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Note

Strongly simplicial vertices of powers of trees

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Abstract

For a tree T and an integer $k \geqslant 1$, it is well known that the kth power T^k of T is strongly chordal and hence has a strong elimination ordering of its vertices. In this note we obtain a complete characterization of strongly simplicial vertices of T^k, thereby characterizing all strong elimination orderings of the vertices of T^k.

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1. Introduction

Strongly chordal graphs have received much attention since first defined in [9], in particular because they yield polynomial time solvability of the domatic set and the domatic partition problems. For more information on these problems we refer the reader to [4,10,13,6]. In [9] a characterization of strongly chordal graphs in terms of balanced matrices is given. In [12,8] it is shown that any power of a strongly chordal graph is again strongly chordal. We should note that this does not hold for chordal graphs in general: only odd powers of chordal graphs are guaranteed to be again chordal, as was first explicitly shown in [3]. A simpler proof of this result can be found in [2]. Since a tree is strongly chordal, then any power of a tree is again strongly chordal. Although a direct consequence of [12,8], this was also shown explicitly in [7]. Since any power of a tree is strongly chordal, this implies that it has a strong elimination orderings of its vertices.

The purpose of this note is to characterize completely the strongly simplicial vertices of a power of a tree, and thereby give a complete characterization of the strongly elimination orderings of the vertices of a power of a tree. Strongly simplicial vertices of powers of trees can be applied to obtain an optimal greedy vertex coloring of squares of outerplanar graphs, as described in detail in [1]. In particular, we derive here Theorem 2.9 and Corollary 2.11, the latter of which was used to obtain the mentioned optimal greedy coloring algorithm in [1]. We should stress that the main result of this note is not a theorem, but rather Definition 2.2. Namely the mere statement of what we mean by k-strong simplicity of a vertex in a tree, as described in Lemma 2.1.

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General notation: The set $\{1, 2, 3, \ldots\}$ of natural numbers will be denoted by $\mathbb{N}$. All graphs in this note are assumed to be simple and undirected unless otherwise stated. The degree of a vertex u in graph G is denoted by $d_G(u)$. We denote by $N_G(u)$ the *open neighborhood* of u in G, that is the set of all neighbors of u in G, and by $N_G[u]$ the *closed neighborhood* of u in G, that additionally includes u. The distance $\partial_G(u, v)$, between vertices u and v, is the number of edges in a shortest path between them. When the graph in question is clear from context, we omit the subscript in the notation. For a graph G and $k \in \mathbb{N}$, the *power graph* G^k is the simple graph with the same vertex set as G, but where every pair of vertices of distance k or less in G is connected by an edge. In particular, G^2 is the graph in which, in addition to edges of G, every two vertices with a common neighbor in G are also connected with an edge. The closed neighborhood of a vertex u in G^k will be denoted by $N_G^k[u]$ and the degree of vertex u will be denoted by $d_k(u)$.

Recall the following definition:

Definition 1.1. A vertex u in a graph G is *simplicial* if $N_G[u]$ induces a clique in G. If u is simplicial and $\{N_G[v] : v \in N_G[u]\}$ is linearly ordered by set inclusion, then u is *strongly simplicial*.

A graph G is *strongly chordal* if it has a *strong elimination ordering* of its vertices $V(G) = \{u_1, \ldots, u_n\}$, such that each vertex u_i is strongly simplicial in the subgraph of G induced by u_i and the previous vertices $u_1, \ldots, u_{i-1}$. Clearly, a vertex of a tree is strongly simplicial if, and only if, it is a leaf, which gives us a complete description of when exactly an ordering is a strong elimination ordering of the tree. To describe strong simplicity in T^k for arbitrary $k \in \mathbb{N}$, we need to introduce some special notation for trees.

