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A pseudo-differential calculus on non-standard symplectic space; Spectral and regularity results in modulation spaces

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Abstract

The usual Weyl calculus is intimately associated with the choice of the standard symplectic structure on $\mathbb{R}^n \oplus \mathbb{R}^n$. In this paper we will show that the replacement of this structure by an arbitrary symplectic structure leads to a pseudo-differential calculus of operators acting on functions or distributions defined, not on $\mathbb{R}^n$ but rather on $\mathbb{R}^n \oplus \mathbb{R}^n$. These operators are intertwined with the standard Weyl pseudo-differential operators using an infinite family of partial isometries of $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ indexed by $\mathcal{S}(\mathbb{R}^n)$. This allows us to obtain spectral and regularity results for our operators using Shubin's symbol classes and Feichtinger's modulation spaces.

Résumé

Le calcul de Weyl habituel est étroitement associé au choix de la structure symplectique standard sur $\mathbb{R}^n \oplus \mathbb{R}^n$. Dans cet article nous allons montrer que le remplacement de cette structure par une structure symplectique arbitraire mène à un calcul pseudo-différentiel pour des opérators agissant sur des fonctions ou des distributions définis, non pas sur $\mathbb{R}^n$ mais sur $\mathbb{R}^n \oplus \mathbb{R}^n$. Ces opérateurs sont entrelacés avec les opérateurs de Weyl habituels par une famille infinie d'isométries partielles $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ indexées par l'espace de Schwartz $\mathcal{S}(\mathbb{R}^n)$. Ceci nous permet d'obtenir des résultats spectraux, ainsi que des propriétés de régularité pour nos opérateurs, utilisant les classes de symboles de Shubin ainsi que les espaces de modulation de Feichtinger.

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1. Introduction

Every traditional pseudo-differential calculus harks back in one way or another to the physicists' early work on quantum mechanics. Following the founding fathers of quantum mechanics one should associate to a symbol (or "observable") defined on $\mathbb{R}^{2n} \equiv \mathbb{R}^n \oplus \mathbb{R}^n$ an operator obtained by replacing the coordinates x_j by the operator $\widehat{X}_j$ of multiplication by x_j and the dual variable ξ_j by the operator $\widehat{\Xi}_j = -i\partial_{x_j}$. The ordering problem (what is the operator associated with $\xi_j x_j = x_j \xi_j$?) was solved in a satisfactory way by Weyl [45]: one associates to the symbol a the operator $\widehat{A} = \mathrm{Op}^w(a)$ with kernel formally defined by:

$$K(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a\left(\frac{1}{2}(x+y), \xi\right) d\xi. \tag{1}$$

The Weyl correspondence $a \overset{\text{Weyl}}{\longleftrightarrow} \widehat{A}$ plays a somewhat privileged role among the other possible choices $a \overset{\tau}{\longleftrightarrow} A_\tau$ corresponding to the kernels

$$K_\tau(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a\big(\tau x + (1-\tau)y, \xi\big) d\xi \tag{2}$$

with $\tau \in \mathbb{R}$. This is due mainly to two reasons: first of all, the choice (1) ensures us that to real symbols correspond (formally) self-adjoint operators; secondly, among all possible choices (2) the Weyl correspondence $a \overset{\text{Weyl}}{\longleftrightarrow} \widehat{A}$ is the only one which has the symplectic covariance property $a \circ S \overset{\text{Weyl}}{\longleftrightarrow} \widehat{S}^{-1}\widehat{A}\widehat{S}$ where $\widehat{S} \in \mathrm{Mp}(2n, \sigma)$ has projection $S \in \mathrm{Sp}(2n, \sigma)$ ($\mathrm{Sp}(2n, \sigma)$ and $\widehat{S} \in \mathrm{Mp}(2n, \sigma)$ are the symplectic and metaplectic groups, respectively). It turns out that the Weyl correspondence is intimately related to the standard symplectic structure $\sigma(z, z') = \xi \cdot x' - \xi' \cdot x$ on $\mathbb{R}^n \oplus \mathbb{R}^n$ or, equivalently, to the commutation relations

$$[\widehat{X}_j, \widehat{X}_k] = [\widehat{\Xi}_j, \widehat{\Xi}_k] = 0, \qquad [\widehat{X}_j, \widehat{\Xi}_k] = i\delta_{jk} \tag{3}$$

satisfied by the elementary Weyl operators $\widehat{X}_j$, $\widehat{\Xi}_k$. Setting $\widehat{Z}_\alpha = \widehat{X}_\alpha$ if $1 \leqslant \alpha \leqslant n$ and $\widehat{Z}_\alpha = \widehat{\Xi}_{\alpha-n}$ if $n+1 \leqslant \alpha \leqslant 2n$ these relations can be rewritten

$$[\widehat{Z}_\alpha, \widehat{Z}_\beta] = i j_{\alpha\beta} \quad \text{for } 1 \leqslant \alpha, \beta \leqslant 2n, \tag{4}$$

where

$$J = (j_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant 2n} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the matrix of the symplectic form σ. Here $I, 0$ denote the $n \times n$ identity and zero matrices, respectively.

We now make the two following essential observations:

- There are many operators satisfying the commutation relations (3)–(4). For instance, they are preserved if one replaces $\widehat{X}_j$ and $\widehat{\Xi}_j$ with the operators

$$\widetilde{X}_j = x_j + \frac{1}{2} i\partial_{\xi_j}, \qquad \widetilde{\Xi}_j = \xi_j - \frac{1}{2} i\partial_{x_j}, \tag{5}$$

  (these are the "Bopp shifts" [9] familiar from the physical literature). Notice that $\widetilde{X}_j$ and $\widetilde{\Xi}_j$ act not on functions defined on $\mathbb{R}^n$ but rather on functions defined on $\mathbb{R}^n \oplus \mathbb{R}^n$. Indeed, in recent papers de Gosson [22], de Gosson and Luef [26,28], Dias et al. [14] it has been shown that the operators $\widetilde{X}_j$ and $\widetilde{\Xi}_j$ can be used to reformulate the Moyal product familiar from deformation quantization [5,6] in terms of a phase-space pseudo-differential calculus, which also intervenes in the study of certain magnetic operators ("Landau calculus" [22]).
- The second observation takes us to the subject of this paper. The choice of the standard symplectic structure, associated with the commutation relations (4), is to a great extent arbitrary. So one could wonder what happens if we replace the matrix $J = (j_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant 2n}$ with some other non-degenerate skew-symmetric matrix Ω. This question is not only academic: the study of non-commutative field theories and their connections with quantum gravity [1–4,15,42] has led physicists to consider more general commutation relations of the type

$$[\widetilde{Z}_\alpha, \widetilde{Z}_\beta] = i\omega_{\alpha\beta} \quad \text{for } 1 \leqslant \alpha, \beta \leqslant 2n, \tag{6}$$

where the numbers $\omega_{\alpha\beta}$ are defined by

$$\Omega = (\omega_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant 2n} = \begin{pmatrix} \Theta & I \\ -I & N \end{pmatrix}, \tag{7}$$

where $\Theta = (\theta_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant n}$ and $N = (\eta_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant n}$ are antisymmetric matrices (see [3,7,12]). The commutation relations (6) are satisfied by the operators

$$\widetilde{X}_j = x_j + \frac{1}{2}i\partial_{\xi_j} + \frac{1}{2}i\sum_k \theta_{jk}\partial_{x_k}, \tag{8}$$

$$\widetilde{\Xi}_j = \xi_j - \frac{1}{2}i\partial_{x_j} + \frac{1}{2}i\sum_k \eta_{jk}\partial_{\xi_k} \tag{9}$$

which reduce to the "Bopp shifts" (5) when $\Omega = J$. The relation of these operators with a deformation quantization has been made explicit in Dias et al. [14].

