Short containers in Cayley graphs

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A B S T R A C T

The star diameter of a graph measures the minimum distance from any source node to several other target nodes in the graph. For a class of Cayley graphs from abelian groups, a good upper bound for their star diameters is given in terms of the usual diameters and the orders of elements in the generating subsets. This bound is tight for several classes of graphs including hypercubes and directed n-dimensional tori. The technique used is the so-called disjoint ordering for a system of subsets, due to Gao, Novick and Qiu [S. Gao, B. Novick, K. Qiu, From Hall’s matching theorem to optimal routing on hypercubes, J. Comb. Theory B 74 (1998) 291–301].

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1. Introduction

A graph models a communication network, for example a computer system, a parallel computer, or a telephone system. A node of the graph represents a processor or a switch, and an edge corresponds to a link between two processors or switches. In several applications, it is desirable to send messages from one node to several other nodes simultaneously in the network in minimum delay time. This applies in particular to Rabin’s information dispersal algorithm (IDA) [1] for efficient and accurate transmission of large files in a parallel computer or a distributed network. This motivates us to studying the star diameter of a graph, which measures the minimum delay time.

Suppose G is a graph (without self-loops and multiple edges). Let w be a positive integer. For any vertices x, y1, . . . , yw of G with x ̸= yi, 1 ≤ i ≤ w, a \textit{w-star container} from x to y1, . . . , yw is a collection of w internally node-disjoint paths from x to y1, . . . , yw, one for each yi. Note that the vertices y1, . . . , yw may have repetition, thus if y1 appears r times then the container has r disjoint paths from x to y1. Here and throughout the paper, by “disjoint paths” we mean “internally node-disjoint paths”. In the case that y1 = ··· = yw = y, a \textit{w-star container} is also called a \textit{w-wide container} from x to y. The length of a container is the maximum length of its paths. The \textit{w-star distance} from x to y1, . . . , yw, denoted by d(x; y1, . . . , yw), is the minimum length among all the w-star containers from x to y1, . . . , yw. When there are no disjoint paths from x to y1, . . . , yw, we define d(x; y1, . . . , yw) = ∞. When y1 = ··· = yw = y, d(x; y1, . . . , yw) is simply denoted as dw(x, y). The \textit{w-wide diameter} [2] of G, denoted by dw(G), is defined to be the maximum of dw(x, y) for all pairs of distinct vertices x and y in G. The \textit{w-star diameter} of G, denoted by Dw(G), is defined to be the maximum of d(x; y1, . . . , yw) for all vertices x, y1, . . . , yw (possibly with repetition) of G with x ̸= yi, 1 ≤ i ≤ w. Certainly, dw(G) ≤ Dw(G).

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Note that $D_1(G)$ is just the usual diameter of $G$. Obviously, $D_1(G) \leq D_2(G) \leq \cdots \leq D_w(G) \leq \cdots$. If $G$ has connectivity $k$, then Menger’s theorem implies that $D_w(G) < \infty$ if $w \leq k$. A natural question is to quantize Menger’s theorem, that is, to give a good bound on $D_k(G)$.

The above definition of $w$-star diameter is slightly different from that in the literature [2] where it is required that the target nodes be distinct. The benefit of our definition is that the $w$-star diameter bounds both the star diameter in [2] and the wide diameter $d_w(G)$, thus allows a uniform treatment for these two parameters. For more information on containers and wide diameters, see [3–13]. In general, it seems more difficult to determine star diameters than wide diameters due to the possibly complicated configuration of the target nodes.

The concept of star diameter applies to both directed and undirected graphs. We view undirected graphs as special cases of directed graphs where each undirected edge represents two directed edges one in each direction.

In this paper, we study a class of Cayley graphs that are defined over abelian groups. We give a good upper bound for their star diameters in terms of the usual diameters and the orders of the elements in the generating subsets. This bound is tight for several classes of graphs including hypercubes and directed $n$-dimensional tori.

The rest of the paper is organized as follows. In the next section, we define Cayley graphs and state our main results. In Section 3, we present the concept of disjoint ordering for a system of finite sets and the related results from Gao et al. [14], together with some minor improvements. These results will be useful for construction of short disjoint paths later. Section 4 is the technical part of the paper where we show how to construct the desired containers in Cayley graphs from abelian groups via disjoint ordering of sets and thus proves our main results. We conclude in Section 5 with some comments and open problems for future studies.

2. Main results

Let $G$ be a group with its binary operation written multiplicatively, and let $S$ be a subset of $G$ not containing the identity element 1. The Cayley graph $\Gamma(G, S)$ is defined to be the (directed) graph whose vertices are the elements of $G$ and, for $x, y \in G$, there is an edge $x \rightarrow y$ iff $x \cdot g = y$ for some $g \in S$. When $S$ contains the inverses of all its elements, the Cayley graph $\Gamma(G, S)$ is an undirected graph.

For example, the $n$-dimensional hypercube $H_n$ has a vertex set $Z_2^n = \{(a_1, \ldots, a_n) : a_i = 0 \text{ or } 1\}$ and two vertices are adjacent if and only if they differ by exactly one coordinate. This is an undirected graph and can be viewed as a Cayley graph as follows. We know that $G = Z_2^n$ is a group under componentwise addition modulo 2. Take $S$ to be the set of unit vectors $s_i = (0, \ldots, 1, \ldots, 0)$ of length $n$, which is 1 at ith position and 0 otherwise, $1 \leq i \leq n$. Then the Cayley graph $\Gamma(G, S)$ is the hypercube $H_n$.

An $n$-dimensional torus is a generalized hypercube. For a positive integer $m$, $Z_m = \{0, 1, \ldots, m - 1\}$ denotes the ring of integers modulo $m$, a cyclic group of order $m$ under addition. Let $m_1, \ldots, m_n$ be integers $\geq 2$. Define

$$H(m_1, \ldots, m_n) = Z_{m_1} \times \cdots \times Z_{m_n},$$

the set of all $n$-tuples $(a_1, \ldots, a_n)$ with $a_i \in Z_{m_i}$ for $1 \leq i \leq n$. Note that $G = H(m_1, \ldots, m_n)$ is a group under componentwise addition. Let $S$ be the set of unit vectors as above. Then the Cayley graph $\Gamma(G, S)$ is called a directed $n$-dimensional torus. Let $S_1 = S \cup \{-S\}$. Then $\Gamma(G, S_1)$ is the undirected version of $\Gamma(G, S)$ and is simply called an $n$-dimensional torus. Note that an $n$-dimensional torus is also called a generalized hypercube or a toroidal mesh in the literature. When $m_1 = \cdots = m_n = k$, it is also called a $k$-ary $n$-cube.

