Integral-based constitutive equation for rubber at high strain rates

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Abstract

High-speed experiments were conducted to characterize the deformation and failure of Styrene Butadiene Rubber at impact rates. Dynamic tensile stress–strain curves of uniaxial strip specimens and force–extension curves of thin sheets were obtained from a Charpy tensile impact apparatus. Results from the uniaxial tension tests indicated that although the rubber became stiffer with increasing strain rates, the stress–strain curves remained virtually the same above 280 s$^{-1}$. Above this critical strain rate, strength, fracture strain and toughness decreased with increasing strain rates. When strain rates were below 180 s$^{-1}$, the initial modulus, tensile strength and breaking extension increased as the strain rate increased. Between strain rates of 180 and 280 s$^{-1}$, the initial modulus and tensile strength increased with increasing strain rates but the extension at break decreased with increasing strain rates. A hyper-viscoelastic constitutive relation of integral form was used to describe the rate-dependent material behavior of the rubber. Two characteristic relaxation times, 5 ms and 0.25 ms, were needed to fit the proposed constitutive equation to the data. The proposed constitutive equation was implemented in ABAQUS Explicit via a user-defined subroutine and used to predict the dynamic response of the rubber sheets in the experiments. Numerical predictions for the transient deformation and failure of the rubber sheet were within 10% of experimental results.

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1. Introduction

Elastomers or rubber-like materials are often used to mitigate damage caused by impulsive or impact loads because of their low modulus, high damping characteristics and large extensibility. Rubber isolation bearings, for example, protect buildings and bridges from earthquakes by imposing a layer of low shear modulus between the structure and ground and allowing lateral, almost rigid body motion of the structure. The isolation bearing deflects the earthquake energy through the dynamics of the system (Kelly, 1997). Rubber shock absorbers arrest impacting bodies with minimum force transmission by undergoing a sufficiently large
deflection before bringing the body to rest. They also reduce large-amplitude vibrations and minimize rebound by dissipating energy via internal damping (Gent, 2001). Polyurea coatings on the inside walls of concrete buildings offer significant protection from exterior terrorist’s bomb blasts by inhibiting fragmentation and allowing an otherwise brittle wall to deform under the blast (Knox et al., 2003). Security films, which are tough but pliable polyester layers bonded to window glass, reduce glass fragmentation and retain glass shards when the window is subjected to gust wind and impact.

The mechanical behavior of all the above-mentioned elastomeric structures is dominated by large strain (greater than 10%), high strain rate (more than 10 s\(^{-1}\)) and nonlinear viscoelastic response. Yet testing and modeling of elastomers have been confined to quasi-static deformation, creep, relaxation or small strain, linear viscoelastic vibration response. There are several recent studies on high strain rate compression of elastomers (low impedance materials) with modified versions of the Split Hopkinson Pressure Bar apparatus (Chen et al., 2002; Rao et al., 1997; Quintavalla and Johnson, 2005; Shim et al., 2004; Song and Chen, 2003, 2004a,b; Yang et al., 2000) but high strain rate tension tests on elastomers are scarce. The paper by Shim et al. (2004) was the only study to present tensile (in addition to compression) Split Hopkinson Pressure Bar material stress–strain curves. An alternative test to a tensile SHPB is the expanding ring technique (Kuhn and Medlin, 2000). However, this technique suffers from two major drawbacks: the strain rate in the expanding ring is usually not constant and the specimen receives a compressive preload in the radial stress. Since the stress history can affect the subsequent stress–strain behavior of the material, data from expanding ring tests do not always agree with results from other tests.

