30], Galerkin method [31] and other methods [32–34].

Orthogonal functions have received considerable attention in dealing with various problems. The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. In this method, a truncated orthogonal series is used for numerical integration of differential equations, with the goal of obtaining efficient computational solutions. Several papers have appeared in the literature concerned with the application of shifted Legendre polynomials [35–38].

In the present paper we intend to extend the application of Legendre polynomials to solve fractional differential equations. Our main aim is to generalize Legendre operational matrix to fractional calculus. It is worthy to mention here that, the method based on using the operational matrix of an orthogonal function for solving differential equations is computer oriented.

The article is organized as follows: We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory and Legendre polynomials which are required for establishing our results. In Section 3 the Legendre operational matrix of fractional derivative is obtained. Section 4 is devoted to applying the Legendre operational matrix of fractional derivative for solving multi-order fractional differential equation. In Section 5 the proposed method is applied to several examples. Also a conclusion is given in Section 6.

2. Preliminaries and notations

2.1. The fractional derivative in the Caputo sense

In this section, let us start with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration [4,5]. There are various definitions of fractional integration and differentiation, such as Grunwald–Letnikov's definition and Riemann–Liouville's definition. The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity [39].

Definition 2.1. The Caputo definition of the fractional-order derivative is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} \mathrm{d}t, \quad n-1 < \alpha \le n, n \in \mathbb{N}, \tag{1}$$

where $\alpha > 0$ is the order of the derivative and n is the smallest integer greater than α. For the Caputo derivative we have [40]

$$D^\alpha C = 0, \quad (C \text{ is a constant}), \tag{2}$$

$$D^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \ge \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \alpha \rfloor. \end{cases} \tag{3}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α, and the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α. Also $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of an integer order.

Similar to the integer-order differentiation, the Caputo fractional differentiation is a linear operation:

$$D^\alpha (\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x), \tag{4}$$

where λ and μ are constants. In the present work, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition, as pointed by [20], is as follows: to solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of the Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are therefore familiar to us. In contrast, for the Riemann–Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details see [41].

2.2. Properties of shifted Legendre polynomials

The well-known Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae:

$$L_{i+1}(z) = \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z), \quad i = 1, 2, \ldots,$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $x \in [0, 1]$ we define the so-called shifted Legendre polynomials by introducing the change of variable $z = 2x - 1$. Let the shifted Legendre polynomials $L_i(2x - 1)$ be denoted by $P_i(x)$. Then $P_i(x)$ can be obtained as follows:

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{(i+1)}P_i(x) - \frac{i}{i+1}P_{i-1}(x), \quad i = 1, 2, \dots, \tag{5}$$

where $P_0(x) = 1$ and $P_1(x) = 2x - 1$. The analytic form of the shifted Legendre polynomial $P_i(x)$ of degree i given by

$$P_i(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{x^k}{(k!)^2}. \tag{6}$$

Note that $P_i(0) = (-1)^i$ and $P_i(1) = 1$. The orthogonality condition is

$$\int_0^1 P_i(x)P_j(x)\mathrm{d}x = \begin{cases} \dfrac{1}{2i+1} & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{7}$$

A function $y(x)$, square integrable in $[0, 1]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j P_j(x),$$

where the coefficients c_j are given by

$$c_j = (2j+1)\int_0^1 y(x)P_j(x)\mathrm{d}x, \quad j = 1, 2, \dots.$$

In practice, only the first $(m + 1)$-terms shifted Legendre polynomials are considered. Then we have

$$y(x) = \sum_{j=0}^{m} c_j P_j(x) = C^{\mathrm{T}}\Phi(x),$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi(x)$ are given by

$$C^{\mathrm{T}} = [c_0, \dots, c_m],$$

$$\Phi(x) = [P_0(x), P_1(x), \dots, P_m(x)]^{\mathrm{T}}. \tag{8}$$

The derivative of the vector $\Phi(x)$ can be expressed by

$$\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} = \mathbf{D}^{(1)}\Phi(x), \tag{9}$$