Writing formulas (8)–(9) in compact form as

$$\widetilde{Z} = z + \frac{1}{2}i\Omega\partial_z \tag{10}$$

this suggests that one should be able to give a sense to pseudo-differential operators formally written as

$$\widetilde{A}_\omega = a(\widetilde{Z}) = a\left(z + \frac{1}{2}i\Omega\partial_z\right). \tag{11}$$

We set out in this paper to justify formula (11); more generally we define a pseudo-differential calculus arising from the choice of an arbitrary symplectic form ω with constant coefficients on $\mathbb{R}^n \oplus \mathbb{R}^n$ associated to an antisymmetric matrix $\Omega \in GL(2n;\mathbb{R})$ by the formula

$$\omega(z, z') = z \cdot \Omega^{-1} z'.$$

This symplectic form obviously coincides with the standard symplectic form σ when $\Omega = J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. The consideration of such operators $\widetilde{A}_\omega$ leads to a class of Weyl operators with symbols defined on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$.

In this article we will show that

- The formal definition (11) of the operators $\widetilde{A}_\omega$ and their Weyl symbols can be made rigorous.
- The operators $\widetilde{A}_\omega$ are intertwined with the usual Weyl operators $\widehat{A}$ using a family of partial isometries $u \longmapsto W_{f,\phi}u$ of $L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^{2n})$ parametrized by $\phi \in \mathcal{S}(\mathbb{R}^n)$.
- The spectral properties of the operators $\widetilde{A}_\omega$ can be recovered from those of $\widehat{A}$ using these intertwining relations; in particular the consideration of Shubin's classes of globally hypoelliptic symbols will allow us to state a very precise result when $\widehat{A}$ is formally self-adjoint.

Our results show that the study of the physicist's "non-commutative quantum mechanics" can be reduced to the study of a particular Weyl calculus with symbols defined on a double phase space.

We want to mention that the connections between symbol classes and non-commutative harmonic analysis have also been explored (from a different point of view) by Unterberger [43] and Unterberger and Upmeier [44]; it would perhaps be interesting to analyze their results from the point of view of the methods and tools introduced in the present paper.

Notation 1. The generic point of $\mathbb{R}^n \oplus \mathbb{R}^n \equiv \mathbb{R}^{2n}$ is denoted by $z = (x, \xi)$ and that of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \equiv \mathbb{R}^{4n}$ by (z, ζ). The standard symplectic form σ on $\mathbb{R}^{2n}$ is defined by $\sigma(z, z') = \xi \cdot x' - \xi' \cdot x$ and the corresponding symplectic group is denoted $\mathrm{Sp}(2n, \sigma)$. Given an arbitrary symplectic form ω on $\mathbb{R}^n \oplus \mathbb{R}^n$ we denote by $\mathrm{Sp}(2n, \omega)$ the corresponding symplectic group.

Notation 2. Functions (or distributions) on $\mathbb{R}^n$ are denoted by small Latin or Greek letters $u, v, \phi, \dots$ while those defined on $\mathbb{R}^{2n}$ by capitals $U, V, \Phi, \dots$. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$; its dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. The scalar product of two functions $u, v \in L^2(\mathbb{R}^n)$ is denoted by $(u|v)$ and that of $U, V \in L^2(\mathbb{R}^{2n})$ by $((U|V))$. The corresponding norms are written $\|u\|$ and $|||U|||$.

2. Phase space Weyl operators

Let us begin by giving a short review of the main definitions and properties from standard Weyl calculus as exposed (with fluctuating notation) in for instance [21,34,38,40,46]; this will allow us to list some useful formulas we will need in the forthcoming sections.

2.1. Standard Weyl calculus

Given a function $a \in \mathcal{S}(\mathbb{R}^{2n})$ the Weyl operator $\widehat{A}$ with symbol a is defined by:

$$\widehat{A}u(x) = \left(\frac{1}{2\pi}\right)^n \iint_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a\left(\frac{1}{2}(x+y), \xi\right) u(y)\, dy\, d\xi \tag{12}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$. This definition makes sense for more general symbols a provided that the integral interpreted in some "reasonable way" (oscillatory integral, for instance) when a is in a suitable symbol class, for instance the Hörmander classes $S^m_{\rho,\delta}$, or the global Shubin spaces $H\Gamma^{m_1,m_0}_\rho$. A better definition is, no doubt, the operator integral

$$\widehat{A} = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} F_\sigma a(z) \widehat{T}(z)\, dz \tag{13}$$

because it immediately makes sense for arbitrary symbols $a \in \mathcal{S}'(\mathbb{R}^{2n})$; here F_σ is the symplectic Fourier transform:

$$F_\sigma a(z) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} e^{-i\sigma(z,z')} a\big(z'\big)\, dz', \tag{14}$$

$\widehat{T}(z_0)$ is the Heisenberg–Weyl operator $\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ formally defined by

$$\widehat{T}(z_0) = e^{-i\sigma(\widehat{Z}, z_0)} \quad \text{with } \widehat{Z} = (x, -i\partial_x); \tag{15}$$

the action of $\widehat{T}(z_0)$ on $u \in \mathcal{S}(\mathbb{R}^n)$ is given by the explicit formula

$$\widehat{T}(z_0)u(x) = e^{i(\xi_0\cdot x - \frac{1}{2}\xi_0\cdot x_0)} u(x - x_0) \tag{16}$$

if $z_0 = (x_0, \xi_0)$. We note that F_σ is an involution which extends into an involutive automorphism $\mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$. The Weyl correspondence, written $a \overset{\text{Weyl}}{\longleftrightarrow} \widehat{A}$ or $\widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a$, between an element $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and the Weyl operator it defines is bijective; in fact the Weyl transformation is one-to-one from $\mathcal{S}'(\mathbb{R}^{2n})$ onto the space $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^{2n}))$ of continuous maps $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ (see e.g. Maillard [36], Wong [46] or [43]). This can be proved using Schwartz's kernel theorem and the fact that the Weyl symbol a of the operator $\widehat{A}$ is related to the distributional kernel of that operator by the partial Fourier transform with respect to the y variable

$$a(x, \xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot y} K\left(x + \frac{1}{2}y, x - \frac{1}{2}y\right) dy, \tag{17}$$

where $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ and the Fourier transform is defined in the usual distributional sense. Conversely (cf. formula (12)) the kernel K is expressed in terms of the symbol a by the inverse Fourier transform

$$K(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} a\left(\frac{1}{2}(x+y), \xi\right) d\xi.$$

Assuming that the product $\widehat{A}\,\widehat{B}$ exists (which is the case for instance if $\widehat{B}:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n)$) the Weyl symbol c of $\widehat{C}=\widehat{A}\,\widehat{B}$ and its symplectic Fourier transform $F_\sigma c$ are given by the formulas

$$c(z)=\left(\frac{1}{4\pi}\right)^{2n}\iint\limits_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{\frac{i}{2}\sigma(u,v)}a\left(z+\frac{1}{2}u\right)b\left(z-\frac{1}{2}v\right)du\,dv, \tag{18}$$

$$F_\sigma c(z)=\left(\frac{1}{2\pi}\right)^{n}\int\limits_{\mathbb{R}^{2n}} e^{\frac{i}{2}\sigma(z,z')}F_\sigma a\bigl(z-z'\bigr)F_\sigma b\bigl(z'\bigr)\,dz'. \tag{19}$$

The first formula is often written $c=a\#b$ and $a\#b$ is called the “twisted product” or “Moyal product” (see e.g. [46]).

