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INVARIANT IMBEDDING, METHOD OF CHARACTERISTICS, AND PARAMETER ESTIMATION*

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Abstract—Parameter estimation in a physical system from observed data is carried out by a method which combines invariant imbedding formalism and the method of characteristics. This avoids the necessity of making use of an initial guess for the parameter necessary if one adopts quasilinearization techniques. Numerical results for a simple one parameter system is presented.

1. INTRODUCTION

Both invariant imbedding and quasilinearization have been used to derive equations for parameter estimation (Bellman *et al.*, 1966; Lee, 1968). When quasilinearization is used, the estimation problem is treated as a two-point or multi-point boundary value problem. One difficulty with this approach is that the problem is frequently ill-conditioned. Furthermore, in order for the problem to converge, the initially guessed values for the parameters must be fairly close to the correct value.

Invariant imbedding has been used to derive the sequential estimators in nonlinear filtering. However, in order to obtain the filtering equation, several approximations must be used. (See Bellman *et al.* 1966; Lee, 1968). In this work, these approximations are avoided by directly solving the invariant imbedding equations by the method of characteristics (Abbott, 1966). Furthermore, since the resulting integral equation can be minimized directly by a suitable search technique, the problem with equality or inequality constraints on the parameters can be solved by this approach. The resulting integral equations form an ideal sequential estimator.

Although only one dimensional problem is considered in this work, the approach can be extended easily to multidimensional problems.

2. THE ESTIMATION PROBLEM

Consider a system whose behavior can be represented by the scalar equation

$$\frac{dx}{dt} = f(x, p), \tag{1}$$

with the condition

$$x(t_f) = c. \tag{2}$$

Let us assume that although the above equations represent a certain physical system, the parameter p and the initial condition c are not known exactly and must be determined from experimental data. In most practical situations, only x at various t can be measured. Let these measured values be represented by

$$z(t) = x(t) + (\text{measurement noise}). \tag{3}$$

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By using these experimental values, we wish to determine the unknown parameters p and c so that the following least squares criterion is minimized.

$$J = \int_0^{t_f} (x(t) - z(t))^2 \, dt, \tag{4}$$

we have assumed that the observation interval is $0 \le t \le t_f$.

3. INVARIANT IMBEDDING

Let us define a new variable, $y(t)$, with

$$y(t) = \int_0^{t} (x(t) - z(t))^2 \, dt. \tag{5}$$

The integral equation (4) can be written as (Lee, 1968)

$$\frac{dy}{dt} = (x(t) - z(t))^2, \tag{6}$$

with

$$y(t_f) = J. \tag{7}$$

Consider the system represented by equations (1) and (6) and let us first obtain an expression for the final condition $y(t_f)$ using invariant imbedding. Define

$$r(c, a) = y(a), \tag{8}$$

where the process ends at $t = a$ with

$$x(a) = c. \tag{9}$$

The invariant imbedding equation can be obtained easily

$$f(c, a, p)\frac{\partial r}{\partial c} + \frac{\partial r}{\partial a} = (c - z(a))^2. \tag{10}$$

4. THE METHOD OF CHARACTERISTICS

Various techniques have been developed to solve equation (10) and, at the same time, to estimate p and c so that equation (4) is minimized. (See Bellman, 1966; Lee, 1968). In this work, the method of characteristics will be used. The characteristic equations of equation (10) are

$$\frac{dc}{f(c, a, p)} = \frac{da}{1} = \frac{dr}{(c - z(a))^2}. \tag{11}$$

For the moment, let us assume that the equations in (11) can be integrated. Integrate the left-hand equation in (11), we have

$$c = g(c, a_f, p, \psi) \tag{12}$$

where ψ is the integration constant and a_f denotes the a value after integration. The other set of the equation from (11) leads to

$$r = \int_0^{a} (c - z(a'))^2 \, da'. \tag{13}$$

Using equation (12), equation (13) becomes

$$r = \int_0^a [g(c, a, p, \psi) - z(a)]^2 \, da. \tag{14}$$

The problem now reduces to the minimization of the ingegral in equation (14). Notice that if equation (14) is minimized, then the least squares expression, equation (4), is also minimized. Thus, our problem is to obtain c and p so that equation (14) is minimized. Furthermore, this minimization can be carried in a sequential fashion.

5. AN EXAMPLE

To illustrate the approach, we shall consider the following simple scalar equation

$$\frac{dx}{dt} = -kx^2. \tag{15}$$

With given experimental values for x, we wish to estimate the parameter k. Using the same approach discussed in Section 3, the invariant imbedding equation is

$$\frac{\partial r}{\partial a} - kc^2 \frac{\partial r}{\partial c} = (c - z(a))^2. \tag{16}$$

Equation (12) becomes

$$c = \frac{1}{k(a_f + \psi)} \tag{17}$$

or

$$\psi = \frac{1}{kc} - a_f. \tag{18}$$

Equation (13) now becomes

$$r(c, a) = \int_0^a (c - z(a'))^2 \, da'. \tag{19}$$

Using equation (17) and (18), equation (19) becomes

$$r(c, a_f) = \int_0^{a_f} \left[\frac{1}{k[a' + (1/kc) - a_f]} - z(a') \right]^2 da'. \tag{20}$$

The noisy measurements for the concentration are generated by the computer in two steps. First, equation (15) is integrated numerically. The numerical values used are

$$x(0) = 1.0$$
$$k = 0.05$$
$$\Delta t = 0.1$$
$$t_f = 50,$$

where Δt is the integration step size. Second, the results from this integration are corrupted with noise by the equation

$$z(t_i) = x(t_i) + R(t_i), \quad i = 1, 2, \ldots, N, \tag{21}$$

with $t_0 = 0$, $t_f = N$, and $t_{i+1} - t_i = \Delta t$. The $R(t_i)$'s represent random numbers with Gaussian distribution. The mean of the distribution is zero and the standard deviation is 0.05.

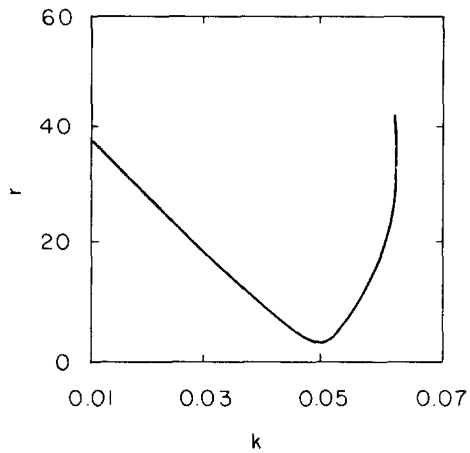

Fig. 1. The invarient imbedding variable as a function of k.

Using $z(t_i)$ and the following numerical values for equation (20)

$$a_f = 50$$
$$c = x(50)$$
$$a = 0, 0.1, 0.2, \ldots, 50.$$

Equation (20) can be minimized with respect to k. Any search technique can be used for this minimization. A simple enumeration search is used. The results are shown in Fig. 1. As can be seen that at $k = 0.05$, the function r does reach the minimum.

If k is not a constant but varies with time, in a suitable flashion, sequential determination of k in various regions can be made by the application of dynamic programming methods. The details of this procedure and numerical computations will be presented later.

REFERENCES

Bellman R., H. Kagiwada and R. Kalaba, Invariant imbedding and nonlinear filtering theory. *J. Astronaut. Sci.* 13(3), 110–115 (1966).

Lee E. S., *Quasilinearization and Invariant Imbedding.* Academic Press, New York (1968).

Abbott, M. B., *An Introduction to the Method of Characteristics*, Elsevier, New York (1966).