The groups used in hypercube and torus graphs above are abelian. There is a large literature on Cayley graphs from nonabelian groups, see [15–19,8,11,13] for more information. In this paper, we shall focus mainly on Cayley graphs over abelian groups.

Let $G$ be any finite group, written multiplicatively. An ordered subset $B = \{b_1, \ldots, b_n\}$ is called a generating basis, or simply a basis, of $G$ if each element $g \in G$ can be written as a product

$$g = b_1^{e_1}b_2^{e_2}\cdots b_n^{e_n},$$

and the $e_i$’s are unique in the range $0 \leq e_i < e_i, 1 \leq i \leq n$, where $e_i$ is the order of $b_i$ (that is, $e_i$ is the smallest positive integer such that $b_i^{e_i} = 1$). If such a basis exists then $G$ has exactly $e_1e_2\cdots e_n$ elements. For example, the unit vectors form a generating basis for $Z_2^n$. For another example, consider the additive group of $Z_{30}$. Then the subset $\{1\}$ is a generating basis for $Z_{30}$, as $1$ has additive order $30$ in $Z_{30}$. Also, the subsets $\{4, 15\}, \{6, 10, 15\}$ and $\{12, 15, 20\}$ are generating bases of $Z_{30}$ for its additive group.

Theorem 1. Let $G$ be an abelian group and $S$ a subset of $G$ not containing the identity. Suppose $B \subseteq S \subseteq B \cup B^{-1}$ for some generating basis $B$ of $G$. Denote by $k$ the cardinality of $S$ and $e$ the maximum order of elements in $S \cap B^{-1}$ ($e = 1$ when $S \cap B^{-1}$ is empty). Then the Cayley graph $\Gamma(G, S)$ has connectivity $k$ and

$$D_k(\Gamma(G, S)) \leq \begin{cases} d + 1, & \text{if } e \leq 2 \\ d + \lfloor (e - 1)/2 \rfloor, & \text{if } e > 2 \end{cases},$$

where $d$ is the usual diameter of $\Gamma(G, S)$. 

Consider the special case when \( S = B \). If all elements in \( B \) have order 2, the graph \( \Gamma (G, B) \) is the \( n \)-dimensional hypercube and has diameter \( d = n \). In this case, the upper bound is tight as the \( n \)-star diameter is known to be \( d + 1 \) [1]. If all elements in \( B \) have order larger than 2, then \( \Gamma (G, B) \) is a directed \( n \)-dimensional torus. We will show that the star diameter is \( d + 1 \), so the bound is again tight.

**Corollary 2** (Directed \( n \)-dimensional Torus). Let \( G \) be an abelian group with a generating basis \( B \) of \( n \) elements. Then the (directed) Cayley graph \( \Gamma (G, B) \) has connectivity \( n \) and

\[
D_n (\Gamma (G, B)) = d + 1
\]

where \( d \) is the diameter of \( \Gamma (G, B) \).

On the other extreme, consider \( S = B \cup B^{-1} \). Then \( \Gamma (G, S) \) is undirected.

**Corollary 3** (Undirected \( n \)-dimensional Torus). Let \( G \) be an abelian group with a basis \( B \) with \( n \) elements and \( S = B \cup B^{-1} \). Let \( e \) be the maximum order of elements in \( B \). Suppose each element in \( B \) has order \( > 2 \) (so \( e > 2 \)). Then the Cayley graph \( \Gamma (G, S) \) has connectivity \( 2n \) and

\[
D_{2n} (\Gamma (G, S)) \leq d + [(e - 1)/2]
\]

where \( d \) is the diameter of \( \Gamma (G, S) \).

### 3. Disjoint ordering

The concept of disjoint ordering for a collection of subsets was introduced by Gao, Novick and Qiu [14]. We describe the definition and the main result from [14]. We also give some minor improvements that will be useful for the construction in the next section.

A permutation of the elements of a finite set is called an ordering. Suppose \( X \) and \( Y \) are two sets ordered as \( O_1 = (x_1, x_2, \ldots, x_k) \) and \( O_2 = (y_1, y_2, \ldots, y_\ell) \) where \( k = |X| \) and \( \ell = |Y| \). We say that \( O_1 \) and \( O_2 \) are disjoint if for every \( 1 \leq t \leq \min(k, \ell) \)

\[
\{x_1, x_2, \ldots, x_t\} \neq \{y_1, y_2, \ldots, y_t\}
\]

as sets, except for \( t = k = \ell \). Note that \( X \) and \( Y \) may be the same set which is why we need to exclude the case \( t = k = \ell \). For instance, if \( X = Y = \{1, 2, 3\} \) then \((1, 2, 3)\) and \((2, 3, 1)\) are disjoint but \((1, 2, 3)\) and \((2, 1, 3)\) are not. Also, if \( X = Y = \{1\} \) then the trivial ordering \((1)\) is disjoint to itself.

A collection of finite sets is said to have a disjoint ordering if each set has an ordering and all the orderings are pairwise disjoint. In particular, as long as all singletons in the collection are distinct, the elements in the first position of a disjoint ordering form a system of distinct representatives. So for a disjoint ordering to exist, the conditions in Hall’s matching theorem [20] must be satisfied. The converse is also true.

**Theorem 4** (Gao et al. [14]). For any finite collection of nonempty finite sets in which all singletons are distinct, there is a disjoint ordering if and only if there is a system of distinctive representatives.

Recall that a system of distinctive representatives (SDR) for \( k \) sets consists of \( k \) distinct elements with one from each set. A partial SDR is an SDR for a subcollection of the sets. When an SDR does not exist, one needs to add elements to the sets so that an SDR and thus disjoint ordering exists. By using this technique, Gao et al. [14] show how to construct short containers on hypercube graphs. In the next section, we adapt this method to a class of Cayley graphs over abelian groups.

For the construction of short containers in the next section we need disjoint ordering under further constraints as specified by the following lemmas.