Two important properties of the Weyl correspondence already mentioned in the Introduction are the following:

Proposition 3. *Let* $\widehat{A}\overset{\text{Weyl}}{\longleftrightarrow}a$:

(i) *The operator $\widehat{A}$ is formally self-adjoint if and only the symbol a is real*; *more generally the symbol of the formal adjoint of an operator with Weyl symbol a is its complex conjugate $\bar{a}$*;
(ii) *Let $\widehat{S}\in\mathrm{Mp}(2n,\sigma)$. We have $\widehat{S}^{-1}\widehat{A}\,\widehat{S}\overset{\text{Weyl}}{\longleftrightarrow}a\circ S$.*

Here $\mathrm{Mp}(2n,\sigma)$ is the metaplectic group, that is the unitary representation of the double cover of $\mathrm{Sp}(2n,\sigma)$. To every $S\in\mathrm{Sp}(2n,\sigma)$ thus corresponds, via the natural projection $\pi:\mathrm{Mp}(2n,\sigma)\longrightarrow\mathrm{Sp}(2n,\sigma)$, two operators $\pm\widehat{S}\in\mathrm{Mp}(2n,\sigma)$. We note that property (ii) is *characteristic* of the Weyl pseudo-differential calculus (see Stein [40], Wong [46]). We notice that Unterberger and Upmeier [44] have studied similar covariance formula for more general operators (pseudo-differential operators of Fuchs type) which occur in the study of boundary problems with edges or corners.

A related well-known object is the cross-Wigner transform $W(u,v)$ of $u,v\in\mathcal{S}(\mathbb{R}^n)$; it is defined by

$$W(u,v)(z)=\left(\frac{1}{2\pi}\right)^{n}\int\limits_{\mathbb{R}^{n}} e^{-i\xi\cdot y}u\left(x+\frac{1}{2}y\right)\overline{v\left(x-\frac{1}{2}y\right)}\,dy \tag{20}$$

(it is thus, up to a constant, the Weyl symbol of the operator with kernel $u\otimes\bar{v}$). We note, for further use, that $W(u,v)$ can alternatively be defined by the formula

$$W(u,v)(z)=\pi^{-n}\bigl\langle\widehat{T}_{\mathrm{GR}}(z)u,\bar{v}\bigr\rangle, \tag{21}$$

where $\widehat{T}_{\mathrm{GR}}(z)$ is the Grossmann–Royer operator:

$$\widehat{T}_{\mathrm{GR}}(z_0)u(x)=e^{2i\xi_0\cdot(x-x_0)}u(2x_0-x). \tag{22}$$

Formula (21) allows us to define $W(u,v)$ when $u\in\mathcal{S}'(\mathbb{R}^n)$ and $v\in\mathcal{S}(\mathbb{R}^n)$; one can actually extend the mapping $(u,v)\longrightarrow W(u,v)$ into a continuous mapping $\mathcal{S}'(\mathbb{R}^n)\times\mathcal{S}'(\mathbb{R}^n)\longrightarrow\mathcal{S}'(\mathbb{R}^{2n})$. The cross-Wigner transform enjoys the following symplectic-covariance property: if $S\in\mathrm{Sp}(2n,\sigma)$ then

$$W(u,v)\bigl(S^{-1}z\bigr)=W(\widehat{S}u,\widehat{S}v)(z), \tag{23}$$

where $\widehat{S}\in\mathrm{Mp}(2n,\sigma)$ has projection S. Let $u,v\in\mathcal{S}(\mathbb{R}^n)$. The following important property is sometimes taken as the definition of the Weyl operator $\widehat{A}$:

$$(\widehat{A}u|v)=\int\limits_{\mathbb{R}^{2n}} a(z)W(u,v)(z)\,dz=\bigl\langle a,W(u,v)\bigr\rangle. \tag{24}$$

Also note that the cross-Wigner transform satisfies the Moyal identity

$$\bigl(\bigl(W(u,v)|W\bigl(u',v'\bigr)\bigr)\bigr)=\left(\frac{1}{2\pi}\right)^{n}\bigl(u|u'\bigr)\overline{\bigl(v|v'\bigr)}. \tag{25}$$

The following formula describes the action of the Heisenberg–Weyl operators:

$$W\big(\widehat{T}(z_0)u, \widehat{T}(z_1)v\big)(z) = e^{i[-\sigma(z,z_0-z_1)-\frac{1}{2}\sigma(z_0,z_1)]} W(u,v)\big(z - \langle z\rangle\big), \tag{26}$$

where $\langle z\rangle = \frac{1}{2}(z_0 + z_1)$; the particular case

$$W\big(\widehat{T}(z_0)u, v\big)(z) = e^{-i\sigma(z,z_0)} W(u,v)\left(z - \frac{1}{2}z_0\right) \tag{27}$$

will be used in our study of intertwining operators.

2.2. *Definition of the operators $\widetilde{A}_\omega$*

In what follows Ω denotes an arbitrary (real) invertible antisymmetric $2n \times 2n$ matrix. The formula

$$\omega\big(z, z'\big) = z \cdot \Omega^{-1} z' = -\Omega^{-1} z \cdot z' \tag{28}$$

defines a symplectic form on $\mathbb{R}^{2n}$ which coincides with the standard symplectic form σ when $\Omega = J$.

Let us introduce the following variant of the symplectic Fourier transform:

Definition 4. For $a \in \mathcal{S}(\mathbb{R}^{2n})$ we set:

$$F_\omega a(z) = \left(\frac{1}{2\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-i\omega(z,z')} a\big(z'\big)\, dz'. \tag{29}$$

Obviously F_ω is a continuous automorphism of $\mathcal{S}(\mathbb{R}^{2n})$. Moreover,

Lemma 5. *The automorphism F_ω extends into a unitary automorphism of $L^2(\mathbb{R}^{2n})$ and into a continuous automorphism of $\mathcal{S}'(\mathbb{R}^{2n})$. Moreover, F_ω is related to the usual unitary Fourier transform F on $\mathbb{R}^{2n}$ by the formula*

$$Fa(z) = |\det \Omega|^{1/2} F_\omega a(-\Omega z). \tag{30}$$

In particular F_ω is involutive, that is

$$F_\omega F_\omega a = a. \tag{31}$$

Remark 6. Notice that we are using the normalization of the Fourier transform according to the rule $(2\pi)^{-\text{dimension}/2}$. Since we are working in the phase-space (dimension $= 2n$), we have a factor $(2\pi)^{-n}$ rather than the usual factor $(2\pi)^{-n/2}$.

Proof. From $\omega(-\Omega z, z') = z \cdot z'$, we immediately recover (30). From (30) and the unitarity of the Fourier transform, we have in $L^2(\mathbb{R}^{2n})$:

$$\begin{aligned} \|a\| = \|Fa\| &= |\det \Omega|^{1/2} \left(\int_{\mathbb{R}^{2n}} \big|F_\omega a(-\Omega z)\big|^2 dz \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^{2n}} \big|F_\omega a\big(z'\big)\big|^2 dz' \right)^{\frac{1}{2}} = \|F_\omega a\|, \end{aligned} \tag{32}$$

where we performed the substitution $z' = -\Omega z$. Consequently, F_ω extends into a unitary automorphism of $L^2(\mathbb{R}^{2n})$. The symplectic Fourier transform F_ω also extends into a continuous automorphism of $\mathcal{S}'(\mathbb{R}^{2n})$ in the usual way by defining $F_\omega a$ for $a \in \mathcal{S}'(\mathbb{R}^{2n})$ by the formula $\langle F_\omega a, b\rangle = \langle a, F_\omega b\rangle$ for all $b \in \mathcal{S}(\mathbb{R}^{2n})$ (or, alternatively, by using the relation (30) above). Note that when $\Omega = J$ we have $F_\omega = F_\sigma$ (the ordinary symplectic Fourier transform) since $\det J = 1$. Using formula (30) the symplectic Fourier transform F_ω can thus be written:

$$F_\omega = U_\Omega I F, \tag{33}$$

where U_Ω and I are the transformations defined by

$$(U_\Omega a)(z) = |\det \Omega|^{1/2} a\big(\Omega^{-1} z\big), \qquad (Ia)(z) = a(-z) \tag{34}$$

for which trivially:

$$((U_\Omega a|U_\Omega b)) = ((a|b)), \qquad ((Ia|Ib)) = ((a|b)) \tag{35}$$

for all $a, b \in L^2(\mathbb{R}^{2n})$. From (35) and the Parseval identity, it follows that for all $a, b \in L^2(\mathbb{R}^{2n})$:

$$((F_\omega F_\omega a|b)) = ((F_\omega a|F_\omega b)) = ((U_\Omega I F a|U_\Omega I F b)) = ((a|b)) \tag{36}$$

which proves (31). □

In the sequel we will also need the operators

$$\widetilde{T}_\omega(z_0) : \mathcal{S}'\big(\mathbb{R}^{2n}\big) \longrightarrow \mathcal{S}'\big(\mathbb{R}^{2n}\big)$$

defined by the formula

$$\widetilde{T}_\omega(z_0) U(z) = e^{-i\omega(z,z_0)} U\left(z - \frac{1}{2} z_0\right). \tag{37}$$

