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Some Expansions in Theta Functions

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1. Consider the product

$$F(a, x, q) = \prod_{-\infty}^{\infty} (1 + ax^{2n}q^{n^2}), \tag{1}$$

where $|q| < 1$. Clearly the product is convergent for all a and all $x \neq 0$. We may put

$$F(a, x, q) = \sum_{n=0}^{\infty} a^n A_n(x, q). \tag{2}$$

It follows from (1) that

$$F(ax^2q, xq, q) = F(a, x, q). \tag{3}$$

But by (2)

$$F(ax^2q, xq, q) = \sum_{n=0}^{\infty} a^n x^{2n} q^n A_n(xq, q)$$

so that

$$x^{2n}q^n A_n(xq, q) = A_n(x, q). \tag{4}$$

If we put

$$A_n(x, q) = \sum_{r=-\infty}^{\infty} x^{2r} C_r^{(n)}, \tag{5}$$

where $C_r^{(n)} = C_r^{(n)}(q)$, then it follows from (4) that

$$x^{2n}q^n \sum_{r=-\infty}^{\infty} x^{2r}q^{2r}C_r^{(n)} = \sum_{r=-\infty}^{\infty} x^{2r}C_r^{(n)},$$

so that

$$C_r^{(n)} = q^{2r-n}C_{r-n}^{(n)}. \tag{6}$$

It is evident from (1) that $A_0(x, q) = 1$ so that

$$C_0^{(0)} = 1, \qquad C_r^{(0)} = 0 \qquad (r \neq 0). \tag{7}$$

Also it is clear from (1) that

$$A_1(x, q) = \sum_{r=-\infty}^{\infty} x^{2r} q^{r^2}, \tag{8}$$

so that

$$C_r^{(1)} = q^{r^2}. \tag{9}$$

For $n = 2$, (6) implies

$$C_{2r}^{(2)} = q^{4r-2} C_{2r-2}^{(2)}, \qquad C_{2r+1}^{(2)} = q^{4r} C_{2r-1}^{(2)},$$

from which it follows that

$$C_{2r}^{(2)} = q^{2r^2} C_0^{(2)}, \tag{10}$$

$$C_{2r+1}^{(2)} = q^{2r(r+1)} C_1^{(2)}. \tag{11}$$

These formulas hold for all values of r.

Next, from (2) and (5), C_0 is the coefficient of a^2x^0 in $F(a, x, q)$ while $C_1^{(2)}$ is the coefficient of a^2x^2. Hence, by (1),

$$C_0^{(2)} = \sum_{r=1}^{\infty} q^{2r^2}, \tag{12}$$

$$C_1^{(2)} = q \sum_{r=0}^{\infty} q^{2r(r+1)}. \tag{13}$$

It follows that

$$A_2(x, q) = C_0^{(2)} \sum_{r=-\infty}^{\infty} x^{4r} q^{2r^2} + C_1^{(2)} \sum_{r=-\infty}^{\infty} x^{4r+2} q^{2r(r+1)}. \tag{14}$$

To evaluate $C_r^{(n)}$ for arbitrary n we again use (6). Replacing r by $rn + s$ we get

$$C_{rn+s}^{(n)} = q^{(2r-1)n+2s} C_{(r-1)n+s}^{(n)},$$

which implies

$$C_{rn+s} = q^{r(rn+2s)} C_s^{(n)}. \tag{15}$$

This formula holds for all values of r. Thus (5) becomes

$$A_n(x, q) = \sum_{s=0}^{n-1} C_s^{(n)} \sum_{r=-\infty}^{\infty} x^{2rn+2s} q^{r(rn+2s)}. \tag{16}$$

It remains to determine $C_s^{(n)}$ for $s = 0, 1, \cdots, n-1$. It is clear from (1) that the coefficient of $a^n x^0$ in $F(a, x, q)$ is equal to

$$\sum_{\substack{r_1+\cdots+r_n=0\\ r_1<\cdots<r_n}} q^{r_1^2+\cdots+r_n^2},$$

where each r_j ranges from $-\infty$ to ∞. Therefore

$$\begin{aligned} C_0^{(n)} &= \sum_{\substack{r_1+\cdots+r_n=0\\ r_1<\cdots<r_n}} q^{r_1^2+\cdots+r_n^2} \\ &= \sum_{\substack{r_1+\cdots+r_n=0\\ r_1<\cdots<r_n}} q^{r_1^2+\cdots+r_{n-1}^2+(r_1+\cdots+r_{n-1})^2}. \end{aligned} \tag{17}$$