Bekar et al. (2002) have developed a simple high-speed tensile experiment to measure deformation and fracture properties of unfilled SBR vulcanizates at room temperature and strain rates between 10 and 10\(^3\) s\(^{-1}\). They found that the modulus of the stress–strain curve increased with increasing strain rates but remained the same when the strain rate was above 280 s\(^{-1}\). In this paper, test data from these experiments are used to develop a three-dimensional constitutive model for simulating large strain, nonlinear, dynamic response of rubber. One-dimensional, rate-dependent constitutive equations for elastomers can be found by Rao et al. (1997) and Song and Chen (2004a,b). These equations may be adequate for an elastomer in the uniaxial stress state but cannot be used to simulate rubber in a three-dimensional stress state. Three-dimensional, hyper-viscoelastic constitutive models for high strain rate response of rubber were proposed by Yang et al. (2000) and Shim et al. (2004) using BKZ-type hereditary laws and very limited test data. The term BKZ was used in a broad sense to describe a single integral form of nonlinear viscoelastic constitutive equation. The original BKZ integral constitutive equation can be found in Bernstein et al. (1963). Recently, the Bergstrom–Boyce viscoelastic constitutive equation (Bergstrom and Boyce, 1998), which is not of integral form, has been extended to high strain rate regime (Quintavalla and Johnson, 2005). Material parameters for the Bergstrom–Boyce model were estimated from a nonlinear least-square FORTRAN program. The above-mentioned proposed constitutive equations are very specialized and material parameters for them are based on a few stress–strain curves at rates exceeding 10\(^3\) s\(^{-1}\).

A three-dimensional, hyper-viscoelastic integral constitutive equation will be developed in this paper using test data from a high-speed tensile experiment. Rate-dependent material behavior of SBR is assumed to take place because of the lack of adequate stress relaxation time at high rates of loading or under small load duration. The hyper-viscoelastic constitutive equations are based on an integral hereditary law for the material. Hyper-viscoelastic constitutive equations can be derived using integral or differential equations. In the differential approach, evolution laws for viscous dissipation are assumed from either micro-mechanism inspired flow rules (Bergstrom and Boyce, 1998) or finite strain, linear viscoelasticity (Huber and Tsakmakis, 2000). The time scales of our experiments are very short (under 100 msec) and it is unclear what viscoelastic mechanisms actually occur during such a short amount of time. By using an integral approach, we make no assumption on a viscous flow rule. The hereditary law will be assumed for the material based on empirical data.

A brief description of the experimental setup and results from uniaxial tension and sheet experiments will be given in the following section. More details of the tensile impact experiments can be found in Hoo Fatt and Bekar (2004). This paper focuses on the development of a suitable constitutive model for monotonic loading of the SBR at high strains and under high strain rates. This constitutive model can be used to predict the transient response of rubber structures under blast or impact. It is not intended for use in vibratory or cyclic load-
ing situations. The constitutive equation will be implemented in ABAQUS Explicit using a user-defined material subroutine (VUMAT). To test the accuracy of this constitutive equation, the dynamic response of the rubber sheets under tension will be simulated with ABAQUS Explicit and compared to experimental results.

2. Tensile impact experiments

The tensile impact apparatus shown in Fig. 1a and b was developed for obtaining the deformation and fracture characteristics of rubber-like materials under impact rates (Bekar et al., 2002; Hoo Fatt and Bekar, 2004). This apparatus can be used to obtain both dynamic stress–strain curves of uniaxial strip specimens and force–extension curves for thin sheets. As shown in Fig. 1a, the Charpy impact pendulum impacts a specially designed slider bar connected to two copper cables. The copper cables are directed around pulleys and attached to guided bases with grips that hold opposite ends of the specimen (see Fig. 1b). The guided bases in Fig. 2 are allowed to slide freely along steel rails when pulled by the copper cables. Each end of the specimen is gripped into a guide base, which also carries load cells. PCB 200-B01 piezoelectric dynamic load cells with a linear range of 0–44.5 N (0–10 lb) are mounted on each guided base and record the impact tensile force as the copper cables pull on the guided base. The tensile force is actually measured as a compressive force as the piezoelectric load cells hit the backside of rectangular slots. The extension of the specimen is recorded by two RDP D5 Linear Variable Differential Transformers (LVDT Displacement transducers), which have a range.

Fig. 1. Schematic diagram of tensile impact test: (a) side view of Charpy hammer impact and (b) top view of tensile impact apparatus.

Fig. 2. Guided bases.
of ±150 mm and are mounted on each of the guided bases. Designing the experiment so that there are equal grip separation velocities on both sides of the specimen isolates the middle of the specimen so the fracture process in the center of the rubber sheet can be captured with a high-speed camera. A Photron Ultima APX FASTCAM monochrome camera, which is capable of recording 2000 frames per sec at full resolution and up to 120,000 frames per sec at reduced resolution, is used for the high-speed video photography. Halogen light sources, not shown in Fig. 1b, are used to illuminate the specimen during the video recording.