These operators satisfy the same commutation relations as the usual Heisenberg–Weyl operators $\widehat{T}(z_0)$ when $\omega = \sigma$. In fact, a straightforward computation shows that

$$\widetilde{T}_\omega(z_0 + z_1) = e^{-\frac{i}{2}\omega(z_0,z_1)} \widetilde{T}_\omega(z_0) \widetilde{T}_\omega(z_1), \tag{38}$$

$$\widetilde{T}_\omega(z_0) \widetilde{T}_\omega(z_1) \ = e^{i\omega(z_0,z_1)} \widetilde{T}_\omega(z_1) \widetilde{T}_\omega(z_0). \tag{39}$$

Let us justify the introduction of the operators $\widetilde{T}_\omega(z_0)$ with an informal discussion; after all it is not obvious at this stage why they should allow us to implement the "quantization" (10)–(11)! Recall [21] that the introduction of the usual Heisenberg–Weyl operator $\widehat{T}(z_0) = e^{-i\sigma(\widehat{z},z_0)}$ can be motivated as follows: consider the translation Hamiltonian $H_{z_0}(z) = \sigma(z, z_0)$; the operator with this Weyl symbol is $\widehat{H}_{z_0}(z) = \sigma(\widehat{Z}, z_0)$ and the solution of the corresponding Schrö dinger equation

$$i \frac{\partial}{\partial t} u = \widehat{H}_{z_0} u, \quad u(x, 0) = u_0(x)$$

is formally given by $u(x,t) = e^{-it\sigma(\widehat{Z},z_0)} u_0(x)$; a direct calculation shows that we have the explicit formula

$$u(x,t) = e^{-it\sigma(\widehat{Z},z_0)} u_0(x) = e^{i(t\xi_0\cdot x - \frac{1}{2}t^2 \xi_0 \cdot x_0)} u_0(x - t x_0)$$

hence $\widehat{T}(z_0) u(x, 0) = u(x, 1)$. To define the operators $\widetilde{T}_\omega(z_0)$ one proceeds exactly in the same way: replacing the Hamiltonian operator $\widehat{H}_{z_0}(z) = \sigma(\widehat{Z}, z_0)$ with

$$\widetilde{H}_{z_0}(z) = \omega(\widetilde{Z}, z_0) = \omega\left(z + \frac{1}{2} i \Omega \partial_z, z_0\right)$$

we are led to the "phase space Schrödinger equation"

$$i \frac{\partial}{\partial t} U = \omega(\widetilde{Z}, z_0) U, \quad U(z, 0) = U_0(z)$$

whose solution is

$$U(z,t) = e^{-it\omega(\widetilde{Z},z_0)} U_0(z) = e^{-it\omega(z,z_0)} U_0\left(z - \frac{1}{2} t z_0\right).$$

We thus have

$$U(z, 1) = \widetilde{T}_\omega(z_0) U_0(z) = e^{-i\omega(\widetilde{Z},z_0)} U_0(z).$$

Let us now define the operators $\widetilde{A}_\omega$. Comparing with the definition (13) of the usual Weyl operators these considerations suggest that we define $\widetilde{A}_\omega = a(\widetilde{Z})$ by the formula

$$\widetilde{A}_\omega U = \left(\frac{1}{2\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\omega a(z)\widetilde{T}_\omega(z)U\,dz. \tag{40}$$

This "guess" is justified by the following result which identifies the Weyl symbol of the operator $\widetilde{A}_\omega$ defined by the formula above:

Proposition 7. *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $U \in \mathcal{S}(\mathbb{R}^{2n})$. The operator $\widetilde{A}_\omega : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ defined by*:

$$\widetilde{A}_\omega U = \left(\frac{1}{2\pi}\right)^n |\det \Omega|^{-1/2} \langle F_\omega a(\cdot), \widetilde{T}_\omega(\cdot)U\rangle, \tag{41}$$

that is, formally, by (40) *is continuous and its Weyl symbol $\widetilde{a}_\omega$ is given by the formula*

$$\widetilde{a}_\omega(z,\zeta) = a\left(z - \frac{1}{2}\Omega\zeta\right), \tag{42}$$

and we have $\widetilde{a}_\omega \in \mathcal{S}'(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n})$. When $a = 1$ the operator $\widetilde{A}_\omega$ is the identity on $\mathcal{S}(\mathbb{R}^{2n})$.

Proof. Since $\widetilde{T}_\omega(z)U \in \mathcal{S}(\mathbb{R}^{2n})$ for every z and $F_\omega a \in \mathcal{S}'(\mathbb{R}^{2n})$ the operator $\widetilde{A}_\omega$ is well-defined. We have, setting $u = z - \frac{1}{2}z_0$,

$$\begin{aligned}\widetilde{A}_\omega U(z) &= \left(\frac{1}{2\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\omega a(z_0)\widetilde{T}_\omega(z_0)U(z)\,dz_0 \\ &= \left(\frac{1}{2\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\omega a(z_0)e^{-i\omega(z,z_0)}U\left(z - \frac{1}{2}z_0\right)dz_0 \\ &= \left(\frac{2}{\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\omega a\big[2(z-u)\big]e^{2i\omega(z,u)}U(u)\,du\end{aligned}$$

hence the kernel of $\widetilde{A}_\omega$ is given by the formula

$$K(z,u) = \left(\frac{2}{\pi}\right)^n |\det \Omega|^{-1/2} F_\omega a\big[2(z-u)\big]e^{2i\omega(z,u)}.$$

It follows from formula (17) that the symbol $\widetilde{a}_\omega$ is given by:

$$\begin{aligned}\widetilde{a}_\omega(z,\zeta) &= \int_{\mathbb{R}^{2n}} e^{-i\zeta\cdot\zeta'}K\left(z + \frac{1}{2}\zeta', z - \frac{1}{2}\zeta'\right)d\zeta' \\ &= \left(\frac{2}{\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-i\zeta\cdot\zeta'} F_\omega a\big(2\zeta'\big)e^{-2i\omega(z,\zeta')}\,d\zeta',\end{aligned}$$

that is, using the obvious relation

$$\zeta\cdot\zeta' + 2\omega\big(z,\zeta'\big) = \omega\big(2z - \Omega\zeta, \zeta'\big)$$

together with the change of variables $z' = 2\zeta'$,

$$\begin{aligned}\widetilde{a}_\omega(z,\zeta) &= \left(\frac{2}{\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-i\omega(2z-\Omega\zeta,\zeta')} F_\omega a\big(2\zeta'\big)\,d\zeta' \\ &= \left(\frac{1}{2\pi}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-i\omega(z-\frac{1}{2}\Omega\zeta,z')} F_\omega a\big(z'\big)\,dz'.\end{aligned}$$

Formula (42) immediately follows using the Fourier inversion formula (31). That $\widetilde{A}_\omega = I$ when $a = 1$ immediately follows from the fact that $F_\omega a = (2\pi)^n |\det \Omega|^{1/2}\delta$ where δ is the Dirac measure on $\mathbb{R}^{2n}$. The continuity statement follows from the fact that $\widetilde{A}_\omega$ is a Weyl operator. □

Two immediate consequences of this result are:

Corollary 8. *The operators $\widetilde{A}_\omega$ have the following properties*:

(i) *The operator $\widetilde{A}_\omega$ defined by* (40) *is formally self-adjoint if and only if a is real.*
(ii) *The formal adjoint $\widetilde{A}_\omega^*$ of $\widetilde{A}_\omega$ is obtained by replacing a with its complex conjugate $\bar{a}$.*
(iii) *The symbol $\widetilde{c}$ of $\widetilde{C}_\omega = \widetilde{A}_\omega \widetilde{B}_\omega$ is given by $\widetilde{c}_\omega(z,\zeta) = c(z - \frac{1}{2}\Omega\zeta)$, where $c = a\#b$ is the Weyl symbol of the operator $\widehat{C} = \widehat{A}\widehat{B}$.*

Proof. (i) The property is obvious since $\widetilde{A}_\omega$ is formally self-adjoint if and only if its Weyl symbol $\widetilde{a}_\omega$ is real, that is if and only if a itself is real. (ii) Similarly, the Weyl symbol of $\widetilde{A}_\omega^*$ is the function

$$(z,\zeta) \longmapsto \overline{a\left(z - \frac{1}{2}\Omega\zeta\right)}.$$

(iii) The property is an immediate consequence of the definition of $\widetilde{C}_\omega$ since $a\#b \overset{\text{Weyl}}{\longleftrightarrow} \widehat{A}\widehat{B}$. □

2.3. Symplectic transformation properties

Let ω be the symplectic form (28) on $\mathbb{R}^n \oplus \mathbb{R}^n$. The symplectic spaces $(\mathbb{R}^n \oplus \mathbb{R}^n, \omega)$ and $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$ are linearly symplectomorphic. That is, there exists a linear automorphism f of $\mathbb{R}^{2n}$ such that $f^*\omega = \sigma$ that is

$$\omega(fz, fz') = \sigma(z, z') \tag{43}$$

for all $(z,z') \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ (this can be viewed as a linear version of Darboux's theorem). The proof is straightforward: choose a symplectic basis $\mathcal{B}$ of $(\mathbb{R}^n \oplus \mathbb{R}^n, \omega)$ and a symplectic basis $\mathcal{B}'$ of $(\mathbb{R}^n \oplus \mathbb{R}^n, \sigma)$. Then any linear automorphism f of $\mathbb{R}^n \oplus \mathbb{R}^n$ such that $f(\mathcal{B}') = \mathcal{B}$ satisfies (43). Identifying the automorphism f with its matrix in the canonical basis, the relation (43) is equivalent to the matrix equality

$$\Omega = fJf^T. \tag{44}$$

Such a symplectomorphism $f : (\mathbb{R}^{2n}, \sigma) \longrightarrow (\mathbb{R}^{2n}, \omega)$ is by no means unique; we can in fact replace it by any automorphism $f' = fS_\sigma$ where $S_\sigma \in \mathrm{Sp}(2n, \sigma)$; note however that the determinant is an invariant because we have

$$\det f' = \det f \det S_\sigma = \det f$$

since $\det S_\sigma = 1$. The symplectic groups $\mathrm{Sp}(\mathbb{R}^{2n}, \omega)$ and $\mathrm{Sp}(\mathbb{R}^{2n}, \sigma)$ are canonically isomorphic.

We are going to see that the study of the operators $\widetilde{A}_\omega$ is easily reduced to the case where $\omega = \sigma$, the standard symplectic form on $\mathbb{R}^{2n}$. This result is closely related to the symplectic covariance of Weyl operators under metaplectic conjugation as we will see below.

For f a linear automorphism of $\mathbb{R}^{2n}$ we define the operator

$$M_f : \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

by the formula

$$M_f U(z) = \sqrt{|\det f|}\, U(fz). \tag{45}$$

Clearly M_f is unitary: we have $|||M_f U||| = |||U|||$ for all $U \in L^2(\mathbb{R}^{2n})$.

Notation 9. When $\Omega = J$ we write $\widetilde{T}(z_0) = \widetilde{T}_\sigma(z_0)$ and $\widetilde{A} = \widetilde{A}_\sigma$.

Proposition 10. *Let $f:(\mathbb{R}^{2n},\sigma)\longrightarrow(\mathbb{R}^{2n},\omega)$ be a linear symplectomorphism.*

(i) *We have the conjugation formulas*

$$M_f\widetilde{T}_\omega(z_0)=\widetilde{T}\big(f^{-1}z_0\big)M_f,\quad M_fF_\omega=F_\sigma M_f, \tag{46}$$

$$M_f\widetilde{A}_\omega=\widetilde{A}'M_f\quad \textit{with } a'(z)=a(fz). \tag{47}$$

(ii) *When f is replaced by an automorphism $f'=fS_\sigma$ with $S_\sigma\in\mathrm{Sp}(2n,\sigma)$ then $\widetilde{A}'$ is replaced by the operator*

$$\widetilde{A''}=M_{S_\sigma}\widetilde{A}'M_{S_\sigma}^{-1}, \tag{48}$$

where $M_{S_\sigma}U(z)=U(S_\sigma z)$.

Proof. (i) Since $\omega(fz,z_0)=\sigma(z,f^{-1}z_0)$ we have for all $U\in\mathcal{S}'(\mathbb{R}^{2n})$,

$$\begin{aligned}M_f\big[\widetilde{T}_\omega(z_0)U\big](z)&=\sqrt{|\det f|}e^{-i\omega(fz,z_0)}U\left(fz-\frac{1}{2}z_0\right)\\&=\sqrt{|\det f|}e^{-i\sigma(z,f^{-1}z_0)}U\left(f\left(z-\frac{1}{2}f^{-1}z_0\right)\right)\\&=e^{-i\sigma(z,f^{-1}z_0)}M_fU\left(z-\frac{1}{2}f^{-1}z_0\right)\\&=\widetilde{T}\big(f^{-1}z_0\big)M_fU(z)\end{aligned}$$

which is equivalent to the first equality (46). We have likewise for $a\in\mathcal{S}'(\mathbb{R}^{2n})$

$$\begin{aligned}M_fF_\omega a(z)&=\sqrt{|\det f|}F_\omega a(fz)\\&=\left(\frac{1}{2\pi}\right)^n|\det\Omega|^{-1/2}\sqrt{|\det f|}\int_{\mathbb{R}^{2n}}e^{-i\omega(fz,z')}a\big(z'\big)\,dz'\\&=\left(\frac{1}{2\pi}\right)^n|\det\Omega|^{-1/2}\sqrt{|\det f|}\int_{\mathbb{R}^{2n}}e^{-i\sigma(z,f^{-1}z')}a\big(z'\big)\,dz'\\&=\left(\frac{1}{2\pi}\right)^n|\det\Omega|^{-1/2}|\det f|\int_{\mathbb{R}^{2n}}e^{-i\sigma(z,z'')}M_fa\big(z''\big)\,dz''\end{aligned}$$

hence the second equality (46) because

$$|\det\Omega|^{-1/2}|\det f|=1 \tag{49}$$

in view of the equality (44). To prove that $M_f\widetilde{A}_\omega=\widetilde{A}'M_f$ it suffices to use the relations (46) together with definition (40) of $\widetilde{A}_\omega$:

$$\begin{aligned}M_f\widetilde{A}_\omega&=\left(\frac{1}{2\pi}\right)^n|\det\Omega|^{-1/2}\int_{\mathbb{R}^{2n}}F_\omega a(z)M_f\widetilde{T}_\omega(z)\,dz\\&=\left(\frac{1}{2\pi}\right)^n|\det\Omega|^{-1/2}\int_{\mathbb{R}^{2n}}F_\omega a(z)\widetilde{T}\big(f^{-1}z\big)M_f\,dz;\end{aligned}$$

performing the change of variables $z\longmapsto fz$ we get, using again (49), and noting that $|\det f|^{-1/2}M_fa(z)=a(fz)$,

$$M_f\widetilde{A}_\omega=\left(\frac{1}{2\pi}\right)^n|\det\Omega|^{-1/2}|\det f|\int_{\mathbb{R}^{2n}}F_\omega a(fz)\widetilde{T}(z)M_f\,dz$$

$$= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} F_\omega a(fz)\widetilde{T}(z)M_f\,dz$$
$$= \left(\frac{1}{2\pi}\right)^n |\det f|^{-1/2} \int_{\mathbb{R}^{2n}} M_f F_\omega a(z)\widetilde{T}(z)M_f\,dz$$
$$= \left(\frac{1}{2\pi}\right)^n |\det f|^{-1/2} \int_{\mathbb{R}^{2n}} F_\sigma M_f a(z)\widetilde{T}(z)M_f\,dz$$
$$= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} F_\sigma(a\circ f)(z)\widetilde{T}(z)M_f\,dz$$
$$= \widetilde{A}'M_f.$$