More generally the coefficient of $a^n x^{2s}$ in $F(a, s, q)$ is equal to

$$\sum_{\substack{r_1+\cdots+r_n=s\\ r_1<\cdots<r_n}} q^{r_1^2+\cdots+r_n^2},$$

where each r_j ranges from $-\infty$ to ∞. It follows that

$$\begin{aligned} C_s^{(n)} &= \sum_{\substack{r_1+\cdots+r_n=s\\ r_1<\cdots<r_n}} q^{r_1^2+\cdots+r_n^2} \\ &= \sum_{\substack{r_1+\cdots+r_n=s\\ r_1<\cdots<r_n}} q^{r_1^2+\cdots+r_{n-1}^2+(s-r_1-\cdots-r_{n-1})^2}. \end{aligned} \tag{18}$$

If we let $N_{n-1}(r, s)$ denote the number of solutions of the equation

$$r = r_1^2 + \cdots + r_{n-1}^2 - s(r_1 + \cdots + r_{n-1}),$$

subject to the conditions

$$r_1 + \cdots + r_n = s, \qquad r_1 < \cdots < r_n,$$

then (18) becomes

$$C_s^{(n)} = q^{s^2} \sum_{r=0}^{\infty} N_{n-1}(r, s)\, q^{2r}. \tag{19}$$

Returning to (16) it is clear from the foregoing that we may express $A_n(x, q)$ as a linear combination of theta functions. Indeed put

$$x = e^{\pi iu}, \qquad q = e^{\pi i\tau}, \qquad \vartheta_3(u, q) = \sum_{r=-\infty}^{\infty} e^{2r\pi iu} q^{r^2}$$

in the usual notation for theta functions. Then (16) becomes

$$A_n(e^{\pi iu}, q) = \sum_{s=0}^{n-1} C_s^{(n)} \sum_{r=-\infty}^{\infty} e^{2\pi iu(rn+s)} e^{2rs\pi i\tau} q^{r^2 n}$$

$$= \sum_{s=0}^{n-1} C_s^{(n)} e^{2s\pi iu} \sum_{r=-\infty}^{\infty} e^{2r\pi i(nu+s\tau)} q^{r^2 n},$$

so that

$$A_n(e^{\pi iu}, q) = \sum_{s=0}^{n-1} C_s^{(n)} e^{2s\pi iu} \cdot \vartheta_3(nu + s\tau, q^n). \tag{20}$$

2. In place of (1) we may consider the product

$$G(a, x, q) = \prod_{-\infty}^{\infty} (1 - ax^{2n}q^{n^2})^{-1} \tag{21}$$

which evidently satisfies

$$G(ax^2q, xq, q) = G(a, x, q). \tag{22}$$

Thus, if

$$G(a, x, q) = \sum_{n=0}^{\infty} a^n B_n(x, q) \tag{23}$$

we get

$$x^{2n}q^n B_n(xq, q) = B_n(x, q). \tag{24}$$

Hence, if we put

$$B_n(x, q) = \sum_{r=-\infty}^{\infty} x^{2r} D_r^{(n)}, \tag{25}$$

where $D_r^{(n)} = D_r^{(n)}(q)$, it follows from (24) that

$$D_r^{(n)} = q^{2r-n} D_{r-n}^{(n)}. \tag{26}$$

Then

$$D_0^{(0)} = 1, \qquad D_r^{(0)} = 0 \qquad (r \neq 0), \tag{27}$$

while

$$B_1(x, q) = \sum_{r=-\infty}^{\infty} x^{2r} q^{r^2}, \tag{28}$$

so that

$$D_r^{(1)} = q^{r^2}. \tag{29}$$

For arbitrary n, it follows from (26) that

$$D_{rn+s}^{(n)} = q^{(2r-1)n+2s} D_{(r-1)n+s}^{(n)}$$

so that

$$D_{rn+s}^{(n)} = q^{r(rn+2s)} D_s^{(n)}. \tag{30}$$

Thus (25) becomes

$$B_n(x, q) = \sum_{s=0}^{n-1} D_s^{(n)} \sum_{r=-\infty}^{\infty} x^{2rn+2s} q^{r(rn+2s)}. \tag{31}$$