**Lemma 5.** Let \( X_1, \ldots, X_w \) be subsets of a finite set \( S \) where \( w \leq k = |S| \). Suppose \( t_i \in X_i \), \( 1 \leq i \leq m \), form a partial SDR of maximum size. Pick any distinct elements \( t_i \in S \setminus \{t_1, \ldots, t_m\} \), \( m < i \leq w \). Then, for any disjoint ordering of the system

\[
X_1, \ldots, X_m, X_{m+1} \cup \{t_{m+1}\}, \ldots, X_w \cup \{t_w\}
\]

the element \( t_i \) must be the initial element in the ordering of \( X_i \cup \{t_i\} \) for all \( m < i \leq w \).

**Proof.** Suppose for some \( i > m \) the initial element \( a \) in the ordering of \( X_i \cup \{t_i\} \) is different from \( t_i \). Then \( a \in X_i \). Note that the initial elements of the ordering form an SDR for the system \((1)\). Particularly, \( X_1, \ldots, X_m \) have representatives different from \( a \). This means that the sets \( X_1, \ldots, X_m, X_i \) have an SDR, contradicting the maximality of \( m \). \( \square \)

**Lemma 6.** Let \( S = \{g_1, \ldots, g_k\} \) be any finite set and \( X_i \subseteq S \), \( 1 \leq i \leq w \). For each pair \( 1 \leq i \leq w \) and \( 1 \leq j \leq k \), let there be a real number \( e_{ij} \). Suppose the system \( X_1, \ldots, X_w \) has an SDR. Then there is a disjoint ordering for the system satisfying the following condition:

Let \( e_{ij} \) be the last element in the ordering of \( X_i \), \( 1 \leq i \leq w \). For any pair \( 1 \leq i < j \leq w \) with \( X_i = X_j \), if \( e_{ir}(i) \geq e_{jr}(i) \) and \( e_{ir}(j) \geq e_{jr}(j) \) then \( e_{ir}(i) = e_{jr}(i) \) and \( e_{ir}(j) = e_{jr}(j) \).

**Proof.** Suppose for some \( i > m \) the initial element \( a \) in the ordering of \( X_i \cup \{t_i\} \) is different from \( t_i \). Then \( a \in X_i \). Note that the initial elements of the ordering form an SDR for the system \((1)\). Particularly, \( X_1, \ldots, X_m \) have representatives different from \( a \). This means that the sets \( X_1, \ldots, X_m, X_i \) have an SDR, contradicting the maximality of \( m \). \( \square \)
Proof. By Theorem 4, the system \( X_i, 1 \leq i \leq w \), has a disjoint ordering, say \( O_i \), for the ordering of \( X_i, 1 \leq i \leq w \). We show how to rearrange the ordering so that the condition in the lemma is satisfied. Suppose it is violated by some pair \( i_0 \) and \( j_0 \) with \( X_{i_0} = X_{j_0} \). We consider all the sets \( X_i \)'s that are equal to \( X_{i_0} \). For convenience of notation, we may assume that they are \( X_1, \ldots, X_m \) for some \( 1 < m \leq w \). That is, \( X_1 = \cdots = X_m \neq X_j \) for \( m < j \leq w \). Let \( g_{u_i} \) be the last element in the ordering \( O_i \) of \( X_i \) where \( 1 \leq u_i \leq k \) and \( 1 \leq i \leq m \). Take any bijection

\[ \eta : \{1, \ldots, m\} \rightarrow \{u_1, \ldots, u_m\} \]

(the latter is viewed as a multiset) that minimizes (among all the bijections) the sum \( \sum_{i=1}^{m} e_{\eta(i)} \). We claim that, for any pair \( 1 \leq i < j \leq m \),

\[ e_{\eta(i)} \geq e_{\eta(j)} \quad \text{and} \quad e_{\eta(j)} \geq e_{\eta(i)} \]

then \( e_{\eta(i)} = e_{\eta(j)} \) and \( e_{\eta(j)} = e_{\eta(i)} \). Suppose otherwise, namely, one of the inequalities is strict. Then

\[ e_{\eta(i)} + e_{\eta(j)} > e_{\eta(i)} + e_{\eta(j)} \]

Switching the values \( \eta(i) \) and \( \eta(j) \) of \( \eta \) would yield a bijection with a smaller sum, contradicting the choice of \( \eta \).

Now we rearrange the orderings \( O_1, \ldots, O_m \) as follows. Suppose \( \eta(i) = u_{\tau(i)} \), for \( 1 \leq i \leq m \), where \( \tau(1), \ldots, \tau(m) \) is a permutation of \( 1, \ldots, m \). This means that \( \eta(i) \) is the index of the last element in the ordering \( O_{\tau(i)} \) of \( X_{\tau(i)} \). To get the desired new ordering of the system, let \( O_{\tau(i)} \) be the new ordering of \( X_i \) for \( 1 \leq i \leq m \), with the orderings of other sets \( X_i, i > m \), unchanged. Then the condition in the lemma is satisfied for all pairs \( 1 \leq i < j \leq m \). Certainly, the new ordering for the system \( X_i, 1 \leq i \leq w \), is still disjoint and no new violating pairs are introduced. Repeat this process if the condition in the lemma is violated by any other pair among \( X_{m+1}, \ldots, X_k \). The condition is satisfied after finitely many steps. \( \square \)

4. Short containers

Let \( G \) be a group and \( S \) a subset of \( G \) not containing the identity 1. Suppose \( S \) generates \( G \) as a group. Then the Cayley graph \( \Gamma(G, S) \) is connected and the left multiplication by any element of \( G \) induces an automorphism of \( \Gamma(G, S) \). Hence \( \Gamma(G, S) \) is vertex transitive. This implies in particular that, for any two vertices \( x \) and \( y \), the set of all the paths from \( x \) to \( y \) in \( \Gamma(G, S) \) is in 1-1 correspondence to that from 1 to \( x^{-1}y \) with length preserved. Similarly, for any \( y_1, \ldots, y_w \), the star containers from \( x \) to \( y_1, \ldots, y_w \) are in 1-1 correspondence with those from 1 to \( x^{-1}y_1, \ldots, x^{-1}y_w \), with lengths preserved. Because of this correspondence, we only discuss below how to construct short \( w \)-star containers that start at 1.