Notation and terminology of trees: The leaves of a tree T will be denoted by $L(T)$. The *diameter* of T is the number of edges in a longest path in T and will be denoted by $\mathrm{diam}(T)$. For a tree T with $\mathrm{diam}(T) \geqslant 1$ we can form the *pruned* tree $\mathrm{pr}(T) = T - L(T)$. For two vertices u and v of a tree T, the unique path between them will be denoted by $p_T(u, v)$ or by $p(u, v)$ when there is no danger of ambiguity. The vertices of this path, including both u and v, is given by $V(p(u, v))$. A *center* of T is a vertex of distance at most $\lceil \mathrm{diam}(T)/2 \rceil$ from all other vertices of T. A center of T is either unique or one of two unique adjacent vertices. Clearly, the power graph T^k of a tree T is only interesting when $k \in \{1, \ldots, \mathrm{diam}(T) - 1\}$. For $U \subseteq V(T)$ the *join* of U in T is the unique smallest subtree of T connecting all of U together. The *connector* of three leaves of T is the unique vertex of degree three in the join of the leaves.

Let T be rooted at $r \in V(T)$. The *kth ancestor* of $u \in V(T)$, if it exists, is the vertex on $p(u, r)$ of distance k from u, and is denoted by $a_r^k(u)$. An *ancestor* of u is a vertex of the form $a_r^k(u)$ for some $k \geqslant 0$. Note that u is viewed as an ancestor of itself. The *descendants* of u, denoted by $D_r[u]$, is the collection of all the vertices having u as an ancestor. For $u \in V(T)$, the distance $\partial_T(u, r)$ to the root r will be referred to as the *level* of u and denoted by $l_T(u)$ or by $l(u)$ when there is no danger of ambiguity. For $U \subseteq V(T)$ the *least common ancestor* of U, denoted by $\mathrm{lca}(U)$, is the unique common ancestor of U on the largest level. For a vertex u in T the subset $R_u \subseteq V(T)$ contains all $r \in V(T)$ with $\partial_T(u, r)$ maximum. Note that this definition of R_u works for unrooted trees T. Also note that for each $u \in V(T)$ we necessarily have $R_u \subseteq L(T)$. In fact, we have the following observations that will be useful in the next section.

Claim 1.2. *Let T be a tree and $u \in V(T)$. Then the center(s) of T is(are) on the path $p(u, r)$ for each $r \in R_u$.*

Proof. Root T at a center c. If $r \in V(T)$ is such that c is not on the path $p(u, r)$ then there is a longer path going from u through c, and hence $r \notin R_u$. □

Claim 1.3. *Let T be a tree, $u \in V(T)$ and $r \in R_u$. Then r is an endvertex of a path of T of maximum length* $\mathrm{diam}(T)$.

Proof. By Claim 1.2 the center(s) of T is(are) on $p(u, r)$. If c is a center of T, then there is a vertex x that is (i) an endvertex of a maximum length path of T and (ii) such that $c \in V(p(u, x))$. By definition we have $\partial_T(u, x) \leqslant \partial_T(u, r)$ and hence $\partial_T(c, x) \leqslant \partial_T(c, r)$. Therefore, equality holds and r is also an endvertex of a maximum length path in T. □

By the above two Claims 1.2 and 1.3 we have the following.

Claim 1.4. *Let T be a tree with* $\mathrm{diam}(T) = d$ *and $u \in V(T)$. If $u \in V(T)$ and $r \in R_u$, then $\partial_T(u, r) \geqslant \lceil d/2 \rceil$.*

With this setup we can start to discuss our first results, the characterization of simplicial vertices of powers of trees.

2. Characterization of simplicial vertices

We start with the following lemma.

Lemma 2.1. *Let T be a tree, $u, r \in V(T)$ and $k \in \mathbb{N}$. If T is rooted at r, let $\mathbf{P}_{u;k}(r)$ be the following statement*:

> $\mathbf{P}_{u;k}(r)$: *If $a_r^k(u)$ exists, then $\partial_T(v, a_r^{k-1}(u)) \leqslant k-1$ for all $v \in D_r[a_r^{k-1}(u)]$. Otherwise (if $a_r^k(u)$ does not exist), T^k is a complete graph.*

Then the truth value of $\mathbf{P}_{u;k}(r)$ is independent on $r \in R_u$.