Comparing (31) with (16) we see that the only difference is in the coefficients $D_s^{(n)}$, where $0 \leqslant s < n$. For $n = 0, 1$ we have from (7), (9), (27), (29)

$$D_r^{(0)} = C_r^{(0)}, \qquad D_r^{(1)} = C_r^{(1)} \qquad (r = 0, \pm 1, \pm 2, \cdots), \tag{32}$$

but for $n > 1$ this is no longer the case. For example it is easily verified that

$$D_0^{(2)} = 1 + \sum_{r=1}^{\infty} q^{2r^2} = 1 + C_0^{(2)}, \tag{33}$$

$$D_1^{(2)} = q^2 + q \sum_{r=0}^{\infty} q^{2r(r+1)} = q^2 + C_1^{(2)}. \tag{34}$$

If we rewrite (21) as

$$G(a, x, q) = \prod_{-\infty}^{\infty} (1 + a x^{2n} q^{n^2} + a^2 x^{4n} q^{2n^2} + \cdots)$$

it follows readily that

$$D_s^{(n)} = \sum q^{k_1 r_1^2 + \cdots + k_m r_m^2} \tag{35}$$

where the summation is over all

$$r_j = 0, \pm 1, \pm 2, \cdots; \qquad k_j = 1, 2, 3, \cdots; \qquad m = 1, 2, 3, \cdots$$

such that

$$r_1 < r_2 < \cdots < r_m, \qquad k_1 r_1 + \cdots + k_m r_m = s.$$

If we compare (23) with (2) we get

$$\sum_{r=0}^{n} (-1)^r A_r(x, q) B_r(x, q) = 0 \qquad (n \geq 1).$$

By means of this relation it is easily verified that (24) is equivalent to (4).

3. The results obtained above can be generalized by considering the product

$$\Phi(a, b, c, x, q) = \prod_{i=1}^{h} F(ab_i, x, q) \cdot \prod_{j=1}^{k} G(ac_i, x, q), \tag{36}$$

where $F(a, x, q)$, $G(a, x, q)$ are defined by (1) and (21), respectively. It follows at once from (3) and (22) that

$$\Phi(ax^2q, b, c, xq, q) = \Phi(a, b, c, x, q). \tag{37}$$

Hence if we put

$$\Phi(a, b, c, x, q) = \sum_{n=0}^{\infty} a^n A_n(b, c, x, q),$$

we have

$$x^{2n} q^n A_n(b, c, xq, q) = A_n(b, c, x, q). \tag{38}$$

Now put

$$A_n(b, c, x, q) = \sum_{r=-\infty} x^{2r} C_r^{(n)}(b, c, q). \tag{39}$$

Using (38) we get

$$C_r^{(n)}(b, c, q) = q^{2r-n} C_r^{(n)}(b, c, q). \tag{40}$$

Then as above we find that

$$C_{rn+s}^{(n)}(b, c, q) = q^{r(rn+2s)} C_s^{(n)}(b, c, q), \tag{41}$$

so that (39) becomes

$$A_n(b, c, x, q) = \sum_{s=0}^{n-1} C_s^{(n)}(b, c, q) \cdot \sum_{r=-\infty}^{\infty} x^{2rn+2s} q^{r(rn+2s)}. \tag{42}$$

Thus again, comparing (42) with (16) and (31), the only difference is in the coefficients $C_s^{(n)}(b, c, q)$, $0 \leqslant s < n$. It is evident that $A_0(b, c, x, q) = 1$ and

$$A_1(b, c, x, q) = \left(\sum_{i=1}^{h} b_i + \sum_{j=1}^{k} c_j\right) \sum_{r=-\infty}^{\infty} x^{2r} q^{r^2},$$

so that

$$C_r^{(1)}(b, c, q) = q^{r^2} \left(\sum_{i=1}^{h} b_i + \sum_{j=1}^{k} c_j\right).$$

It is not difficult to verify that $C_r^{(n)}(b, c, q)$ is a homogeneous polynomial in b_i, c_j of degree n; moreover it is symmetric in the b_i and the c_j separately. A formula like (18) can be stated but it is too complicated to be of much interest. However a somewhat simpler formula can be obtained that expresses $C_r^{(n)}(b, c, q)$ in terms of $C_t^{(s)}(q)$. For example, when no c_j are present, we have

$$\begin{aligned} C_0^{(2)}(b, *, q) &= \sum b_1 b_2 (1 + 2C_0^{(2)}(q)) + \sum b_1^2 C_0^{(2)}(q) \\ &= \sum b_1 b_2 + \left(\sum b_1\right)^2 C_0^{(2)}(q), \end{aligned}$$

$$C_1^{(2)}(b, *, q) = 2\sum b_1 b_2 C_1^{(2)}(q) + \sum b_1^2 C_1^{(2)}(q) = \left(\sum b_1\right)^2 C_1^{(2)}(q).$$