2.1. Materials and specimens

The above-mentioned apparatus was used to obtain stress–extension ratio curves of strip specimens and force–stretch response of thin sheet specimens. The specimens were cut from 2.54-mm thick sheets of cross-linked SBR, which was provided by the Akron Rubber Development Laboratory (ARDL) in 2.54-mm thick sheets. The recipe for the SBR is given in Table 1. During preparation, the SBR was cured at 320 F for 11.5 min.

An ASTM D412 strip specimen (ASTM, 1998) was chosen to obtain stress–extension ratio curves in the experiment. Rectangular strips, 6.35 mm wide and 50.8 mm long, were cut from SBR sheets using a razor and hammer. These were clamped into grips with a clamping distance of 12.7 mm on either end so the effective geometry of the specimen was 6.35 mm wide × 25.4 mm long × 2.54 mm thick (see Fig. 3a). For the sheet experiments, 50.8 mm × 50.8 mm squares were cut from the SBR sheet at a distance of 12.7 mm and were clamped into the grips at two opposing sides. The sheet was thus clamped fully along the 50.8 mm length of the specimen and the effective geometry of the sheet specimen was 25.4 mm long × 50.8 mm wide × 2.54 mm thick (see Fig. 3b).

2.2. Dynamic stress–extension ratio curves from strip specimen

The strain rate of the material tension test was controlled by adjusting the drop height of the Charpy hammer. Each material tension test was repeated 4–10 times to ensure accuracy and repeatability of the data. Those tests for which the specimen did not break in the middle were discarded. After signal conditioning, the PCB 200-B01 load cell had an error of band of ±0.02 lb and the error band associated with the long travel LVDT was ±2.54 mm. These error bands were obtained from the amplitude of the load and extension signals in the no load (zero) position. Data filtering was then used to smooth out the force and extension.

Table 1
Recipe for SBR used in the tensile specimens

<table>
<thead>
<tr>
<th></th>
<th>SBR 1502</th>
<th>ZnO</th>
<th>Sulfur</th>
<th>St. acid</th>
<th>MBTS</th>
<th>TMTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>PHr</td>
<td>100</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td>1.6</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Fig. 3. Test specimens: (a) strip specimen and (b) sheet specimens.
As an example, the time variation of the tensile load and extension for the test performed at a strain rate of 280 s\(^{-1}\) (508 mm drop height) are given in Fig. 4a and b, respectively. These curves were typical of tests performed at other strain rates, and were used to calculate the Cauchy stress–extension ratio curve. As seen in Fig. 4b, the extension rate and resulting engineering strain rate of the test specimen remains constant after an initial transient associated with the initial inertial resistance of the material and grips. Once constant extension rate is achieved, these inertial forces are essentially zero and the force measured by the load cell is due only to the stiffness and damping of the material. The Cauchy stress–extension ratio curve at any given strain rate was taken as the median values from all the tests done at the same strain rate or drop height. Experimental errors and variability in the material itself resulted in a 5–10% scatter band of the Cauchy stress–extension

Fig. 4. Time variation of load and extension in tension test at 280 s\(^{-1}\): (a) transient load and (b) transient extension.
ratio curves. The tensile strength and fracture strain (fracture points) were taken as average values of all acceptable tests (test where the specimen broke in the gage).

The Cauchy stress–extension ratio curves at various strain rates are shown in Fig. 5. The strain rates quoted in Fig. 5 are the engineering strain rate, i.e., velocity imparted to specimen divided by original gage length. The quasi-static tensile test result at 0.1 s\(^{-1}\) from an Instron universal testing machine, is also provided in Fig. 5 for comparison. Three regions can be identified in this figure. In Region 1 (curves below 180 s\(^{-1}\)) the modulus, tensile strength and fracture strain increase with increasing strain rates. In rubber elasticity, the term modulus is used to describe the magnitude of the stress and not its stiffness or tangent modulus. In Region 2 (curves between 180 and 280 s\(^{-1}\)) only the modulus and strength increase with increasing strain rate but the fracture strain decreases with higher strain rates. At strain rates above 280 s\(^{-1}\) (Region 3), all the curves approach a limiting case. In Region 3, the modulus remains virtually the same but the tensile strength and fracture strain decrease with increasing strain rates.