Let \( y \in G \). Suppose \( y \) is represented as

\[ y = g_1g_2 \cdots g_{\ell}, \quad g_i \in S. \]

Then there is a natural induced path from 1 to \( y \):

\[ 1 \cdot \underbrace{g_1} \cdot \underbrace{g_2} \cdot \cdots \underbrace{g_{\ell}} \Rightarrow \cdot y. \]

Note that the number \( \ell \) of elements in \( y \) is equal to the length of the induced path. We call \( \ell \) the length of the representation of \( y \), denoted by \( |y| \). Let \( y_1 = g_1g_2 \cdots g_{\ell}, \) and \( y_2 = h_1h_2 \cdots h_{\ell} \) be two representations where \( g_i, h_j \in S \). We say that \( y_1 \) and \( y_2 \) are disjoint if their induced paths are disjoint, namely,

\[ g_1 \cdots g_{\ell} \neq h_1 \cdots h_{\ell} \]

as elements of \( G \), for all \( 1 \leq i \leq \ell \) and \( 1 \leq j \leq k \), except when \( i = \ell \) and \( j = k \). The exception allows \( y_1 \) and \( y_2 \) being the same vertex of \( \Gamma(G, S) \).

When \( G \) is abelian, one can change the order of the elements in \( y \) in any fashion, and \( y \) is still the same element of \( G \) (thus the same node of \( \Gamma(G, S) \)) but the induced path will likely be different. It is exactly this flexibility of reordering that allows us to construct short \( w \)-containers in \( \Gamma(G, S) \). In the following, we view a product representation of \( y \) as ordered and identify it with its induced path from 1 to \( y \). It should be clear from the context whether \( y \) is viewed as an element of \( G \) (thus a node of \( \Gamma(G, S) \)) or a path from 1 to \( y \).

We assume from now on that \( G \) is abelian and \( B \subseteq S \subseteq B \cup B^{-1} \) for some basis \( B \) of \( G \). For convenience of discussion, we fix that

\begin{equation}
B = \{b_1, \ldots, b_n\} \quad \text{and} \quad S = \{b_1, b_1^{-1}, \ldots, b_s, b_s^{-1}, b_{s+1}, \ldots, b_n\}
\end{equation}

where \( b_i \neq b_i^{-1} \) for \( 1 \leq i \leq s \), and for \( s < i \leq n \), either \( b_i = b_i^{-1} \) or \( b_i^{-1} \notin S \). Denote by \( e_i \) the order of \( b_i \) for \( 1 \leq i \leq n \).

Since \( B \) is a basis of \( G \), any \( y \in G \) can be written uniquely as \( y = b_1^{\ell_1} \cdots b_n^{\ell_n} \) where \( 0 \leq \ell_i < e_i \) for \( 1 \leq i \leq n \). When \( b_i^{-1} \in S \), we may replace \( b_i^{\ell_i} \) by \( b_i^{-\ell_i} \), which yields a shorter path if \( e_i - \ell_i < \ell_i \). So \( y \) is better written in the form

\begin{equation}
y = b_1^{\ell_1} \cdots b_n^{\ell_n}
\end{equation}
where
\[-\frac{e_i}{2} < \ell_i \leq \frac{e_i}{2}, \quad \text{if } 1 \leq i \leq s \]
\[0 \leq \ell_i < e_i, \quad \text{if } s < i \leq n.\] (4)
(5)

It is straightforward to check that this representation of \( y \) is unique; that is, different values of the \( \ell_i \)'s in (4) and (5) give different \( y \)'s in (3) as elements of \( G \).

**Lemma 7.** Suppose that \( y \) is written in the form (3)–(5). Then the distance from 1 to \( y \) in \( \Gamma(G, S) \) is \( d(1, y) = \sum_{i=1}^{n} |\ell_i| \).

**Proof.** Certainly, the induced path of \( y \) has length \( \sum_{i=1}^{n} |\ell_i| \). Suppose that \( P \) is any path from 1 to \( y \) in \( \Gamma(G, S) \). We need to show that \( |P| \geq \sum_{i=1}^{n} |\ell_i| \). The path \( P \) corresponds to writing \( y \) as a product of elements in \( S \). Since \( G \) is abelian, we may reorder the elements in the product and write \( y \) in the following form
\[
y = b_1^{\ell_1}(b_1^{-1})^{v_1} \cdots b_s^{\ell_s}(b_s^{-1})^{v_s} b_{s+1}^{\ell_{s+1}} \cdots b_n^{\ell_n}
\]
where \( u_i \) and \( v_i \) are nonnegative integers counting the number of times \( b_i \) and \( b_i^{-1} \) are used, respectively, in forming the edges of \( P \). Reducing the exponents of \( b_i \) modulo \( e_i \), appropriately, we can write \( y \) as
\[
y = b_1^{\ell_1} \cdots b_n^{\ell_n}
\]
where \( \ell_i \) satisfy (4) and (5). The length \( \sum_{i=1}^{n} |\ell_i| \) is never larger than \( |P| = u_1 + v_1 + \cdots + u_s + v_s + u_{s+1} + \cdots + u_n \). By the uniqueness of the representation of \( y \) in (3)–(5), we have that \( \ell_i = \ell_i \) for \( 1 \leq i \leq n \). Therefore \( |P| \geq \sum_{i=1}^{n} |\ell_i| \) as desired. \( \square \)

**Corollary 8.** Let \( S \) be as in (2). The diameter of \( \Gamma(G, S) \) is
\[
d = \sum_{i=1}^{s} |e_i/2| + \sum_{i=s+1}^{n} (e_i - 1).\]

**Proof.** Since \( G \) is vertex transitive, we just need to compare \( d(1, y) \) for \( y \in G \). The corollary follows from Lemma 7. \( \square \)

A representation \( y = \prod_{i=1}^{t} g_{i}^{\ell_{i}} \), where \( g_i \in S \) and \( \ell_i \geq 0 \), is said to be **minimal** if \( \sum_{i=1}^{t} |\ell_i| \) is equal to the distance from 1 to \( y \) in \( \Gamma(G, S) \). By Lemma 7, the representation of \( y \) in (3)–(5) is a minimal representation by rewriting \( b_1^{\ell_1} = (b_1^{-1})^{-\ell_1} \) if \( \ell_1 < 0 \). Thus we also call (3)–(5) a minimal representation of \( y \). Note that a minimal representation may not be unique. For instance, if \( b_1 \) has order 2, for some \( \ell > 1 \) and if \( b_1, b_1^{-1} \in S \) then \( b_1^{\ell} = (b_1^{-1})^{-\ell} \) are both minimal but \( b_1 \neq b_1^{-1} \). In any case, a representation \( y = \prod_{i=1}^{t} g_{i}^{\ell_{i}} \), where \( g_i \in S \) and \( \ell_i \geq 0 \), is minimal iff the following two conditions are satisfied:

(a) \( 0 \leq \ell_i \leq \tilde{e}_i - 1 \) where \( \tilde{e}_i \) denotes the order of \( g_i \), \( 1 \leq i \leq t \); and
(b) \( g_1, \ldots, g_t \) are distinct elements in \( S \), and if both \( g_i \) and \( g_i^{-1} \) are in \( S \) then \( \ell_i \leq \tilde{e}_i/2 \) and only one of \( g_i, g_i^{-1} \) appears in the list \( g_1, \ldots, g_t \).