where $\mathbf{D}^{(1)}$ is the $(m + 1) \times (m + 1)$ operational matrix of derivative given by

$$\mathbf{D}^{(1)} = (d_{ij}) = \begin{cases} 2(2j+1), & \text{for } j = i-k, \quad \begin{cases} k = 1, 3, \dots, m, & \text{if } m \text{ odd}, \\ k = 1, 3, \dots, m-1, & \text{if } m \text{ even}, \end{cases} \\ 0, & \text{otherwise}, \end{cases}$$

for example for even m we have

$$\mathbf{D}^{(1)} = 2\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \cdots & 2m-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2m-1 & 0 \end{pmatrix}.$$

3. Generalized Legendre operational matrix to fractional calculus

By using Eq. (9), it is clear that

$$\frac{\mathrm{d}^n\Phi(x)}{\mathrm{d}x^n} = (\mathbf{D}^{(1)})^n\Phi(x), \tag{10}$$

where $n \in \mathbb{N}$ and the superscript, in $\mathbf{D}^{(1)}$, denotes matrix powers. Thus

$$\mathbf{D}^{(n)} = (\mathbf{D}^{(1)})^n, \quad n = 1, 2, \ldots. \tag{11}$$

Lemma 1. *Let $P_i(x)$ be a shifted Legendre polynomial then*

$$D^{\alpha} P_i(x) = 0, \quad i = 0, 1, \ldots, \lceil \alpha \rceil - 1, \alpha > 0. \tag{12}$$

Proof. Using Eqs. (2)–(4) in Eq. (6) the lemma can be proved. ∎

In the following theorem we generalize the operational matrix of derivative of shifted Legendre polynomials given in (9) for fractional derivative.

Theorem 1. *Let $\Phi(x)$ be shifted Legendre vector defined in (8) and also suppose $\alpha > 0$ then*

$$D^{\alpha} \Phi(x) \simeq \mathbf{D}^{(\alpha)} \Phi(x), \tag{13}$$

where $\mathbf{D}^{(\alpha)}$ is is the $(m+1) \times (m+1)$ operational matrix of fractional derivative of order α in the Caputo sense and is defined as follows:

$$\mathbf{D}^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \theta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m} \theta_{m,m,k} \end{pmatrix}, \tag{14}$$

where $\theta_{i,j,k}$ is given by

$$\theta_{i,j,k} = (2j+1) \sum_{\ell=0}^{j} \frac{(-1)^{i+j+k+\ell}(i+k)!(\ell+j)!}{(i-k)!k!\Gamma(k-\alpha+1)(j-\ell)!(\ell!)^2(k+\ell-\alpha+1)}. \tag{15}$$

Note that in $\mathbf{D}^{(\alpha)}$, the first $\lceil \alpha \rceil$ rows, are all zero.

Proof. Using Eqs. (3), (4) and (6) we have

$$D^{\alpha} P_i(x) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} D^{\alpha}(x^k) = \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k-\alpha+1)} x^{k-\alpha}, \quad i = \lceil \alpha \rceil, \ldots, m. \tag{16}$$

Now, approximate $x^{k-\alpha}$ by $(m+1)$ terms of shifted Legendre series, we have

$$x^{k-\alpha} \simeq \sum_{j=0}^{m} b_{k,j} P_j(x), \tag{17}$$

where

$$\begin{aligned} b_{k,j} &= (2j+1) \int_0^1 x^{k-\alpha} P_j(x) \mathrm{d}x = (2j+1) \sum_{\ell=0}^{j} \frac{(-1)^{j+\ell}(j+\ell)!}{(j-\ell)!(\ell!)^2} \int_0^1 x^{k+\ell-\alpha} \mathrm{d}x \\ &= (2j+1) \sum_{\ell=0}^{j} \frac{(-1)^{j+\ell}(j+\ell)!}{(j-\ell)!(\ell!)^2(k+\ell-\alpha+1)}. \end{aligned} \tag{18}$$