(ii) To prove formula (48) it suffices to note that

$$\begin{aligned} M_{f'}\widetilde{A}_\omega &= \bigl(M_{f'}M_f^{-1}\bigr)M_f\widetilde{A}_\omega \\ &= M_{S_\sigma}\bigl(\widetilde{A}'M_f\bigr) \\ &= \bigl(M_{S_\sigma}\widetilde{A}'M_{S_\sigma}^{-1}\bigr)M_{S_\sigma}M_f \\ &= \bigl(M_{S_\sigma}\widetilde{A}'M_{S_\sigma}^{-1}\bigr)M_{f'}. \end{aligned}$$

That we have $M_{S_\sigma}U(z) = U(S_\sigma z)$ is clear since $\det S_\sigma = 1$. □

We note that formula (48) can be interpreted in terms of the symplectic covariance property of Weyl calculus. To see this, let us equip the double phase space $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ with the symplectic structure $\sigma^\oplus = \sigma\oplus\sigma$. In view of formula (42) with $\Omega = J$ the Weyl symbols of operators $\widetilde{A''}$ and $\widetilde{A}'$ are, respectively,

$$\widetilde{a}'(z,\zeta) = a\left[f\left(z-\frac{1}{2}J\zeta\right)\right], \qquad \widetilde{a''}(z,\zeta) = a\left[f'\left(z-\frac{1}{2}J\zeta\right)\right],$$

and hence, using the identities $f^{-1}f' = S_\sigma \in \mathrm{Sp}(2n,\sigma)$ and $S_\sigma J = J(S_\sigma^T)^{-1}$,

$$\widetilde{a''}(z,\zeta) = a'\left[S_\sigma\left(z-\frac{1}{2}J\bigl(S_\sigma^T\bigr)^{-1}\zeta\right)\right] = \widetilde{a}'\bigl(S_\sigma z,\bigl(S_\sigma^T\bigr)^{-1}\zeta\bigr).$$

Let now m_{S_σ} be the automorphism of $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ defined by

$$m_{S_\sigma}(z,\zeta) = \bigl(S_\sigma^{-1}z, S_\sigma^T\zeta\bigr);$$

formula (48) can thus be restated as

$$\widetilde{A''} = M_{S_\sigma}\widetilde{A}'M_{S_\sigma}^{-1} \quad \text{with } a'' = a'\circ m_{S_\sigma}^{-1}. \tag{50}$$

Recall now (see for instance [21], Chapter 7) that each automorphism f of $\mathbb{R}^{2n}$ induces an element m_f of $\mathrm{Sp}(4n,\sigma^\oplus)$ defined by $m_f(z,\zeta) = (f^{-1}z, f^T\zeta)$ and that m_f is the projection of the metaplectic operator $\widehat{M}_f \in \mathrm{Mp}(\mathbb{R}^{2n}\oplus\mathbb{R}^{2n},\sigma^\oplus)$ (with $\sigma^\oplus = \sigma\oplus\sigma$) defined by (45). Formula (50) thus reflects the symplectic covariance property of Weyl calculus mentioned in Section 2.1.

We finally note that if we equip $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ with the symplectic form $\omega^\oplus = \omega\oplus\omega$, the symplectomorphism $f:(\mathbb{R}^{2n},\sigma)\longrightarrow(\mathbb{R}^{2n},\omega)$ induces a natural symplectomorphism

$$f\oplus f:\bigl(\mathbb{R}^{2n}\oplus\mathbb{R}^{2n},\sigma^\oplus\bigr)\longrightarrow\bigl(\mathbb{R}^{2n}\oplus\mathbb{R}^{2n},\omega^\oplus\bigr).$$

3. The intertwining property

In this section we show that the operators $\widetilde{A}_\omega$ can be intertwined with the standard Weyl operator $\widehat{A}$ using an infinite family of partial isometries $(W_{f,\phi})_{\phi\in\mathcal{S}(\mathbb{R}^n)}$ of $L^2(\mathbb{R}^n)$ (depending on Ω) into $L^2(\mathbb{R}^{2n})$. Each $W_{f,\phi}$ maps isomorphically $L^2(\mathbb{R}^n)$ onto a closed subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n})$.

3.1. The partial isometries $W_{f,\phi}$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\|\phi\| = 1$; ϕ will be hereafter called a *window*. In [26] two of us have studied the linear mapping $W_\phi : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ defined by the formula

$$W_\phi u = (2\pi)^{n/2} W(u, \phi), \tag{51}$$

where $W(u, \phi)$ is the cross-Wigner distribution (20). Notice that

$$W_\phi u(z) = \left(\frac{2}{\pi}\right)^{n/2} \big(\widehat{T}_{\mathrm{GR}}(z)u|\phi\big), \tag{52}$$

where $\widehat{T}_{\mathrm{GR}}(z)$ is the Grossmann–Royer transform (22).

Proposition 11. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a window.*

(i) *The mapping $W_\phi : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ extends into a mapping*

$$W_\phi : \mathcal{S}'\big(\mathbb{R}^n\big) \longrightarrow \mathcal{S}'\big(\mathbb{R}^{2n}\big)$$

whose restriction to $L^2(\mathbb{R}^n)$ is an isometry onto a closed subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n})$.

(ii) *The inverse of W_ϕ is given by the formula $u = W_\phi^{-1} U$ with*

$$u(x) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^{2n}} U(z_0)\widehat{T}_{\mathrm{GR}}(z_0)\phi(x)\, dz_0, \tag{53}$$

and the adjoint W_ϕ^* of W_ϕ is given by the formula

$$W_\phi^* U = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^{2n}} U(z_0)\widehat{T}_{\mathrm{GR}}(z_0)\phi(x)\, dz_0. \tag{54}$$

(iii) *The operator $P_\phi = W_\phi W_\phi^*$ is the orthogonal projection of $L^2(\mathbb{R}^{2n})$ onto the Hilbert space $\mathcal{H}_\phi$.*

Proof. (i) In view of Moyal's identity (25) the operator W_ϕ extends into an isometry of $L^2(\mathbb{R}^n)$ onto a subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n})$:

$$\big(\big(W_\phi u|W_\phi u'\big)\big) = \big(u|u'\big).$$

The subspace $\mathcal{H}_\phi$ is closed, being homeomorphic to $L^2(\mathbb{R}^n)$. (ii) The inversion formula (53) is verified by a direct calculation: let us set

$$w(x) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^{2n}} U(z_0)\widehat{T}_{\mathrm{GR}}(z_0)\phi(x)\, dz_0$$

and choose an arbitrary function $v \in \mathcal{S}(\mathbb{R}^n)$. We have

$$\begin{aligned}
(w|v) &= \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^{2n}} U(z_0)\big(\widehat{T}_{\mathrm{GR}}(z_0)\phi|v\big)\, dz_0 \\
&= (2\pi)^{n/2} \int_{\mathbb{R}^{2n}} U(z_0)\overline{W(v,\phi)}(z_0)\, dz_0 \\
&= \int_{\mathbb{R}^{2n}} W_\phi u(z_0)\overline{W_\phi v(z_0)}\, dz_0 \\
&= (u|v)
\end{aligned}$$

hence $w = u$ which proves (53); formula (54) for the adjoint follows since $W_\phi^* W_\phi$ is the identity on $L^2(\mathbb{R}^n)$. (iii) We have $P_\phi = P_\phi^*$ and $P_\phi P_\phi^* = P_\phi$ hence P_ϕ is an orthogonal projection. Since $W_\phi^* W_\phi$ is the identity on $L^2(\mathbb{R}^n)$ the range of W_ϕ^* is $L^2(\mathbb{R}^n)$ and that of P_ϕ is therefore precisely $\mathcal{H}_\phi$. □