A minimal representation \( y = \prod_{i=1}^{t} g_{i}^{\ell_{i}} \) is said to be **canonical** with respect to the basis \( B \) if the following condition is satisfied:

(c) if \( g_i \notin B \) then \( \ell_i < \tilde{e}_i/2, \ 1 \leq i \leq t \).

The minimal representation \( y \) in (3)–(5) is certainly canonical. By the proof of Lemma 7, any canonical minimal representation can be obtained from (3) by permuting the elements \( b_i \)'s. So canonical minimal representation is unique up to order.

We next define the supports of elements in \( G \). For any element \( y \in G \), write \( y \) in a canonical minimal representation
\[
y = \prod_{i=1}^{t} g_{i}^{\ell_{i}} \quad \text{where } g_i \in S \text{ and } \ell_i \geq 0.
\]
The **support** of \( y \) is defined to be
\[
\text{Supp}(y) = \{ g_i : \ell_i > 0, \ 1 \leq i \leq t \}
\]
which is a subset of \( S \). For example, assuming that \( b_1 \) has order 5 and \( b_1, b_1^{-1} \in S \), we have \( \text{Supp}(b_1^3) = \{b_1\} \) but \( \text{Supp}(b_1^5) = \{b_1^{-1}\} \), as \( b_1^5 = (b_1^{-1})^2 \). Also, \( \text{Supp}(b_2 b_1 b_2^{-1}) = \text{Supp}(b_1) \). Certainly, if \( y \) is of the form (3)–(5) then
\[
\text{Supp}(y) = \{ b_i : \ell_i > 0 \} \cup \{ b_i^{-1} : \ell_i < 0 \}.
\]
Lemma 9. Let $x = g_1^{u_1} \cdots g_s^{u_s}$ and $y = h_1^{v_1} \cdots h_t^{v_t}$ be two canonical minimal representations with $\text{Supp}(x) = \{g_1, \ldots, g_s\}$ and $\text{Supp}(y) = \{h_1, \ldots, h_t\}$. Suppose that $g_i \neq h_1$ and the ordering $(g_1, \ldots, g_s)$ of $\text{Supp}(x)$ is disjoint from the ordering $(h_1, \ldots, h_t)$ of $\text{Supp}(y)$. Then the induced paths of $x$ and $y$ are internally node disjoint, provided that the condition in Lemma 6 is satisfied, namely, if $\text{Supp}(x) = \text{Supp}(y)$ and $g_i = h_m$ for some $m < s$ and $h_t = g_r$ for some $r < t$, then $u_i \leq v_m$ and $v_t \leq u_r$ imply that $u_i = v_m$ and $v_t = u_r$.

**Proof.** A node, other than 1, on the induced path of $x$ is of the form

$$x_1 = g_1^{u_1} \cdots g_{i-1}^{u_{i-1}} g_i^{u_i}$$

for some $1 \leq i \leq s$ and $1 \leq u_i \leq u$. Similarly a node, other than 1, on the induced path of $y$ is of the form

$$y_1 = h_1^{v_1} \cdots h_{j-1}^{v_{j-1}} h_j^{v_j}$$

for some $1 \leq j \leq t$ and $1 \leq v_j \leq v$. Then $x_1$ and $y_1$ are both canonical minimal representations with

$$\text{Supp}(x_1) = \{g_1, \ldots, g_i\} \quad \text{and} \quad \text{Supp}(y_1) = \{h_1, \ldots, h_j\}.$$

Suppose that $x_1 = y_1$. Since canonical minimal representation is unique up to order, we have $\text{Supp}(x_1) = \text{Supp}(y_1)$ and the exponents of the $g$'s and $h$'s must be equal accordingly. Hence $\{g_1, \ldots, g_i\} = \{h_1, \ldots, h_j\}$, thus $i = j$. However, since $(g_1, \ldots, g_s)$ is disjoint from $(h_1, \ldots, h_t)$, we have $i = s$ and $j = t$, so $i = j = s = t$. Since $g_1 \neq h_1$, we have $s = t > 1$. Also, since $\{g_1, \ldots, g_{i-1}\} \neq \{h_1, \ldots, h_{j-1}\}$, we see that $g_i \neq h_t$. Thus $g_i = h_m$ for some $m < t$ and $h_t = g_r$ for some $r < t$. Comparing the exponents of $g_i$ and $h_t$, in $x_1$ and $y_1$, we have $u_i \geq u = v_m$ and $v_t \geq v = u_r$. The condition of the lemma implies that $u_i = v_m$ and $v_t = u_r$. Therefore, $x_1$ and $y_1$ are not internal nodes, and the induced paths of $x$ and $y$ are node disjoint. □

We define a partial ordering on the elements of $G$, which is needed in the proof of the next theorem. Let $y_1, y_2 \in G$. Represent them in canonical minimal form, say

$$y_1 = g_1^{u_1} \cdots g_t^{u_t}, \quad y_2 = g_1^{v_1} \cdots g_t^{v_t}$$

where $g_i \in S$, $u_i \geq 0$ and $v_i \geq 0$. We say that $y_1 \prec y_2$ if $u_i \leq v_i$ for $1 \leq i \leq t$. We note that if $y_1 \prec y_2$ and $y_1 \neq y_2$ then $|y_1| < |y_2|$.

**Theorem 10.** Let $B$ and $S$ be as in (2) where $B$ is a generating basis of $G$. Let $x, y_1, \ldots, y_w$ be any vertices of $\Gamma(G, S)$ with $x \neq y_1, 1 \leq i \leq w$. Suppose that $d_i$ is the distance from $x$ to $y_i$, $1 \leq i \leq w$. Then there is a container from $x$ to $y_1, \ldots, y_w$ with the path from $x$ to $y_i$ having length at most $d_i + \tilde{e}$ where $\tilde{e} = \max\{e_1, \ldots, e_n\}$.