Employing Eqs. (16)–(18) we get

$$D^{\alpha}P_i(x) \simeq \sum_{k=\lceil\alpha\rceil}^{i}\sum_{j=0}^{m}\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k-\alpha+1)}b_{k,j}P_j(x)$$
$$= \sum_{j=0}^{m}\left(\sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,j,k}\right)P_j(x), \quad i=\lceil\alpha\rceil,\ldots,m, \tag{19}$$

where $\theta_{i,j,k}$ is given in Eq. (15). Rewrite Eq. (19) as a vector form we have

$$D^{\alpha}P_i(x) \simeq \left[\sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,0,k}, \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,1,k}, \ldots, \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,m,k}\right]\Phi(x), \quad i=\lceil\alpha\rceil,\ldots,m. \tag{20}$$

Also according to Lemma 1, we can write

$$D^{\alpha}P_i(x) = [0,0,\ldots,0]\Phi(x), \quad i=0,1,\ldots,\lceil\alpha\rceil-1. \tag{21}$$

A combination of Eqs. (20) and (21) leads to the desired result. ■

Remark. If $\alpha = n \in \mathbb{N}$, Then Theorem 1 gives the same result as Eq. (11).

4. Applications of the operational matrix of fractional derivative

In this section, in order to show the high importance of operational matrix of fractional derivative, we apply it to solve multi-order fractional differential equation. The existence and uniqueness and continuous dependence of the solution to this problem are discussed in [42].

4.1. Linear multi-order fractional differential equation

Consider the linear multi-order fractional differential equation

$$D^{\alpha}y(x) = a_1D^{\beta_1}y(x) + \cdots + a_kD^{\beta_k}y(x) + a_{k+1}y(x) + a_{k+2}g(x), \tag{22}$$

with initial conditions

$$y^{(i)}(0) = d_i, \quad i=0,\ldots n, \tag{23}$$

where a_j, for $j=1,\ldots,k+2$ are real constant coefficients and also $n<\alpha\le n+1$, $0<\beta_1<\beta_2<\cdots<\beta_k<\alpha$, and D^{α} denotes the Caputo fractional derivative of order α.

To solve problem (22) and (23) we approximate $y(x)$ and $g(x)$ by the shifted Legendre polynomials as

$$y(x) \simeq \sum_{i=0}^{m} c_iP_i(x) = C^{\mathrm{T}}\Phi(x), \tag{24}$$

$$g(x) \simeq \sum_{i=0}^{m} g_iP_i(x) = G^{\mathrm{T}}\Phi(x), \tag{25}$$

where vector $G=[g_0,\ldots,g_m]^{\mathrm{T}}$ is known but $C=[c_0,\ldots,c_m]^{\mathrm{T}}$ is an unknown vector. By using Eqs. (13) and (24) we have

$$D^{\alpha}y(x) \simeq C^{\mathrm{T}}D^{\alpha}\Phi(x) \simeq C^{\mathrm{T}}\mathbf{D}^{(\alpha)}\Phi(x), \tag{26}$$
$$D^{\beta_j}y(x) \simeq C^{\mathrm{T}}D^{\beta_j}\Phi(x) \simeq C^{\mathrm{T}}\mathbf{D}^{(\beta_j)}\Phi(x), \quad j=1,\ldots k. \tag{27}$$

Employing Eqs. (24)–(27) the residual $R_m(x)$ for Eq. (22) can be written as

$$R_m(x) \simeq \left(C^{\mathrm{T}}\mathbf{D}^{(\alpha)} - C^{\mathrm{T}}\sum_{j=1}^{k}a_j\mathbf{D}^{(\beta_j)} - a_{k+1}C^{\mathrm{T}} - a_{k+2}G^{\mathrm{T}}\right)\Phi(x). \tag{28}$$