Proof. By definition of R_u the kth ancestor $a_r^k(u)$ exists for one particular $r \in R_u$ iff $a_r^k(u)$ exists for all $r \in R_u$. Hence, we can assume that $a_r^k(u)$ exists for all $r \in R_u$. In this case we may further assume u to be a leaf of T, since otherwise $\mathbf{P}_{u;k}(r)$ is false regardless of $r \in R_u$. For $r, r' \in R_u$ it suffices to show that if $\mathbf{P}_{u;k}(r)$ does not hold, then $\mathbf{P}_{u;k}(r')$ does not hold either:

Since all three vertices r, r' and u are leaves of T, we have the connector $u' \in V(T)\backslash\{u, r, r'\}$. Clearly we have $\partial_T(u', r) = \partial_T(u', r')$. Looking at the join of r, r' and u, there are two cases to consider.

If $a_r^{k-1}(u) \in V(p(u, u'))$, then $a_r^{k-1}(u) = a_{r'}^{k-1}(u)$ and $D_r[a_r^{k-1}(u)] = D_{r'}[a_{r'}^{k-1}(u)]$ so $\mathbf{P}_{u;k}(r')$ is also false in this case.

If $a_r^{k-1}(u) \in V(p(u', r))$, then $a_{r'}^{k-1}(u)$ is the corresponding vertex of $V(p(u', r'))$ at the same distance from u' as $a_r^{k-1}(u)$ is. Assume there is a descendant v of $a_r^{k-1}(u)$ with $\partial_T(v, a_r^{k-1}(u)) \geqslant k$. We must consider three separate cases of the location of the connector v' of r, r' and v to be.

First case: $v' \in V(p(u, u'))$. Here $v \in D_r[a_r^{k-1}(u)] \cap D_{r'}[a_{r'}^{k-1}(u)]$ and further $\partial_T(v, a_{r'}^{k-1}(u)) = \partial_T(v, a_r^{k-1}(u)) \geqslant k$, so we have $\mathbf{P}_{u;k}(r')$ to be false in this case.

Second case: $v' \in V(p(u', r))$. Here $\partial_T(v', a_r^{k-1}(u)) \leqslant \partial_T(v', a_{r'}^{k-1}(u))$ and hence

$$
\begin{aligned}
\partial_T(v, a_{r'}^{k-1}(u)) &= \partial_T(v, v') + \partial_T(v', a_{r'}^{k-1}(u)) \\
&\geqslant \partial_T(v, v') + \partial_T(v', a_r^{k-1}(u)) \\
&= \partial_T(v, a_r^{k-1}(u)) \\
&\geqslant k.
\end{aligned}
$$

Since $a_{r'}^{k-1}(u) \in V(p(v, r'))$ the vertex v is a descendant of $a_{r'}^{k-1}(u)$ when T is rooted at r'. Hence $\mathbf{P}_{u;k}(r')$ is false in this case as well.

Third case: $v' \in V(p(u', r'))$. By definition of R_u we have that $\partial_T(u, r') \geqslant \partial_T(u, v)$, and hence $\partial_T(v', r') \geqslant \partial_T(v', v)$. By symmetry we have therefore

$$
\begin{aligned}
\partial_T(r, a_{r'}^{k-1}(u)) &= \partial_T(r', a_r^{k-1}(u)) \\
&= \partial_T(r', v') + \partial_T(v', a_r^{k-1}(u)) \\
&\geqslant \partial_T(v, v') + \partial_T(v', a_r^{k-1}(u)) \\
&= \partial_T(v, a_r^{k-1}(u)) \\
&\geqslant k.
\end{aligned}
$$

Since $a_{r'}^{k-1}(u) \in V(p(r, r'))$ the vertex r is a descendant of $a_{r'}^{k-1}(u)$ when T is rooted at r'. Hence $\mathbf{P}_{u;k}(r')$ is false in this final case. This completes our proof of the lemma. □

Remark. The statement $\mathbf{P}_{u;k}(r)$ can, at first sight, seem complex. By the right view point it is however quite natural and simple:

1. First root T at the vertex u.
2. The vertices at the lowest level constitute the set $R_u \subseteq L(T)$.
3. Pick $r \in R_u$ and re-root T at r.