**Proof.** Since $\Gamma(G, S)$ is vertex transitive, we may assume that $x = 1$, the identity of $G$. Write $y_i$ in the form (3)–(5):

$$y_i = h_1^{v_1} b_2^{v_2} \cdots b_t^{v_t}, \quad 1 \leq i \leq w.$$ 

Then, by Lemma 7, $d_i = |y_i| = \sum_{j=1}^{m} e_j$. Let $X_i = \text{Supp}(y_i)$. The system of subsets $X_i$, $1 \leq i \leq w$, has a partial SDR of maximum size, say $m$. Without loss of generality, we may assume that $t_1 \in X_1, \ldots, t_m \in X_m$ is such a maximum partial SDR. We may further assume that the total length $\sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} d_i$ is minimum among all such maximum SDRs. The later condition implies the following:

(A) There is no $j > m$ and $i \leq m$ such that $y_j < y_i, y_j \neq y_i$, and the system

$$X_1, \ldots, X_{i-1}, X_j, X_{i+1}, \ldots, X_m$$

has an SDR of size $m$.

If this condition were not satisfied, we could replace $X_i$ by $X_j$, and we would still have a maximal SDR for the original system with a smaller total length.

Let $S_0 = S \setminus \{t_1, \ldots, t_m\}$. Since $t_i \in X_i, 1 \leq i \leq m$, form a maximal partial SDR, we have

$$S_0 \cap X_j = \emptyset, \quad m < j \leq w.$$ 

(6)

We want to enlarge each $X_j, m < j \leq w$, by one element from $S_0$. Since complication arises when $S_0^{-1} \cap X_j \neq \emptyset$, we need to be careful. Here $S_0^{-1} = \{t^{-1} : t \in S_0\}$. Define

$$Z_j = S_0^{-1} \cap X_j, \quad m < j \leq w.$$ 

If there are empty sets among them, just discard them. Among all the maximal partial SDRs for the system $Z_j, m < j \leq w$, we take one that maximizes the total length of the $y_j$'s where $Z_j$ have representatives. For convenience of notation, we assume that

$$t_{\ell}^{-1} \in Z_\ell, \quad m_0 < \ell \leq w$$

is such a maximal SDR where $m_0 \geq m$. We claim that the following condition is satisfied:
(B) There is no pair $j$ and $\ell$ with $m < j \leq m_0$ and $m_0 < \ell \leq w$ such that
\[ t_{\ell}^{-1} \in X_j \quad \text{but} \quad y_{\ell} < y_j, \quad y_{\ell} \neq y_j. \]

If (B) is not satisfied for some $j$, $\ell$, we can always let $t_{\ell}^{-1}$ represent $Z_j$ instead of $Z_{t_{\ell}}$. Then the total length of the $y_i$'s with representatives increases by at least one, contradicting the choice of the $t_{\ell}$'s.

Furthermore, we show that the representatives for $Z_j$'s can be chosen so that the following condition is satisfied:

(C) For any pair $m_0 < i < j \leq w$ with
\[ \{t_i^{-1}, t_j^{-1}\} \subseteq X_i \cap X_j, \]
let $u_i$, $u_j$, $v_i$, $v_j$ be the exponents of $t_i^{-1}$, $t_j^{-1}$ in the representations of $y_i$ and $y_j$, namely,
\[ y_i = \cdots (t_i^{-1})^{u_i} (t_j^{-1})^{v_i}, \quad y_j = \cdots (t_j^{-1})^{v_j} (t_j^{-1})^{v_j}. \]

Then $u_i \leq u_j$ and $v_j \leq v_i$ imply that $u_i = u_j$ and $v_i = v_j$.

When (C) is not satisfied, we can switch the representatives so that $t_i^{-1}$ represents $Z_j$ and $t_j^{-1}$ represents $Z_l$. The total sum of the exponents of the representatives increases by at least one. Repeat this process if necessary. Then (C) must be satisfied by the resulting SDR.

Hence we have $t_{\ell} \in S_0$ with $t_{\ell}^{-1} \in X_{t_{\ell}}$, $m_0 < \ell \leq w$. By the maximality of the SDR for the system $Z_j$'s, we have
\[ Z_j \subseteq \{t_{m_0+1}, \ldots, t_w^{-1}\}, \quad m < j \leq m_0. \]
Thus
\[ t_{\ell} \in S_0 \setminus \{t_{m_0+1}, \ldots, t_w\}, \quad t^{-1} \not\in X_j, \quad \text{for all } m < j \leq m_0. \quad (7) \]

Finally, pick distinct $t_{\ell} \in S_0 \setminus \{t_{m_0+1}, \ldots, t_w\}$, $m < j \leq m_0$. By (6) and (7), we have $w$ distinct elements $t_i \in S$, $1 \leq i \leq w$, satisfying the following:
\[ t_i \in X_i, \quad \text{if } 1 \leq i \leq m \]
\[ t_i \not\in X_i, \quad \text{if } m < i, j \leq w \]
\[ t_{\ell}^{-1} \not\in X_j, \quad \text{if } m < i, j \leq m_0 \]
\[ t_{\ell}^{-1} \in X_i, \quad \text{if } m_0 < i \leq w. \]
\[ (8) \quad (9) \quad (10) \quad (11) \]

Also, we already know that the conditions (A), (B) and (C) are satisfied.

Now we are ready to construct the container required by the theorem. Suppose that
\[ y_i = \tilde{y}_i t_{\ell}^{-1} \quad m_0 < i \leq w \]
where $\tilde{y}_i$ is in canonical minimal form and does not contain any power of $t_i$. Also, let $\epsilon_i$ be the order of $t_i$ for $1 \leq i \leq w$. We modify the expressions of $y_i$'s as follows. Define
\[ \tilde{y}_i = y_i, \quad \epsilon_i = 1 \quad \text{if } 1 \leq i \leq m \]
\[ \tilde{y}_i = t_i y_i, \quad \epsilon_i = t_i^{-1} \quad \text{if } m < i \leq m_0 \quad \text{and} \quad t_{\ell}^{-1} \in S \]
\[ \tilde{y}_i = t_i y_i, \quad \epsilon_i = t_i^{\ell_i - 1} \quad \text{if } m < i \leq m_0 \quad \text{and} \quad t_{\ell}^{-1} \not\in S \]
\[ \tilde{y}_i = t_i y_i, \quad \epsilon_i = t_i^{\ell_i - u_i - 1} \quad \text{if } m_0 < i \leq w. \]
\[ (12) \quad (13) \quad (14) \quad (15) \quad (16) \]