As in a typical tau method [43] we generate $m-n$ linear equations by applying

$$\langle R_m(x), P_j(x)\rangle = \int_0^1 R_m(x)P_j(x)\mathrm{d}x = 0, \quad j=0,1,\ldots,m-n-1. \tag{29}$$

Also, by substituting Eqs. (11) and (24) in Eq. (23) we get

$$\begin{aligned} &y(0) = C^{\mathrm{T}}\Phi(0) = d_0, \\ &y^{(1)}(0) = C^{\mathrm{T}}\mathbf{D}^{(1)}\Phi(0) = d_1, \\ &\vdots \\ &y^{(n)}(0) = C^{\mathrm{T}}\mathbf{D}^{(n)}\Phi(0) = d_n. \end{aligned} \tag{30}$$

Eqs. (29) and (30) generate $(m - n)$ and $(n + 1)$ set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $y(x)$ given in Eq. (24) can be calculated.

4.2. Nonlinear multi-order fractional differential equation

Consider the nonlinear multi-order fractional differential equation

$$D^{\alpha}y(x) = F\left(x, y(x), D^{\beta_1}y(x), \ldots, D^{\beta_k}y(x)\right), \tag{31}$$

with initial conditions

$$y^{(i)}(0) = d_i, \quad i = 0, \ldots n, \tag{32}$$

where $n < \alpha \le n + 1,\ 0 < \beta_1 < \beta_2 < \cdots < \beta_k < \alpha$, and D^{α} denotes the Caputo fractional derivative of order α. It should be noted that F can be nonlinear in general.

In order to use shifted Legendre polynomials for this problem, we first approximate $y(x), D^{\alpha}y(x)$ and $D^{\beta_j}y(x)$, for $j = 0, \ldots, k$ as Eqs. (24), (26) and (27) respectively. By substituting these equations in Eq. (31) we get

$$C^{\mathrm{T}}\mathbf{D}^{(\alpha)}\Phi(x) \simeq F\left(x, C^{\mathrm{T}}\Phi(x), C^{\mathrm{T}}\mathbf{D}^{(\beta_1)}\Phi(x), \ldots, C^{\mathrm{T}}\mathbf{D}^{(\beta_k)}\Phi(x)\right). \tag{33}$$

Also, by substituting Eqs. (11) and (24) in Eq. (32) we obtain

$$\begin{aligned} &y(0) = C^{\mathrm{T}}\Phi(0) = d_0, \\ &y^{(i)}(0) = C^{\mathrm{T}}\mathbf{D}^{(i)}\Phi(0) = d_i, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{34}$$

To find the solution $y(x)$, we first collocate Eq. (33) at $(m - n)$ points. For suitable collocation points we use the first $(m - n)$ shifted Legendre roots of $P_{m+1}(x)$. These equations together with Eq. (34) generate $(m + 1)$ nonlinear equations which can be solved using Newton's iterative method. Consequently $y(x)$ given in Eq. (24) can be calculated.

5. Illustrative examples

We applied the method presented in this paper and solved some examples.

Example 1. As the first example, we consider the following initial value problem in the case of the inhomogeneous Bagley–Torvik equation [44]

$$\begin{aligned} &D^2y(x) + D^{\frac{3}{2}}y(x) + y(x) = 1 + x, \\ &y(0) = 1, \qquad y'(0) = 1. \end{aligned}$$

The exact solution of this problem is $y(x) = 1 + x$. By applying the technique described in Section 4.1 with $m = 2$, we approximate solution as

$$y(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) = C^{\mathrm{T}}\Phi(x).$$

Here, we have

$$\mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}, \qquad \mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \end{pmatrix}, \qquad \mathbf{D}^{(\frac{3}{2})} = \left(\frac{16}{\sqrt{\pi}}\right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{3}{5} & -\frac{1}{7} \end{pmatrix}, \qquad G = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Therefore using Eq. (29) we obtain

$$c_0 + \left(12 + \frac{16}{\sqrt{\pi}}\right)c_2 - \frac{3}{2} = 0. \tag{35}$$

Now, by applying Eq. (30) we have

$$c_0 - c_1 + c_2 - 1 = 0, \tag{36}$$

$$2c_1 - 6c_2 - 1 = 0. \tag{37}$$

Table 1
Absolute error for $\alpha = 0.85$ and different values of m for Example 2.