2.3. Dynamic force–extension curves from sheet specimen

Fig. 6 shows the resulting force–extension response of the SBR sheets at different loading rates. In this figure, the loading rate is defined as the ratio of grip velocity to original height of the sheet. Unlike the material tension tests, only one test was done per drop height. The purpose of the sheet experiments was only to provide test data for comparing FEA solutions using the three-dimensional constitutive equation. Images during the experiment at the 105 s\(^{-1}\) loading rate can be seen in Fig. 7a–d. The sheet first deformed without any sign of fracture until a hole suddenly initiated at the center of the sheet. The hole initiated at the peak loads in Fig. 6. The force–extension graphs become unstable when the hole enlarges, and the specimen breaks into two parts at the final load drop.

3. Constitutive modeling

The dynamic stress–extension ratio curves in Fig. 5 are bounded between the quasi-static and an upper limiting curve. In the literature, these two curves are referred to as the equilibrium and instantaneous response curves, respectively. Assume that the material can be described by an elastic spring A in parallel with a Maxwell element B, as shown in Fig. 8. Both springs in A and B are hyperelastic. The total stress at any time is \(\sigma = \sigma_A + \sigma_B\). The Maxwell element allows the force in the elastic spring of B to vary with loading rate. When the loading rate is very slow, deformation is governed only by A since no force is transmitted through the viscous damper in B. The material is assumed to be in a fully relaxed state and the corresponding equilibrium
response can be described by a hyperelastic constitutive equation. When the loading rate is very high, the viscous damper in B does not have time to deform and the adjacent spring force approaches a limiting value, as if the damper were not present. At very fast loading rates, the corresponding instantaneous response can be described again by a hyperelastic constitutive equation. In between very slow and fast loading rates, the material exhibits time-dependent behavior and must be described by a hyper-viscoelastic theory.

3.1. Kinematics for finite deformation analysis

A key quantity in finite deformation analysis is the deformation gradient $F$, which relates quantities before deformations to them after or during deformations. Consider a point at position $X$. If this point is displaced to a new position $x$, then the deformation gradient tensor $F$ is defined as $F = \frac{\partial x}{\partial X}$. The deformation gradient tensor can be decomposed into stretch and rotation components using polar decomposition: $F = RU = VR$, where $R$ denotes a proper orthogonal rotation tensor, and $U$ and $V$ are the right and left stretch tensors, respectively. The stretch tensors are not convenient measures of strain since they require evaluation by polar decomposition. The right Cauchy–Green tensor, $C = F^TF = U^2$, and the left Cauchy–Green tensor, $b = FF^T = V^2$, are often easier to evaluate. The Green strain tensor is $E = \frac{1}{2}(C - I)$, where $I$ is the identity matrix.

3.2. Hyperelasticity

For the incompressible case, the Cauchy stress is given by

$$\sigma = -pI + 2\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)b - 2\frac{\partial W}{\partial I_2}b \cdot b$$

where $p$ is an unspecified hydrostatic pressure, $W = W(I_1, I_2)$ is a strain energy potential function, $I_1 = \text{tr}(C) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \frac{1}{2} \left[ I_1^2 - \text{tr}(C^2) \right] = (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_3\lambda_1)^2$ and $\lambda_i (i = 1,2,3)$ are the principal stretch ratios.

The most commonly used energy potentials for hyperelastic materials are polynomial and reduced polynomial forms (Rivlin, 1948). These functions do not allow representation of an “upturn” in the static stress–strain curve, which is observed at the breaking strain when the molecular chains become nearly fully stretched. The Gent energy potential (Gent, 1996) and the Arruda–Boyce energy potential (Arruda and Boyce, 1993),

![Figure 6. Force–extension curves for SBR sheet at different loading rates.](image-url)

Fig. 6. Force–extension curves for SBR sheet at different loading rates.
Fig. 7. Images of SBR sheet subjected to 105 s⁻¹ loading rate: (a) beginning of the experiment, \( t = 0 \) ms, (b) onset of fracture, \( t = 6.91 \) ms, (c) hole enlargement, \( t = 7 \) ms, (d) final fracture, \( t = 13 \) ms.

Fig. 8. Rheological representation of a hyper-viscoelastic material.
which is based on eight-chain representation of the macromolecular network structure of the rubber, are more recent energy potentials that can accommodate the upturn at large strains.