4. If $a_r^k(u)$ does not exist, then $\mathbf{P}_{u;k}(r)$ is true only if $k \geqslant \mathrm{diam}(T)$.
5. If $a_r^k(u)$ does exist, then $\mathbf{P}_{u;k}(r)$ is true only if u is on the lowest level of the sub-tree of T that is rooted at $a_r^{k-1}(u)$.

By Lemma 2.1 the following definition makes sense.

Definition 2.2. Let T be a tree and $k \in \mathbb{N}$. We say that a vertex $u \in V(T)$ is *k-strongly simple*, or *k-ss* for short, if the statement $\mathbf{P}_{u;k}(r)$ from Lemma 2.1 is true for one (and hence all) $r \in R_u$.

Note that by definition we have $a_r^0(u) = u$ for every vertex $u \in V(T)$. Hence, a vertex u in a proper tree T (with at least one edge) is 1-ss in T if, and only if, u is a leaf of T. Also note that in general for $1 \leqslant k < \mathrm{diam}(T)$, only leaves of T can possibly be k-ss.

By Claim 1.4 and Definition 2.2 we have the following.

Corollary 2.3. *If $1 \leqslant k' \leqslant k \leqslant \lceil d/2 \rceil$ and $u \in V(T)$ is k-ss, then u is also k'-ss.*

The next result describes k-strong simplicity for the remaining interesting values of k:

Theorem 2.4. *Let T be a tree and $u \in V(T)$. If $d = \mathrm{diam}(T)$ and $k \in \{\lceil d/2 \rceil, \ldots, d-1\}$, then the following are equivalent*:

1. *u is an endvertex of a path of maximum length d.*
2. *u is k-ss.*
3. *u is $\lceil d/2 \rceil$-ss.*

Proof. By Definition 2.2 we clearly have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). What remains to show is (3) $\Rightarrow$ (1).

Let $h = \lceil d/2 \rceil$ and assume that u is h-ss in T. Since $a_r^h(u)$ exists for any $r \in R_u$ we have $\partial_T(v, a_r^{h-1}(u)) \leqslant h-1$ for all $v \in D_r[a_r^{h-1}(u)]$. By Claim 1.2 the center(s) is(are) contained in $V(p(u,r))$ and by the value of h, also contained in $D_r[a_r^h(u)]$. Let c be the center that is closest to u:

If $c \in D_r[a_r^{h-1}(u)]$, then $D_r[c] \subseteq D_r[a_r^{h-1}(u)]$ and hence, in particular, $\partial_T(v, a_r^{h-1}(u)) \leqslant h-1$ for all $v \in D_r[c]$. Since $D_r[c]$ contains a vertex x such that $\partial_T(x,r)=d$ and hence $\partial_T(x,c)=\lfloor d/2 \rfloor$, we must have $h-1 \geqslant \partial_T(x, a_r^{h-1}(u)) = \partial_T(x,c) + \partial_T(c, a_r^{h-1}(u)) = \lfloor d/2 \rfloor + \partial_T(c, a_r^{h-1}(u))$, which can only occur if d is odd and $a_r^{h-1}(u) = c$. In this case $\partial_T(u,c) = \lfloor d/2 \rfloor$ and hence u is an endvertex of maximum length path of T.

If $c \in D_r[a_r^h(u)] \backslash D_r[a_r^{h-1}(u)]$, then $c = a_r^h(u)$ and d must be even. Therefore, $h = \partial_T(u, a_r^h(u)) = \partial_T(u,c)$ and u is an endvertex of a maximum length path of T in this case as well. This completes the proof. □

Remarks. What is defined to be an "extreme leaf" of T by Kearney and Corneil in [7] is precisely a vertex that satisfies one condition in Theorem 2.4 (and hence all of them), that is a k-ss vertex of T where $k \in \mathbb{N}$ and $d/2 \leqslant k < d$.

The following is the first step toward a complete description of strongly simplicial vertices of powers of trees:

Theorem 2.5. *Let T be a tree and $k \in \mathbb{N}$. A k-ss vertex of T is strongly simplicial in the power graph T^k.*

To prove Theorem 2.5, we will use the following from [2, Lemma 2.2, p. 45]:

Lemma 2.6. *If T is a tree rooted at $r \in V(T)$ and $u \in V(T)$, then all the vertices of T on levels at most $l(u)$ and of distance at most k from u, form a clique in T^k.*

Proof of Theorem 2.5. We may assume $k < \mathrm{diam}(T)$. Let u be a k-ss vertex of T and let $r \in R_u$ be a fixed root. By Lemma 2.6 $N_T^k[u]$ forms a clique in T^k.