x	$m = 2$	$m = 5$	$m = 8$
0.1	2.6×10^{-2}	2.0×10^{-3}	8.0×10^{-4}
0.2	2.0×10^{-2}	3.0×10^{-3}	1.2×10^{-3}
0.3	8.2×10^{-3}	6.2×10^{-4}	6.6×10^{-4}
0.4	4.8×10^{-3}	2.9×10^{-3}	8.0×10^{-4}
0.5	1.5×10^{-2}	2.0×10^{-3}	7.5×10^{-4}
0.6	2.2×10^{-2}	7.2×10^{-4}	5.9×10^{-4}
0.7	2.4×10^{-2}	2.5×10^{-3}	7.6×10^{-4}
0.8	1.9×10^{-2}	1.3×10^{-3}	1.8×10^{-4}
0.9	8.3×10^{-3}	1.5×10^{-3}	6.2×10^{-4}
1.0	1.1×10^{-2}	5.5×10^{-4}	1.5×10^{-4}

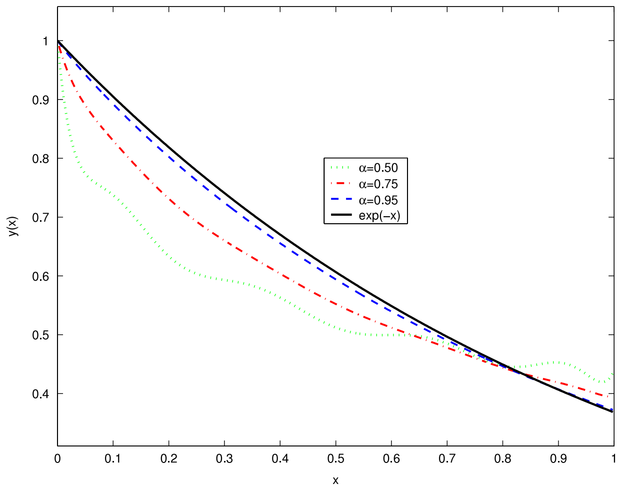

Fig. 1. Comparison of $y(x)$ for $m = 10$ and with $\alpha = 0.5, 0.75, 0.95, 1$, for Example 2.

Finally by solving Eqs. (35)–(37) we get

$$c_0 = \frac{3}{2}, \qquad c_1 = \frac{1}{2}, \qquad c_2 = 0.$$

Thus we can write

$$y(x) = \left(\frac{3}{2}, \frac{1}{2}, 0\right) \begin{pmatrix} 1 \\ 2x - 1 \\ 6x^2 - 6x + 1 \end{pmatrix} = 1 + x,$$

which is the exact solution.

Example 2. Consider the following linear initial value problem [29,32]

$$D^\alpha y(x) + y(x) = 0, \quad 0 < \alpha < 2,$$
$$y(0) = 1, \qquad y'(0) = 0.$$

The second initial condition is for $\alpha > 1$ only. The exact solution of this problem is as follows [45]:

$$y(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)},$$

we solved the problem, by applying the technique described in Section 4.1. The absolute error for $\alpha = 0.85$ and $m = 2, 5$ and 8 are shown in Table 1. From Table 1, we see that we can achieve a good approximation with the exact solution by using a few terms of shifted Legendre polynomials. Also the numerical results for $y(x)$ for $m = 10$ and $\alpha = 0.5, 0.75, 0.95$, and 1 are plotted in Fig. 1. For $\alpha = 1$, the exact solution is given as $y(x) = \exp(-x)$. Note that as α approaches 1, the numerical solution converges to the analytical solution $y(x) = \exp(-x)$. i.e. in the limit, the solution of the fractional differential equations approaches to that of the integer-order differential equations. Now we present results for $\alpha > 1$. Fig. 2 shows the