If one assumes that the SBR is incompressible, the deformation gradient tensor in the uniaxial tensile specimen is

\[
F = \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \frac{1}{\sqrt{\lambda}} & 0 \\
0 & 0 & \frac{1}{\sqrt{\lambda}}
\end{bmatrix}
\]

Thus the left Cauchy–Green strain tensor for the material

\[
b = \begin{bmatrix}
\lambda^2 & 0 & 0 \\
0 & \frac{1}{\lambda} & 0 \\
0 & 0 & \frac{1}{\lambda}
\end{bmatrix}
\]

The Cauchy stress tensor for uniaxial stress is also given by

\[
\sigma = \begin{bmatrix}
\sigma & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Following Eq. (1), one gets

\[
\sigma_{11} = \sigma = -p + 2\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\lambda^2 - 2\lambda^4 \frac{\partial W}{\partial I_2}
\]

where \( I_1 = \lambda^2 + \frac{1}{\lambda} \) and \( I_2 = 2\lambda + \frac{1}{\lambda} \). Setting \( \sigma_2 = \sigma_3 = 0 \) gives the pressure term as

\[
p = 2\lambda \left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right) - 2\lambda^2 \frac{\partial W}{\partial I_2}
\]

Substituting this expression for \( p \) into Eq. (5) allows the following stress–extension relation

\[
\sigma = 2\lambda \left(\frac{\partial W}{\partial I_1} + \lambda \frac{\partial W}{\partial I_2}\right) \left(\lambda^2 - \frac{1}{\lambda}\right)
\]

Following nonlinear regression analysis (OriginLab, 2003) was used to fit the material tension test data with several polynomial and reduced polynomial potential forms. Almost all of the above equations can be fitted to the test data for the equilibrium curves except for the neo-Hookean energy potential. This, however, was not the case for the instantaneous response curve. As shown in Fig. 9a and b, second or higher order polynomial energy potential functions are needed to describe the high modulus in the small strain region of the data for the instantaneous curve while none of the reduced polynomials can describe the steep initial modulus. The polynomial forms also exhibit oscillations. In Hoo Fatt and Bekar (2004), a third-order Ogden form was used to find the instantaneous response curve, and it also oscillated. These oscillations in the stress–strain relations are undesirable in FEA because they lead to numerical instabilities.

To obtain a smooth functional fit, the following energy potential was assumed:

\[
W = \mu (I_1 - 3)^x
\]

where both \( \mu \) and \( x \) may be noninteger material constants. From Eqs. (7) and (8), one gets

\[
\sigma = 2\mu x \left(\lambda^2 - \frac{1}{\lambda}\right) (I_1 - 3)^{x-1}
\]

Fig. 10 shows the equilibrium and the instantaneous response curves with the material coefficients given in Table 2. Eq. (9) is a hyperelastic equation that can describe the apparent yielding which takes place during high-speed loading. Conventional hyperelastic laws were not intended to be used to describe material behavior like this.
3.3. Nonlinear viscoelasticity

Time-dependent or viscoelastic behavior of elastomers is linked to complicated molecular adjustments resulting from macroscopic mechanical deformations (Ferry, 1980). Such behavior is peculiar to elastomers because they consist of flexible, long chain molecules that intertwine with each other and continually change contour due to thermal agitation. Elastomers are polymers with long chains (about $10^4$ monomer units in a molecule) and a glass transition temperature well below room temperature. Alfrey (1948) describes the morphology of an elastomer in terms of convolutions, curls and kinks. Convolutions represent the long-range contour of an entire molecular chain, which forms entanglements (knots). Chains may also be crosslinked with others by other molecules such as sulfur. Curls are shorter range molecular contours that develop between entanglements and crosslinks, and kinks are molecular bonds within a curl. Each molecular bond has rotational freedom that allows the direction of the chain molecule to change at every bond. Thus the entire molecular chain can twist, spiral and tangle with itself or with adjacent chains.

Under stress, molecules in an elastomer slide, except at the entanglements and crosslinks. Relaxation, intermolecular slippage accompanied by some reversible breaking or swapping of bonds, takes place during this time. The various types of slippage are distinguished by different relaxation times. Relaxation on a local scale involves relatively rapid re-adjustments of the kinks in the molecular chains. Relaxation on a long-range scale

![Graph](image-url)
involves very slow rearrangements of the convolutions with respect to each other. The relaxation time on the local scale is very short and the relaxation time on the long-range scale is long. Since there is a range of molecular chain lengths within an elastomer, there is a continuous range of relaxation times. Re-adjustments can be restricted when the elastomer is deforming at very high rates because certain relaxation mechanisms do not have time to occur. During quasi-static loading or at very low loading rates, re-adjustments are not restricted since the loading time is sufficiently long. As the loading rate increases, the time for relaxation processes decreases and the amount of relaxation of the convolutions, curls and kinks vary. This accounts for rate-sensitivity of the modulus and strength of the elastomer.