We now show that $\{N_T^k[v] : v \in N_T^k[u]\}$ is linearly ordered by set inclusion. For $u', u'' \in N_T^k[u]$ we show that if $l(u') \geqslant l(u'')$ then $N_T^k[u'] \subseteq N_T^k[u'']$. Assume $l(u') \geqslant l(u'')$ and let $v \in N_T^k[u']$. If $\mathrm{lca}(u', u'')$ is a descendant of $\mathrm{lca}(u', v)$, then $\mathrm{lca}(u', u'') \in V(p(u', v))$ and hence $\partial_T(u'', v) \leqslant \partial_T(u', v) \leqslant k$ so $v \in N_T^k[u'']$. Otherwise $\mathrm{lca}(u', v)$

must be a descendant of $\mathrm{lca}(u', u'')$. Here we further consider two cases, depending on where u is: If $\mathrm{lca}(u', u'')$ is a descendant of $\mathrm{lca}(u, u'')$, then, by the k-strong simplicity of u, we have $l(v) \leqslant l(u)$ and hence $\partial_T(u'', v) \leqslant \partial_T(u'', u) \leqslant k$ and hence $v \in N_T^k[u'']$. Otherwise, in this case, $\mathrm{lca}(u, u'')$ is a descendant of $\mathrm{lca}(u', u'')$ and hence $\mathrm{lca}(u, u'') \in V(p(u'', \mathrm{lca}(u', u'')))$. Since $l(u') \geqslant l(u'')$ and $l(u) \geqslant l(v)$ we have

$$\begin{aligned}\partial_T(u'', v) &= \partial_T(u'', \mathrm{lca}(u'', u')) + \partial_T(\mathrm{lca}(u'', u'), v)\\ &\leqslant \partial_T(u', \mathrm{lca}(u'', u')) + \partial_T(\mathrm{lca}(u'', u'), u)\\ &= \partial_T(u', u),\\ &\leqslant k,\end{aligned}$$

showing that $v \in N_T^k[u'']$ in this final case. This completes the proof that $\{N_T^k[v] : v \in N_T^k[u]\}$ is linearly ordered by set inclusion. $\square$

Lemma 2.7. *Let T be a tree, $u \in V(T)$ and T be rooted at $r \in R_u$. If $a_r^k(u)$ exists and there is a descendant w of $a_r^{k-1}(u)$ of distance k or more from $a_r^{k-1}(u)$, then u is not strongly simplicial in T^k.*

Proof. We may assume that $l(w) = l(u) + 1$. Going upward from w toward the root r, let v be the first ancestor of w that is contained in $N_T^k[u]$ (such a vertex exists, since w is a descendant of $a_r^{k-1}(u) \in N_T^k[u]$.) Since $\partial_T(w, v) \leqslant \partial_T(w, a_r^{k-1}(u)) = k$ we have

$$w \in N_T^k[v] \backslash N_T^k[a_r^k(u)]. \tag{1}$$

If $\partial_T(u, w) \leqslant k$, then $N_T^k[u]$ is not a clique in T^k since $\partial_T(w, a_r^k(u)) = k + 1$, and hence u is not even simplicial. So we assume $\partial_T(u, w) > k$. In this case, since $v \in V(p(u, w))$, we have by definition of $r \in R_u$ that $\partial_T(w, v) \leqslant \partial_T(a_r^k(u), r)$, and hence there is a unique vertex w' on the path $p(a_r^k(u), r)$ with $\partial_T(w, v) = \partial_T(a_r^k(u), w')$. Therefore we have

$$\begin{aligned}\partial_T(v, w') &= \partial_T(v, a_r^k(u)) + \partial_T(a_r^k(u), w')\\ &= \partial_T(w, v) + \partial_T(v, a_r^k(u))\\ &= \partial_T(w, a_r^k(u))\\ &= k + 1.\end{aligned}$$