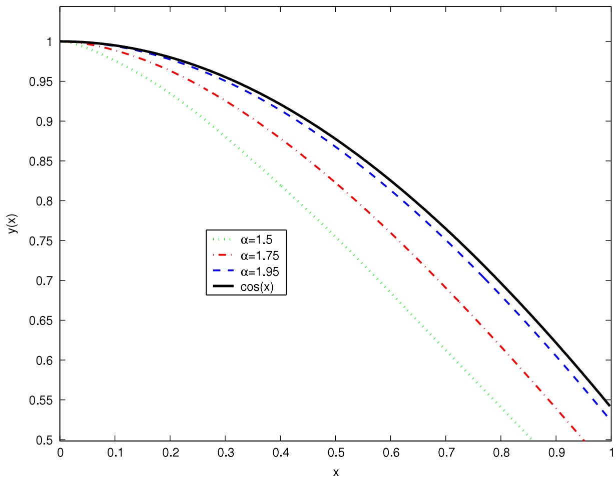

Fig. 2. Comparison of $y(x)$ for $m = 10$ and with $\alpha = 1.5, 1.75, 1.95, 2$, for Example 2.

Table 2
Absolute error for different values of α and for $m = 10$ for Example 2.

α	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
0.2	2.9×10^{-1}	4.5×10^{-1}	7.4×10^{-1}	3.7×10^{-1}	2.0×10^{-1}
0.4	3.9×10^{-1}	5.1×10^{-1}	7.3×10^{-1}	3.3×10^{-1}	2.2×10^{-1}
0.6	6.7×10^{-3}	2.0×10^{-5}	5.2×10^{-3}	4.4×10^{-3}	4.6×10^{-3}
0.8	1.1×10^{-3}	2.1×10^{-4}	8.4×10^{-4}	8.7×10^{-4}	5.8×10^{-4}
1.2	3.1×10^{-3}	2.8×10^{-3}	4.5×10^{-3}	3.6×10^{-3}	1.8×10^{-3}
1.4	1.0×10^{-3}	7.0×10^{-4}	1.3×10^{-3}	1.1×10^{-3}	2.4×10^{-4}
1.6	3.0×10^{-4}	1.3×10^{-4}	3.1×10^{-4}	3.0×10^{-4}	6.2×10^{-7}
1.8	6.1×10^{-5}	1.4×10^{-5}	4.9×10^{-5}	5.3×10^{-5}	8.8×10^{-6}

numerical results for $y(x)$ for $m = 10$ and $\alpha = 1.5,\ 1.75,\ 1.95$, and 2. For $\alpha = 2$, the exact solution is given as $y(x) = \cos(x)$. Once again, from Fig. 2, we see that as α approaches 2, the numerical solution converges to that of integer-order differential equation. The absolute error for different values of α and $m = 10$ are shown in Table 2. From Table 2, we see that as α approaches an integer value the error is reduced, as expected.

Example 3. In this example we consider the following nonlinear initial value problem [32]

$$D^{\alpha}y(x) = \frac{40320}{\Gamma(9-\alpha)}x^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)}x^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1) + \left(\frac{3}{2}x^{\alpha/2} - x^4\right)^3 - [y(x)]^{\frac{3}{2}},$$
$$y(0) = 0, \qquad y'(0) = 0, \quad 0 < \alpha < 2.$$

As before, the second initial condition is for $\alpha > 1$ only. The exact solution of this problem is given as [45]

$$y(x) = x^8 - 3x^{(4+\alpha/2)} + \frac{9}{4}x^{\alpha}.$$

We applied the method presented in Section 4.2 and solved this problem. Fig. 3 shows the analytical and numerical results for $m = 6,\ 8,\ 10$ and $\alpha = 0.75,\ 1.5$.