Almost all of the research concerning nonlinear viscoelastic behavior has been under quasi-static loading conditions. Constitutive equations have been formulated from either creep or stress relaxation experiments based on continuum mechanics or molecular theories. The reader can find comprehensive coverage of the basic principles of viscoelasticity in Lockett (1972) and Carreau et al. (1997). A continuum mechanics approach will be taken in this paper. In particular, the rate-dependent behavior of the rubber is assumed to be related to stress relaxation during loading.

A general constitutive equation for nonlinear viscoelasticity of an incompressible material is as follows (Lockett, 1972):

$$\sigma = -p_v I + F(t) \cdot \int_{\tau=-\infty}^{t} \{C(\tau)\} \cdot F^T(t)$$  \hspace{1cm} (10)

where $p_v$ is an undetermined hydrostatic pressure and $\Omega\{C(\tau)\}$ is a stress functional that describes the strain history on the stress. Green and Rivlin (1957, 1960) and Green et al. (1959) assumed the stress functional to be continuous and expressed it in terms of polynomial integrals using Frechet expansion. Although this form of a nonlinear viscoelastic constitutive equation was not very practical, simplified versions of it have led to several useful integral constitutive equations (Carreau et al., 1997). For instance, Bernstein et al. (1963) proposed a
single integral form of the Green–Rivlin constitutive model, i.e., the BKZ integral, and found it to be capable of predicting stress relaxation under constant strain of plasticized polyvinyl chloride, polyisobutylene and vulcanized butyl rubber. A new single integral form will be developed in this paper since earlier single integral forms, including the BKZ integral, are based on polynomial expansion of strain invariants and do not capture the high modulus due to high strain rates in the small strain region.

Let us assume for an incompressible hyperelastic material

$$\sigma = -p_e I + F \cdot \frac{\partial W}{\partial E} \cdot F^T$$

(11)

where \( W(I_1, I_2) \) and \( \frac{\partial W}{\partial E} = S = 2\left( \frac{\partial W}{\partial I_1} I_1 + \frac{\partial W}{\partial I_2} (I_1 I_2 - C) \right) \). Also assume

$$\frac{\partial W}{\partial E} = \int_{-\infty}^{t} \frac{\partial u}{\partial E} (I_1, I_2, t - \tau) d\tau$$

(12)

where \( u \) describes the history of the strain energy potential. Then the stress is given by

$$\sigma = -p_e I + F \cdot \int_{-\infty}^{t} 2\left[ \frac{\partial u}{\partial I_1} I_1 + \frac{\partial u}{\partial I_2} (I_1 I_2 - C) \right] d\tau \cdot F^T$$

(13)

The above formulation assumes that the stress at any time depends on the time history of the strain energy. It is very similar to the BKZ integral constitutive equation in this respect.

3.4. Hyper-viscoelastic constitutive law

Using Eq. (7) to represent the equilibrium stress and Eq. (13) to represent the time-dependent stress, one gets the following hyper-viscoelastic constitutive law:

$$\sigma = -(p_e + p_i) I + 2\left( \frac{\partial W_e}{\partial I_1} I_1 + \frac{\partial W_e}{\partial I_2} I_2 \right) b - 2 \frac{\partial W_e}{\partial I_2} b \cdot b + F \cdot \int_{-\infty}^{t} 2\left[ \frac{\partial u}{\partial I_1} I_1 + \frac{\partial u}{\partial I_2} (I_1 I_2 - C) \right] d\tau \cdot F^T$$

(14)

where \( p_e \) and \( W_e \) represent the undetermined pressure and energy potential corresponding to equilibrium response.