Certainly, the $\tilde{y}_i$'s are in canonical minimal form and
\[ y_i = \tilde{y}_i \epsilon_i, \quad 1 \leq i \leq w. \]

Let $\tilde{X}_i = \text{Supp}(\tilde{y}_i)$, $1 \leq i \leq w$. Then
\[ \tilde{X}_i = X_i \quad \text{if } 1 \leq i \leq m \]
\[ \tilde{X}_i = X_i \cup \{t_i\} \quad \text{if } m < i \leq m_0 \]
\[ \tilde{X}_i = (X_i \setminus t_{\ell}^{-1}) \cup \{t_i\} \quad \text{if } m_0 < i \leq w. \]

Note that $t_1, \ldots, t_w$ form an SDR for the system $\tilde{X}_1, \ldots, \tilde{X}_w$, and each element in $\tilde{X}_i$ has a positive exponent in $\tilde{y}_i$, $1 \leq i \leq w$. By Theorem 4, there is a disjoint ordering and the disjoint ordering can be chosen so that the exponents of the last elements in the ordering satisfy the condition in Lemma 6.

We rewrite the product $\tilde{y}_i$ according to the ordering of $\tilde{X}_i$, $1 \leq i \leq w$. For instance, if $\tilde{y}_i = b_i^{t_i} b_i^{t_i} b_i^{t_i}$ and $\tilde{X}_i = \{b_1, b_2, b_3\}$ is ordered as $(b_2, b_3, b_1)$ then $\tilde{y}_i$ is rewritten as $b_2^{t_i} b_3^{t_i} b_1^{t_i}$. By Lemma 9, the resulting representations of $\tilde{y}_i$, $1 \leq i \leq w$, are
pairwise disjoint, so the induced paths are pairwise disjoint. For convenience of notation, the new \( \bar{y}_i \) is still denoted by \( \bar{y}_i \), \( 1 \leq i \leq w \). By appending \( \epsilon_i \) to \( \bar{y}_i \), we have a path \( P_i = \gamma_i \epsilon_i \) from \( 1 \) to \( y_i \), \( 1 \leq i \leq w \). Obviously, the length of \( P_i \) is

\[
|\bar{y}_i| + |\epsilon_i| \leq d_i + \bar{e}
\]

for \( 1 \leq i \leq w \).

It remains to show that the paths \( P_i \), \( 1 \leq i \leq w \), are pairwise (internally) node disjoint. We only need to prove that the end node of \( \bar{y}_i \) and the nodes introduced by \( \epsilon_i \) do not become an internal node of any other path. Let \( z \) be any node on \( P_i \), other than \( 1 \). Then

\[
\text{Supp}(z) \subseteq \begin{cases} X_i, & \text{if } 1 \leq i \leq m, \\ X_i \cup \{t_i\}, & \text{if } m < i \leq w. \end{cases}
\]

Let \( a_1, \ldots, a_w \) be the initial elements in the disjoint orderings of \( \bar{X}_1, \ldots, \bar{X}_w \) used above. Then \( a_1, \ldots, a_w \) are distinct and, by Lemma 5, \( a_i = t_i \) for \( m < i \leq w \). Since \( a_i \) is the first node after 1 on \( P_i \), we have

\[
\begin{align*}
\text{Case 1:} & \quad 1 \leq i < j \leq m. \quad \text{Nothing to prove.} \\
\text{Case 2:} & \quad m < i < j \leq m_0. \quad \text{Since } t_i \not\in X_j, \text{we have } t_i \not\in \text{Supp}(z). \text{By (18), } z = y_i. \text{Similarly, we also have } z = y_j. \\
\text{Case 3:} & \quad m_0 < i < j \leq w. \quad \text{Since } t_i \not\in X_j \text{ and } t_i \neq t_j, \text{we have } t_i \not\in \text{Supp}(z) \subset X_j \cup \{t_j\}. \text{By (19), } t_j^{-1} \in \text{Supp}(z). \text{Similarly, } t_j^{-1} \in \text{Supp}(z). \text{So } z \text{ must be of the form}
\end{align*}
\]

\[
z = t_i t_j^{-1} = t_i t_j^{u-v}
\]

with \( e_i/2 - 1 \leq v_i \leq e_i - u_i - 1 \) and \( e_j/2 - 1 \leq u_j \leq e_j - u_j - 1 \). The minimal representation of \( z \) is of the form

\[
z = \bar{y}_i (t_i^{-1})^{e_i - v_i - 1} = \bar{y}_j (t_j^{-1})^{e_j - v_j - 1}.
\]

Hence \( t_j^{-1} \) appears in \( \bar{y}_j \), say with exponent \( c_j \), and \( t_j^{-1} \) appears in \( \bar{y}_i \), say with exponent \( c_i \). We have

\[
\begin{align*}
e_i - v_i - 1 = c_i, \\
e_j - v_j - 1 = c_j.
\end{align*}
\]

As \( v_i \leq e_i - u_i - 1 \) and \( v_j \leq e_j - u_j - 1 \), we have \( c_j \geq u_i \) and \( c_i \geq u_j \). By (C), this implies that \( c_j = u_i \) and \( c_i = u_j \). It follows from (20) that

\[
\begin{align*}
v_i = e_i - u_i - 1 \quad \text{and} \quad v_j = e_j - u_j - 1.
\end{align*}
\]

Thus \( y_i = z = y_j \).

\[
\text{Case 4:} \quad 1 \leq i \leq m \quad \text{and} \quad m < j \leq m_0. \quad \text{As } t_i \neq t_j \text{ and } t_i \in \text{Supp}(z) \subset X_j \cup \{t_j\}, \text{we have } t_i \in X_j. \text{If } z \text{ is an internal node of } P_j \text{ then } t_j \in \text{Supp}(z) \subset X_i, \text{hence we have an SDR}
\end{align*}
\]

\[
t_i \in X_1, \ldots, t_{i-1} \in X_{i-1}, t_i \in X_i, t_{i+1} \in X_{i+1}, \ldots, t_m \in X_m, t_i \in X_j
\]

of size \( m + 1 \), contradicting the maximality of \( m \). So \( z \) must be the end node of \( P_i \), i.e., \( z = y_j \). As \( z = y_j \) is a node on \( P_i \), we have \( y_j \sim y_i \) and \( a_i \in \text{Supp}(z) = \text{Supp}(y_j) = X_j \). Hence the system \( X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_m \) has an SDR. By the condition (A), it follows that \( y_i = y_j \).