Furthermore the numerical results for $y(x)$ for $m = 9$ and $\alpha = 0.5,\ 0.75,\ 0.95$, and 1 are plotted in Fig. 4. Again we see that as α approaches 1, the solution of the fractional differential equations approaches to that of the integer-order differential equations. Also the absolute error for different values of α and $m = 10$ are shown in Table 3. Again, from Table 3, we see that as the α approaches an integer value, the error is reduced, as expected.

Example 4. We next consider the following nonlinear initial value problem [21,28]

$$D^3y(x) + D^{\frac{5}{2}}y(x) + y^2(x) = x^4, \quad y(0) = y'(0) = 0, \qquad y''(0) = 2.$$

We solved the above problem, by applying the technique described in Section 4.2 with $m = 3$, we approximate solution as

$$y(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) = C^T\Phi(x).$$

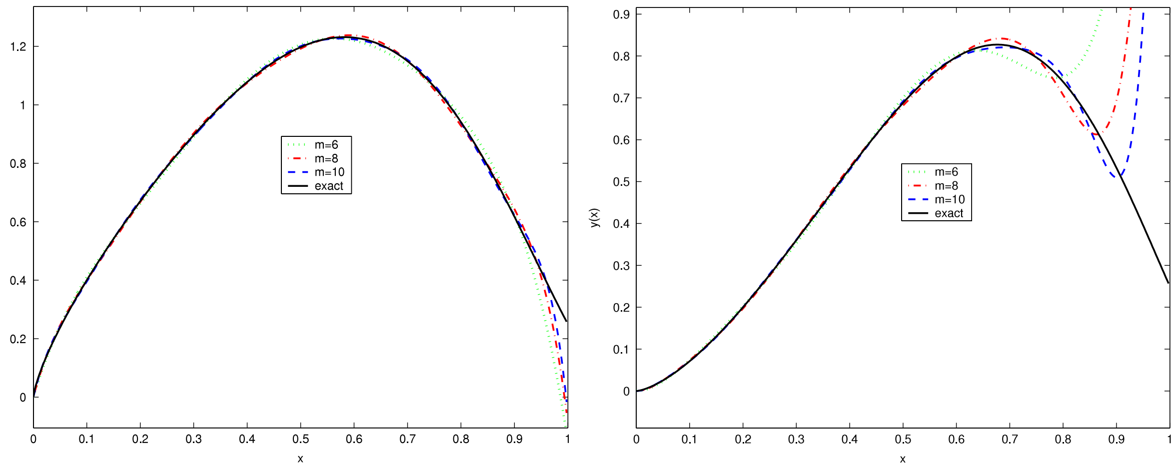

Fig. 3. Comparison of $y(x)$ for $m = 6, 8, 10$ and (Left) $\alpha = 0.75$, (Right) $\alpha = 1.5$, with exact solution, for Example 3.

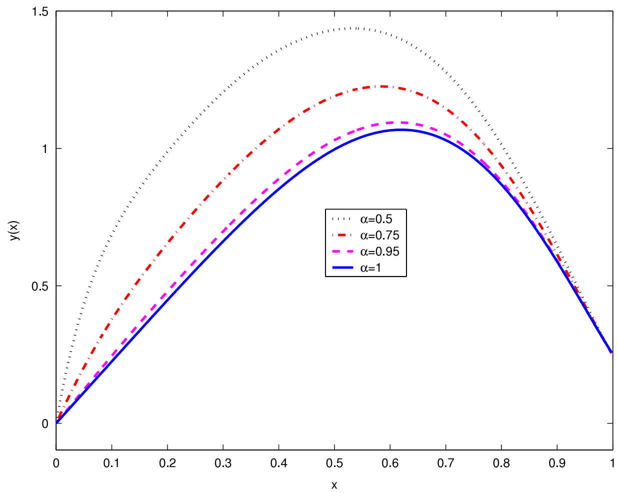

Fig. 4. Comparison of $y(x)$ for $m = 9$ and $\alpha = 0.5, 0.75, 0.95, 1$, for Example 3.

Table 3
Absolute error for different values of α and for $m = 10$ for Example 3.