From Eq. (8), one obtains \( \frac{\partial W_e}{\partial I_1} = \alpha(I_1 - 3)^{-1} \) and \( \frac{\partial W_e}{\partial I_2} = 0 \). We now assume the following conditions:

$$\frac{\partial u}{\partial I_1} = c_1 (I_1 - 3)^{c_2} \dot{I}_1 m(t - \tau)$$

(15)

$$\frac{\partial u}{\partial I_2} = 0$$

(16)

where \( c_1 \) and \( c_2 \) are material constants, \( \dot{I}_1 \) is the time rate of change of the first invariant and \( m(t - \tau) \) is a memory function. The time rate of change of the first invariant is not an independent variable since it depends on \( I_1 \) and \( t \).

Eq. (15) has the form of the time-derivative form of \( \frac{\partial W}{\partial I_1} \) and includes a memory function given by

$$m(t - \tau) = \sum_{i=1}^{N} m_i e^{-(t - \tau)/\theta_i}$$

(17)

where \( m_i \) are weight factors, \( \theta_i \) are relaxation times and \( N \) is some chosen integer. As mentioned earlier, stress relaxation may occur on several length or time scales. The weight factor allows one to adjust the relaxation effect over length scales (extension ratio).

Substituting Eqs. (15)–(17) into Eq. (14) yields

$$\sigma = -(p_e + p_i) I + 2\alpha(I_1 - 3)^{c_1 - 1} b + F \cdot \int_{-\infty}^{t} (2c_1 m(t - \tau)(I_1 - 3)^{c_2} \dot{I}_1 I_1) d\tau \cdot F^T$$

(18)

For the case of uniaxial tension which begins at \( t = 0 \), this reduces to
\[ \sigma_{11} = \sigma = -(p_c + p_s) + 2 \mu \dot{\varepsilon}(I_1 - 3)\dot{\lambda}^2 + \dot{\lambda}^2 \int_0^t [2c_1 m(t - \tau)(I_1 - 3)^2 \dot{I}_1] d\tau \]

where \( I_1 = \dot{\lambda}^2 + \frac{2}{\lambda} \) and \( \dot{I}_1 = 2 (\dot{\lambda} - \frac{1}{\lambda^2}) \dot{\lambda} \). Setting \( \sigma_{22} = \sigma_{33} = 0 \), one finds

\[ (p_c + p_s) = 2 \mu \dot{\varepsilon}(I_1 - 3)\dot{\lambda}^2 + \frac{1}{\lambda} \int_0^t [2c_1 m(t - \tau)(I_1 - 3)^2 \dot{I}_1] d\tau \]

Thus,

\[ \sigma = 2 \mu \varepsilon(I_1 - 3)^{\varepsilon-1} \left( \dot{\lambda}^2 - \frac{1}{\lambda} \right) + \left( \dot{\lambda}^2 - \frac{1}{\lambda} \right) \int_0^t [2c_1 m(t - \tau)(I_1 - 3)^2 \dot{I}_1] d\tau \]  

The first term in the above expression for \( \sigma \) is the equilibrium response, while the second integral represents the time-dependent response.

Eq. (21) is used to determine the viscoelastic material constants from the constant strain rate tests. Denote the strain rate \( \ddot{\varepsilon}_0 \) such that \( \dot{\lambda} = \ddot{\varepsilon}_0 \) \( t + 1 \) and \( \dot{\lambda} = \ddot{\varepsilon}_0 \). The integral in Eq. (21) may be reformulated in terms of extension ratio by transforming \( \ddot{\varepsilon} = \ddot{\varepsilon}_0 \tau + 1 \) and \( d\tau = \frac{d\ddot{\varepsilon}}{\ddot{\varepsilon}_0} \), where \( \ddot{\varepsilon} \) is a dummy variable:

\[ \sigma = 2 \mu \varepsilon(I_1 - 3)^{\varepsilon-1} \left( \dot{\lambda}^2 - \frac{1}{\lambda} \right) + \left( \dot{\lambda}^2 - \frac{1}{\lambda} \right) \int_1^{\ddot{\varepsilon}} \left[ 4c_1 \left( \frac{\ddot{\varepsilon}}{\ddot{\varepsilon}_0} - \ddot{\varepsilon} \right) \left( \ddot{\varepsilon}^2 + \frac{2}{\ddot{\varepsilon}} - 3 \right) \left( \frac{\ddot{\varepsilon} - 1}{\ddot{\varepsilon}^2} \right) \right] d\ddot{\varepsilon} \]