In [26] it was shown that the partial isometries W_ϕ can be used to intertwine the operators $\widetilde{A} = \widetilde{A}_\sigma$ with symbol $\widetilde{a}$ with the usual Weyl operators with symbol a; we reproduce the proof for convenience:

Proposition 12. *Let $\widetilde{T}(z_0) = \widetilde{T}_\sigma(z_0)$. We have the following intertwining properties*:

$$W_\phi \widehat{T}(z_0) = \widetilde{T}(z_0) W_\phi \quad \text{and} \quad W_\phi^* \widetilde{T}(z_0) = \widehat{T}(z_0) W_\phi^*, \tag{55}$$

$$\widetilde{A} W_\phi = W_\phi \widehat{A} \quad \text{and} \quad W_\phi^* \widetilde{A} = \widehat{A} W_\phi^*. \tag{56}$$

Proof. Formula (55) immediately follows from the shift property (27). On the other hand we have

$$W_\phi \widehat{A} u = \left(\frac{1}{2\pi}\right)^n \int\limits_{\mathbb{R}^{2n}} F_\sigma a(z_0) W_\phi \big[\widehat{T}(z_0) u\big]\, dz_0,$$

and hence, in view of (55),

$$W_\phi \widehat{A} u = \left(\frac{1}{2\pi}\right)^n \int\limits_{\mathbb{R}^{2n}} F_\sigma a(z_0) \big[\widetilde{T}(z_0) W_\phi u\big]\, dz_0$$

which is the first equality (56). To prove the second equality (56) it suffices to apply the first to $W_\phi^* \widetilde{A} = (\widetilde{A}^* W_\phi)^*$. □

Let us generalize this result to the case of an arbitrary operator $\widetilde{A}_\omega$.

Proposition 13. *Let ω be a symplectic form* (28) *on $\mathbb{R}^{2n}$ and f a linear automorphism such that $f^*\omega = \sigma$. The mappings $W_{f,\phi} : \mathcal{S}(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^{2n})$ defined by the formula*

$$W_{f,\phi} = M_f^{-1} W_\phi \tag{57}$$

are partial isometries $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$, in fact isometries on a closed subspace $\mathcal{H}_{f,\phi}$ of $L^2(\mathbb{R}^{2n})$, and we have

$$\widetilde{A}_\omega W_{f,\phi} = W_{f,\phi} \widehat{A'} \quad \text{and} \quad W_{f,\phi}^* \widetilde{A}_\omega = \widehat{A'} W_{f,\phi}^*, \tag{58}$$

where $\widehat{A'} \overset{\text{Weyl}}{\longleftrightarrow} a \circ f$.

Proof. We have, using the first formula (56) and (47),

$$\begin{aligned} \widetilde{A}_\omega W_{f,\phi} &= M_f^{-1} \widetilde{A}' M_f \big(M_f^{-1} W_\phi\big) \\ &= M_f^{-1} \big(\widetilde{A}' W_\phi\big) \\ &= M_f^{-1} W_\phi \widehat{A'} \\ &= W_{f,\phi} \widehat{A'}; \end{aligned}$$

the equality $W_{f,\phi}^* \widetilde{A}_\omega = \widehat{A'} W_{f,\phi}^*$ is proven in a similar way. That $W_{f,\phi}$ is a partial isometry is obvious since W_ϕ is a partial isometry and M_f is unitary. □

Let us make explicit the change of the mapping f:

Proposition 14. *Let f and f' be linear automorphisms of $\mathbb{R}^{2n}$ such that $f^*\omega = f'^*\omega = \sigma$. We have*

$$W_{f',\phi} u = W_{f,\widehat{S}_\sigma \phi}(\widehat{S}_\sigma u) \tag{59}$$

where $\widehat{S}_\sigma \in \mathrm{Mp}(2n, \sigma)$ is such that $\pi(\widehat{S}_\sigma) = f^{-1} f'$.

Proof. The relation $f^*\omega = f'^*\omega = \sigma$ implies that $S_\sigma = f^{-1}f' \in \mathrm{Sp}(2n,\sigma)$. We have $M_{f'} = M_{fS_\sigma} = M_{S_\sigma}M_f$ and hence

$$W_{f',\phi} = M_{f'}^{-1}W_\phi = M_f^{-1}M_{S_\sigma}^{-1}W_\phi.$$

Now, taking into account definition (51) of W_ϕ in terms of the cross-Wigner transform and the fact that $\det S_\sigma = 1$ we have, using the symplectic covariance property (23),

$$\begin{aligned} M_{S_\sigma}^{-1}W_\phi u(z) &= (2\pi)^{n/2}W(u,\phi)\big(S_\sigma^{-1}z\big) \\ &= (2\pi)^{n/2}W(\widehat{S}_\sigma u,\widehat{S}_\sigma\phi)(z) \\ &= W_{\widehat{S}_\sigma\phi}(\widehat{S}_\sigma u)(z) \end{aligned}$$

hence formula (59). □

We remark that the union of the ranges of the partial isometries W_ϕ viewed as mappings defined on $\mathcal{S}'(\mathbb{R}^n)$ is in a sense a rather small subset of $\mathcal{S}'(\mathbb{R}^{2n})$ even when ϕ runs over all of $\mathcal{S}'(\mathbb{R}^n)$; this is a consequence of Hardy's theorem on the concentration of a function and its Fourier transform (de Gosson and Luef [24,25]), and is related to a topological formulation of the uncertainty principle (de Gosson [23]). We will discuss these facts somewhat more in detail at the end of the article.

3.2. Action of $W_{f,\phi}$ on orthonormal bases

Let us prove the following important result that shows that orthonormal bases of $L^2(\mathbb{R}^n)$ can be used to generate orthonormal bases of $L^2(\mathbb{R}^{2n})$ using the mappings $W_{f,\phi}$:

Proposition 15. *Let $(\phi_j)_j$ be a complete family of vectors in $L^2(\mathbb{R}^n)$.*

(i) *The family $(\Phi_{j,k})_{j,k}$ with $\Phi_{j,k} = W_{f,\phi_j}\phi_k$ is complete in $L^2(\mathbb{R}^{2n})$.*
(ii) *If $(\phi_j)_j$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ then $(\Phi_{j,k})_{j,k}$ is an orthonormal basis of $L^2(\mathbb{R}^{2n})$.*

Proof. We first note that (ii) follows from (i) since $W_{f,\phi}$ is an isometry of $L^2(\mathbb{R}^n)$ onto its range $\mathcal{H}_{f,\phi}$ in $L^2(\mathbb{R}^{2n})$. Let us show that if $U \in L^2(\mathbb{R}^{2n})$ is orthogonal to the family $(\Phi_{j,k})_{j,k}$ (and hence to all the spaces $\mathcal{H}_{f,\phi_j}$) then $U = 0$. Since by definition $W_{f,\phi} = M_f^{-1}W_\phi$ and the image of a complete system of vectors by M_f^{-1} is also complete, it is sufficient to assume that $W_{f,\phi} = W_\phi$. Suppose now that we have $((U|\Phi_{j,k})) = 0$ for all indices j,k. Since

$$((U|\Phi_{j,k})) = ((U|W_{\phi_j}\phi_k)) = \big(\big(W_{\phi_j}^*U|\phi_k\big)\big)$$

it follows that $W_{\phi_j}^*U = 0$ for all j since $(\phi_j)_j$ is a basis; using the anti-linearity of W_ϕ in ϕ we have in fact $W_\phi^*U = 0$ for all $\phi \in L^2(\mathbb{R}^n)$. Let us show that this property implies that we must have $U = 0$. Recall that the adjoint of the wave-packet transform W_ϕ^* is given by

$$W_\phi^*U = \left(\frac{2}{\pi}\right)^{n/2}\int_{\mathbb{R}^{2n}} U(z_0)\widehat{T}_{\mathrm{GR}}(z_0)\phi\,dz_0,$$

where $\widehat{T}_{\mathrm{GR}}(z_0)$ is the Grossmann–Royer operator (see formula (54) above). Let now u be an arbitrary element of $\mathcal{S}(\mathbb{R}^n)$; we have, using definition (21) of the cross-Wigner transform,

$$\begin{aligned} \big(W_\phi^*U|u\big) &= \left(\frac{2}{\pi}\right)^{n/2}\int_{\mathbb{R}^{2n}} U(z)\big(\widehat{T}_{\mathrm{GR}}(z)\phi|u\big)\,dz \\ &= (2\pi)^{n/2}\int_{\mathbb{R}^{2n}} U(z)W(\phi,u)(z)\,dz. \end{aligned}$$