\[
\text{Case 5:} \quad 1 \leq i \leq m \quad \text{and} \quad m_0 < j \leq w. \quad \text{Since } a_i \neq t_j \text{ and } a_i \in \text{Supp}(z), \text{we have } a_i \in X_j. \text{If } t_j \in \text{Supp}(z) \text{ then } t_j \in X_i, \text{so the system } X_1, \ldots, X_m, X_j \text{ has an SDR of size } m + 1, \text{contradicting the maximality of } m. \text{Hence } t_j^{-1} \in \text{Supp}(z). \text{As a node on } P_j, z \text{ must be of the form}
\end{align*}
\]

\[
z = t_j \bar{y}_j t_j^{u} = \bar{y}_j (t_j^{-1})^{e_j - v_j - 1}
\]

for some \( v \) satisfying \( e_j/2 - 1 \leq v \leq e_j - u_j - 1 \). Since \( z \) is node on \( P_i \), we have \( z < y_i \). As \( v \leq e_j - u_j - 1 \), we have \( u_j \leq e_j - v - 1 \) and so

\[
y_j = \bar{y}_j (t_j^{-1})^{u} < \bar{y}_j (t_j^{-1})^{e_j - v_j - 1} = z < y_i.
\]

Hence \( y_j \sim z \sim y_i \) and the system \( X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_m \) has an SDR. By the condition (B), it follows that \( y_j = y_i \). As \( y_j \sim z \sim y_i \), we have \( y_j = z = y_i \).
Case 6: $m < i \leq m_0$ and $m_0 < j \leq w$. In this case, we have
\[
\text{Supp}(z) \subseteq X_i \cup \{t_i\}, \quad \text{and} \quad t_j \text{ or } t_j^{-1} \in \text{Supp}(z) \subseteq X_j \cup \{t_j\}.
\]
Since $t_j \not\in X_i$ and $t_j \not\equiv t_i$, we see that $t_j \not\in X_i \cup \{t_i\}$, so $t_j \not\in \text{Supp}(z)$. Hence $t_j^{-1} \in \text{Supp}(z)$. It follows that $z$, as a node on $P_j$, must be of the form,
\[
z = t_jy_jt_j^{-1} = \tilde{y}_j(t_j^{-1})^{y_j-1}
\]
for some $v$ satisfying $e_j/2 - 1 \leq v \leq e_j - u_j - 1$. Thus, $\text{Supp}(z) = X_j$. Since $i, j > m$, we have $t_i \not\in X_j$ and so $t_i \not\in \text{Supp}(z)$. By (18), we must have $z = y_i$. As $v \leq e_j - u_j - 1$, we have $u_j \leq e_j - v - 1$, hence
\[
y_j = \tilde{y}_j(t_j^{-1})^{u_j} < \tilde{y}_j(t_j^{-1})^{y_j-1} = z = y_i.
\]
Note that $Z_i$ has the representative $t_j^{-1}$ but $Z_i$ does not. By the condition (B), we have $z = y_i = y_j$. This concludes the proof of the theorem. □

Proof of Theorem 1. The diameter $d$ of $\Gamma(G, S)$ is determined by Corollary 8. Now use Theorem 10, but examine the lengths of the paths $P_i$'s more carefully. Certainly, for $1 \leq i \leq m$, $|P_i| = |y_i| \leq d$. For $m < i \leq m_0$, $X_i = \text{Supp}(y_i)$ does not contain $t_i$ and $t_i^{-1}$. If $t_i^{-1} \in S$ then $|y_i| \leq d - |e_i/2| \leq d - 1$, so $|P_i| \leq |y_i| + 2 \leq d + 1$. If $t_i^{-1} \not\in S$ then $|y_i| \leq d - (e_i - 1)$, so $|P_i| \leq |y_i| + e_i \leq d + 1$. Hence $|P_i| \leq d + 1$ for $m < i \leq m_0$. If $e_1 = 2$ or 3, which means that $s = 0$ in (2), then the proof is finished, as the next case will not happen.

Assume that $e \geq 3$, thus $m_0 < w$. For $m_0 < i \leq w$, $t_i^{-1} \in S$ and $|y_i| \leq d - |e_i/2|$ as $y_i$ does not contain $t_i$ and $t_i^{-1}$. As $u_i \geq 1$, we have
\[
|P_i| = |y_i| + e_i - u_i \leq d - |e_i/2| + e_i - 1 \leq d + e_i - |e_i/2| - 1 = d + \lfloor (e_i - 1)/2 \rfloor,
\]
which is at most $d + \lfloor (e - 1)/2 \rfloor$. This completes the proof. □

5. Comments and open problems

For the class of Cayley graphs considered above, it remains an open problem to determine the true star diameters. We know that hypercubes and directed tori have $w$-star diameters equal to $w$-wide diameters. It would be interesting to determine for which class of graphs this phenomenon holds.

Our bound on star diameters is based on explicit construction of short containers. The main property we used is the commutativity of the group operation. It may be possible that our method could be extended to many other Cayley graphs over abelian groups.

For the class of graphs we discussed, their connectivity is just the cardinality of the generating set (which is assumed to generate the group), and their wide diameter is also easy to determine. For a general Cayley graph, however, the first obstacle is to determine its connectivity which may be much smaller than the cardinality of the generating set. The problem of deciding whether a given Cayley graph is connected is already hard, since testing primitivity of elements in a finite field is a notoriously hard problem and it is just a special instance of connectedness of Cayley graphs (where $G$ is the multiplicative group of the field and $S$ has only one element). Interestingly, it is proved in [21] that if a Cayley graph $\Gamma(G, S)$ is known to be connected then its connectivity (or fault tolerance) can be determined efficiently (i.e. in time polynomial in $|S|$ and $\log |G|$).

Note that, for general Cayley graphs, finding good bounds for the usual diameter is already hard. However, it may not be unreasonable to ask for a good bound for the star and wide diameters in terms of the usual diameter. For the class of graphs we discussed, the star and wide diameters are at most $2d$ where $d$ is the usual diameter. We wonder whether a similar bound can be proved for all Cayley graphs.

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