4. The product (1) can be generalized in another way. Put (cf. [1, p. 64])

$$n = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{bmatrix}, \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}, \qquad W = \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1k} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2k} \\ \vdots & & & \\ \omega_{k1} & \omega_{k2} & \cdots & \omega_{kk} \end{bmatrix},$$

where the n_j are integers while the z_j and ω_{js} are complex numbers. We also put

$$\omega_j = \begin{bmatrix} \omega_{ij} \\ \omega_{2j} \\ \vdots \\ \omega_{kj} \end{bmatrix} \qquad (j = 1, 2, \cdots, k),$$

the jth column of W. It will be assumed that W is a symmetric matrix whose imaginary part is positive.

We define

$$F(a, z, W) = \prod_n [1 + a \exp(2\pi i n' z + \pi i n' W_n)], \tag{43}$$

where n' is the transpose of n and the product is extended over all n_j from $-\infty$ to ∞.

Let e_j denote the vector whose jth component is equal to 1 while all the other components are 0. Since

$$(n + e_j)' W(n + e_j) = n' W n + 2e_j' W_n + \omega_{jj},$$

it is easily verified that

$$F(a, \exp(2\pi i z_j + \pi i \omega_{jj}), W)$$

$$= \prod_n \{1 + a \exp[2\pi i (n + e_j)' z + \pi i (n + e_j)' W(n + e_j)]\}$$

$$= \prod_n [1 + a \exp(2\pi i n' z + \pi i n' W_n)].$$

We have therefore

$$F(a, \exp(2\pi i z_j + \pi i \omega_{jj}), z + \omega_j, W) = F(a, z, W). \tag{44}$$

If we put

$$F(a, z, W) = \sum_{m=0}^{\infty} a^m A_m(z, W)$$

it follows from (44) that

$$\exp(2m\pi i z_j + m\pi i \omega_{jj}) A_m(z + \omega_j, W) = A_m(z, W). \tag{45}$$

We now put

$$A_m(z, W) = \sum_r e^{2\pi i r' z} C_r^{(m)}, \tag{46}$$

where

$$C_r^{(m)} = C_r^{(m)}(W) \qquad \text{and} \qquad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{bmatrix}.$$

The summation on the right side of (46) is over all r_j from $-\infty$ to ∞. Applying (45) to (46) we get

$$C_r^{(m)} = \exp(2\pi i r' \omega_j - m\pi i \omega_{jj}) C_{r - m e_j}^{(m)}. \tag{47}$$

It should be noted that in (45), (46), (47) m is a nonnegative integer (not a vector).

Replacing r by $rm + s$, where

$$s = \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_k \end{bmatrix},$$

(47) becomes

$$C^{(m)}_{rm+s} = \exp\,[2\pi i(rm + s')\,\omega_j - m\pi i\omega_{jj}] \cdot C^{(m)}_{rm+s-me_j}.$$

It follows that

$$C^{(m)}_{rm+s} = \exp\,(m\pi i r' W r + 2\pi i r' W s)\, C^{(m)}_s. \tag{48}$$

Substituting from (48) in (46) we get

$$A_m(z, W) = \sum_s e^{2\pi i s' z} C^{(m)}_s \cdot \sum_r \exp\,(m\pi i r' W r + 2\pi i r' W s + 2m\pi i r' z);$$

the outer sum is over $s_1, \cdots, s_k = 0, 1, \cdots, m - 1$. If we put

$$\vartheta(z, W) = \sum_r \exp\,(2\pi i r' z + \pi i r' W r),$$

it follows that

$$A_m(z, W) = \sum_s e^{2\pi i s' z} C^{(m)}_s\ \vartheta(mz + Ws, mW). \tag{49}$$

Therefore

$$F(a, z, W) = \sum_{m=0}^{\infty} \sum_s e^{2\pi i s' z} C^{(m)}_s\ \vartheta(mz + Ws, mW). \tag{50}$$

Reference

1. Bellman, R. "A Brief Introduction to Theta Functions." Holt, Rinehart and Winston, New York, 1961.