But since we also have $\partial_T(w', a_r^k(u)) = \partial_T(w, v) \leqslant \partial_T(w, a_r^{k-1}(u)) = k$, then $w' \in N_T^k[a_r^k(u)] \backslash N_T^k[v]$, which together with (1) shows that u is not strongly simplicial in T^k. $\square$

Let T be a tree, $u \in V(T)$ and T be rooted at $r \in R_u$. If $a_r^k(u)$ does not exist, then $N_T^k[u] = V(T)$. If $k < \mathrm{diam}(T)$ then $N_T^k[u]$ cannot induce a clique in T and hence u is not simplicial, let alone strongly simplicial in T. We summarize:

Lemma 2.8. *If $a_r^k(u)$ does not exist and $k < \mathrm{diam}(T)$, then u is not strongly simplicial in T^k.*

By Theorem 2.5 and Lemmas 2.7 and 2.8 we have the following.

Theorem 2.9. *For a tree T and $k \in \mathbb{N}$, the vertex $u \in V(T)$ is k-ss in T if, and only if, u is strongly simplicial in T^k.*

Note: Theorem 2.9 yields a procedure to characterize all strong elimination orderings $u_1, \ldots, u_n$ of a power of a tree T on n vertices, and hence an algorithm to list them all in polynomial time.

By Theorems 2.4 and 2.9 we have the following corollary:

Corollary 2.10. *Let T be a tree and $u \in V(T)$. If $d = \mathrm{diam}(T)$ and $k \in \{\lceil d/2 \rceil, \ldots, d-1\}$, then u is strongly simplicial in T^k if, and only if, u is strongly simplicial in $T^{\lceil d/2 \rceil}$.*

For a tree T we can recursively define $T^{(i)}$ by $T^{(0)} = T$ and $T^{(i)} = \mathrm{pr}(T^{(i-1)})$ for $i \in \mathbb{N}$, as long as $T^{(i-1)}$ has leaves, that is, is neither empty nor one vertex. With this notation we obtain the following.

Corollary 2.11. *Let T be a tree with* $\operatorname{diam}(T) = d \geqslant 2$. *For $u \in V(T)$ and $k \in \{1, \ldots, \lceil (d-1)/2 \rceil\}$, the following are equivalent*:

1. *For a center $c \in V(T)$, the vertex $a_c^{k-1}(u)$ is a leaf of $T^{(k-1)}$.*
2. *u is strongly simplicial in T^k.*

Proof. Assume $\partial_T(u, c) \leqslant k-1$ for the(each) center c of T. Then, first of all $a_c^{k-1}(u)$ is either isolated or does not exist, and secondly $a_r^k(u)$ exists for each $r \in R_u$ and further, since $D_r[a_r^k(u)]$ contains c, it also contains an endvertex x of a maximum length path. We therefore have $\partial_T(x, c) \geqslant \lfloor d/2 \rfloor > k-1 \geqslant \partial_T(u, c)$, so by Lemma 2.7 u is not strongly simplicial in T^k. Hence, both statements of the corollary are wrong in this case.

If $\partial_T(u, c) \geqslant k$ for a center c, then $a_c^k(u)$ exists. By Claim 1.2 we have $c \in V(p(u, r))$ for each $r \in R_u$, and since $k \leqslant \lceil (d-1)/2 \rceil$ we have that $a_r^k(u)$ also exists and further

$$a_c^{k-1}(u) = a_r^{k-1}(u) \quad \text{and} \quad D_c[a_c^{k-1}(u)] = D_r[a_r^{k-1}(u)], \tag{2}$$

when T is rooted at c on one hand and at r on the other. By (2) the first statement is equivalent to k-strong simplicity and the corollary easily follows. $\square$

Remark. When determining a strong elimination ordering of the vertices of the graph $G = T^k$ it suffices to only know the graph G, since in [7] a polynomial time algorithm is given to obtain the kth root T from G (provided that we know G is a kth power of a tree $G = T^k$. In fact, Chang et al. [5] claim they can in this case obtain the kth root in linear time). However, for a general graph H it is NP-hard to compute the kth root H of $G = H^k$ as shown in [11].

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