α	$x = 0.1$	$x = 0.3$	$x = 0.5$	$x = 0.7$	$x = 0.9$
0.2	2.2×10^{-1}	2.3×10^{-1}	3.6×10^{-2}	5.3×10^{-1}	1.7×10^{0}
0.4	6.3×10^{-2}	6.0×10^{-2}	2.4×10^{-2}	1.2×10^{-1}	3.0×10^{-1}
0.6	1.5×10^{-2}	1.3×10^{-2}	9.6×10^{-3}	2.1×10^{-2}	3.7×10^{-2}
0.8	2.9×10^{-3}	2.1×10^{-3}	2.3×10^{-3}	2.5×10^{-3}	2.1×10^{-3}
1.2	1.9×10^{-3}	1.6×10^{-3}	2.8×10^{-2}	2.9×10^{-3}	1.6×10^{-2}
1.4	2.0×10^{-4}	1.6×10^{-3}	7.6×10^{-3}	4.9×10^{-3}	3.3×10^{-2}
1.6	6.3×10^{-5}	7.3×10^{-4}	1.7×10^{-3}	2.3×10^{-3}	1.3×10^{-2}
1.8	3.8×10^{-5}	2.0×10^{-4}	2.6×10^{-4}	5.9×10^{-4}	2.8×10^{-3}

Here, we have

$$\mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 2 & 0 & 10 & 0 \end{pmatrix}, \qquad \mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 \end{pmatrix}, \qquad \mathbf{D}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 120 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(\frac{5}{2})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{160}{\sqrt{\pi}} & \frac{96}{\sqrt{\pi}} & -\frac{160}{7\sqrt{\pi}} & \frac{32}{3\sqrt{\pi}} \end{pmatrix}, \qquad C = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Using Eq. (33) we have

$$C^{\mathrm{T}}\mathbf{D}^{(3)}\Phi(x) + C^{\mathrm{T}}\mathbf{D}^{(\frac{5}{2})}\Phi(x) + \left[C^{\mathrm{T}}\Phi(x)\right]^2 - x^4 = 0. \tag{38}$$

Now we collocate Eq. (38) at the first root of $P_4(x)$, i.e.

$$x_0 = \frac{1}{2} + \frac{\sqrt{525 - 70\sqrt{30}}}{70} \simeq 0.069431844.$$

Also by using Eq. (34) we get

$$\begin{aligned}
&C^{\mathrm{T}}\Phi(0) = c_0 - c_1 + c_2 - c_3 = 0,\\
&C^{\mathrm{T}}\mathbf{D}^{(1)}\Phi(0) = 2c_1 + 12c_3 - 6c_2 = 0,\\
&C^{\mathrm{T}}\mathbf{D}^{(2)}\Phi(0) = 12c_2 - 60c_3 = 2.
\end{aligned} \tag{39}$$

By solving Eqs. (38) and (39) we obtain

$$c_0 = \frac{1}{3}, \qquad c_1 = \frac{1}{2}, \qquad c_2 = \frac{1}{6}, \qquad c_3 = 0.$$

Therefore

$$y(x) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}, 0\right)\begin{pmatrix} 1 \\ 2x - 1 \\ 6x^2 - 6x + 1 \\ 20x^3 - 30x^2 + 12x - 1 \end{pmatrix} = x^2,$$

which is the exact solution of this problem.

It is clear that in Examples 1–4 the present method can be considered as an efficient method.

6. Conclusion

A general formulation for the Legendre operational matrix of fractional derivative has been derived. The fractional derivatives are described in the Caputo sense. This matrix is used to approximate numerical solution of a class of fractional differential equations. Our approach was based on the shifted Legendre tau and shifted Legendre collocation methods. In the limit, as α approaches an integer value, the scheme provides solution for the integer-order differential equations. The solution obtained using the suggested method shows that this approach can solve the problem effectively. Moreover, only a small number of shifted Legendre polynomials is needed to obtain a satisfactory result.

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