The values for \( c_1 \) and \( c_2 \) are found from the instantaneous curve, taken from the tests performed at \( 390 \) s\(^{-1}\). At this high strain rate, the loading time is considered small compared to the characteristic relaxation time and \( m(t - \tau) \approx 1 \). Denoting the stress of the instantaneous curve as \( \sigma_i \), one gets that

\[ \sigma_i = 2 \mu \varepsilon(I_1 - 3)^{\varepsilon-1} \left( \dot{\lambda}^2 - \frac{1}{\lambda} \right) + \left( \dot{\lambda}^2 - \frac{1}{\lambda} \right) \int_1^{\ddot{\varepsilon}_i} \left[ 4c_1 \left( \ddot{\varepsilon}^2 + \frac{2}{\ddot{\varepsilon}} - 3 \right) \left( \ddot{\varepsilon} - 1 \right) \right] d\ddot{\varepsilon} \]

A FORTRAN program was written to perform numerical integration and find appropriate values for \( c_1 \) and \( c_2 \). With \( c_1 = 0.035 \) MPa and \( c_2 = -1.5 \), the predicted instantaneous response is shown in Fig. 10 by the uppermost solid line. These two values were then used in Eq. (22) and another FORTRAN program is written to find relaxation times. Two relaxation times were needed to capture the data at low and high strain values. The relaxation times and their weighting factors are as follows: \( \theta_1 = 0.005 \) s, \( m_1 = 0.09 \), \( \theta_2 = 0.00025 \) s, and \( m_2 = 0.91 \). The analytical solution is compared to test data in Fig. 11. Both the equilibrium and instantaneous response are predicted very well, but there are slight discrepancies between the analytical solution and test data. The worst fit is at \( 114 \) s\(^{-1}\), where the analytical solution is about 10% under test data. Improvements can be made by considering more relaxation times but this would lead to a more complicated data fitting procedure. The relaxation constants given above are deemed sufficiently accurate for this study.

4. Finite element analysis of rubber sheet experiments

The sheet experiments are simulated in ABAQUS Explicit. The upper right side (quarter model) of the sheet is shown in Fig. 12. Deformation in the 1-direction is constrained at the grip side and displacement at a constant loading rate is applied. Displacement \( \Delta \) is applied to the side of the plate, like it was done in the experiments. The element type chosen is C3D8R: continuum, three-dimensional elements with 8-node reduced integration points.

The three-dimensional, hyper-viscoelastic equation proposed in Eq. (18) is incorporated via a user-defined material subroutine, VUMAT, using the material constants found from the uniaxial tension tests. Numerical time integration, using a simple trapezoidal rule, is performed to calculate the stress at each time step. The total resultant force to pull the sheet is plotted with respect to sheet extension (displacement) for the various loading rates in Fig. 13. The FEA predictions of the force–extension response of the sheets compare very well to test results. They are within 10% of expected results. Thus the three-dimensional hyper-viscoelastic constitutive equation is adequate for predicting structural response of rubber under impact or blast.
High-speed experiments were conducted to characterize the deformation and failure of Styrene Butadiene Rubber at impact rates. Dynamic tensile stress–strain curves of uniaxial strip specimens and force–extension curves of thin sheets were obtained from a Charpy tensile impact apparatus. Results from the uniaxial tension tests indicated that although the rubber becomes stiffer with increasing strain rates, the stress–strain curves remained virtually the same above 280 s$^{-1}$. Above this critical strain rate, strength, fracture strain and toughness decreased with increasing strain rate. When strain rates were below 180 s$^{-1}$, the initial modulus, tensile strength and breaking extension increased as the strain rate increased. Between strain rates of 180 s$^{-1}$ and 280 s$^{-1}$, the initial modulus and tensile strength increased with increasing strain rates but the extension at break decreased with increasing strain rates.
A hyper-viscoelastic constitutive relation of integral form was proposed to describe the rate-dependent material behavior of the rubber. Memory functions were obtained from the uniaxial test data. Two characteristic relaxation times, 5 ms and 0.25 ms, were necessary to fit the proposed constitutive equation to the data. The proposed constitutive equation was implemented in ABAQUS Explicit via a user-defined subroutine and used to predict the dynamic response of the rubber sheets in the experiments. Numerical predictions for the transient deformation and failure of the rubber sheet compared were within 10% of the experimental results. Future research will focus on the dynamic failure criteria of rubber.

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References