Let us now view $(2\pi)^{n/2}U \in L^2(\mathbb{R}^{2n})$ as the Weyl symbol of an operator $\widehat{A}_U$. In view of formula (24) we have

$$(2\pi)^{n/2} \int_{\mathbb{R}^{2n}} U(z)W(\phi, u)(z)\,dz = (\widehat{A}_U\phi|u),$$

and the condition $W^*_\phi U = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ is thus equivalent to the condition $(\widehat{A}_U\phi|u) = 0$ for all $\phi, u \in \mathcal{S}(\mathbb{R}^n)$. It follows that $\widehat{A}_U\phi = 0$ for all ϕ and hence $\widehat{A}_U = 0$. Since the Weyl correspondence is one-to-one we must have $U = 0$ as claimed. □

We remark that the argument in the proof above in fact allows to show that, more generally, given two orthonormal bases $(\phi_j)_j$ and $(\psi_j)_j$ of $L^2(\mathbb{R}^n)$ the vectors $\Phi_{j,k} = W_{f,\phi_j}\psi_k$ form an orthonormal basis of $L^2(\mathbb{R}^{2n})$.

4. Spectral properties of the operators $\widetilde{A}_\omega$

Particularly useful symbol classes for the study of the spectral properties are the "global" symbol classes $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ introduced in Shubin [38]; also see Buzano et al. [11].

4.1. The Shubin symbol classes $H\Gamma_\rho^{m_1,m_0}$

Let $m_0, m_1 \in \mathbb{R}$ and $0 < \rho \leqslant 1$. Introducing the multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{N}^{2n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$, and $\partial_z^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \partial_{y_1}^{\alpha_{n+1}} \cdots \partial_{y_n}^{\alpha_{2n}}$, we have by definition $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ if:

- *We have $a \in C^\infty(\mathbb{R}^{2n})$.*
- *There exist constants $R, C_0, C_1 \geqslant 0$ and, for every $\alpha \in \mathbb{N}^{2n}$, $|\alpha| \neq 0$, a constant $C_\alpha \geqslant 0$ such that for $|z| \geqslant R$ the following estimates hold*:

$$C_0|z|^{m_0} \leqslant \left|a(z)\right| \leqslant C_1|z|^{m_1}, \qquad \left|\partial_z^\alpha a(z)\right| \leqslant C_\alpha \left|a(z)\right| |z|^{-\rho|\alpha|}. \tag{60}$$

The first condition (60) is an ellipticity condition; observe that $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ is not a vector space.

A simple but typical example is the following: the function a defined by $a(z) = \frac{1}{2}|z|^2$ is in $H\Gamma_1^{2,2}(\mathbb{R}^{2n})$, the same applies, more generally to $a(z) = \frac{1}{2}Mz \cdot z$ when M is a real positive definite matrix.

The interest of these symbol classes comes from the following result (Shubin [38], Chapter 4):

Proposition 16. *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ be real, and $m_0 > 0$. Then the formally self-adjoint operator $\widehat{A}$ with Weyl symbol a has the following properties*:

(i) *$\widehat{A}$ is essentially self-adjoint and has discrete spectrum in $L^2(\mathbb{R}^n)$*;
(ii) *There exists an orthonormal basis of eigenfunctions $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ $(j = 1, 2, \ldots)$ with eigenvalues $\lambda_j \in \mathbb{R}$ such that* $\lim_{j\to\infty} |\lambda_j| = \infty$.

We observe that in the proposition above there exists a basis of eigenfunctions belonging to $\mathcal{S}(\mathbb{R}^n)$; this property follows from the global hypoellipticity of operators with Weyl symbol in $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$:

$$u \in \mathcal{S}'\big(\mathbb{R}^n\big) \text{ and } \widehat{A}u \in \mathcal{S}\big(\mathbb{R}^n\big) \text{ implies } u \in \mathcal{S}\big(\mathbb{R}^n\big),$$

(global hypoellipticity is thus a stronger property than that of the usual hypoellipticity, familiar from the (micro)local analysis of pseudo-differential operators).

We will also need the following elementary result that says that the symbol classes $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ are invariant under linear changes of variables:

Lemma 17. *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ with $m_0 > 0$. For every linear automorphism f of $\mathbb{R}^{2n}$ we have $f^*a = a \circ f \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$.*

Proof. Set $a'(z) = a(fz)$; clearly $a' \in C^\infty(\mathbb{R}^{2n})$. We now note that there exist $\lambda, \mu > 0$ such that $\lambda|z| \leqslant |fz| \leqslant \mu|z|$ for all $z \in \mathbb{R}^n$. Since $m_0 > 0$ it follows that

$$C'_0|z|^{m_0} \leqslant |a'(z)| \leqslant C'_1|z|^{m_1}$$

with $C'_0 = C_0\lambda^{m_0}$ and $C'_1 = C_1\mu^{m_1}$. Next, we observe that for every $\alpha \in \mathbb{N}^{2n}$, $|\alpha| \neq 0$, there exists $B_\alpha > 0$ such that $|\partial_z^\alpha a'(z)| \leqslant B_\alpha |\partial_z^\alpha a(fz)|$ (this is easily seen by induction on $|\alpha|$ and using the chain rule); we thus have

$$|\partial_z^\alpha a'(z)| \leqslant C_\alpha B_\alpha |a'(z)||fz|^{-\rho|\alpha|} \leqslant C'_\alpha |a'(z)||z|^{-\rho|\alpha|}$$

with $C'_\alpha = B_\alpha C_\alpha \mu^{-\rho|\alpha|}$. Hence $a' \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$. $\square$

4.2. Application to the operators $\widetilde{A}_\omega$

Let us now apply the theory of Shubin classes to the study of some spectral properties of the operators $\widetilde{A}_\omega$. We begin by studying the standard case $\Omega = J$; as previously we set $\widetilde{A}_\omega = \widetilde{A}$. The extension to the general case will be done using again the reduction result in Proposition 10.

Proposition 15 is the key to the following general spectral result, which shows how to obtain the eigenvalues and eigenvectors of $\widetilde{A}$ from those of $\widehat{A}$:

Proposition 18. *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ be real, and $m_0 > 0$. Then*:

- (i) *The eigenvalues of the operators $\widehat{A}$ and $\widetilde{A}$ are the same*; *and $\widetilde{A}$ has discrete spectrum $(\lambda_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}|\lambda_j| = \infty$.*
- (ii) *The eigenfunctions of $\widetilde{A}$ are given by $\Phi_{j,k} = W_{\phi_j}\phi_k$ where the ϕ_j are the eigenfunctions of the operator $\widehat{A}$.*
- (iii) *Conversely, if U is an eigenfunction of $\widetilde{A}$, then $u = W_\phi^* U$ is an eigenvector of $\widehat{A}$ corresponding to the same eigenvalue.*

Proof. That every eigenvalue of $\widehat{A}$ also is an eigenvalue of $\widetilde{A}$ is clear: if $\widehat{A}u = \lambda u$ for some $u \neq 0$, then

$$\widetilde{A}(W_\phi u) = W_\phi \widehat{A}u = \lambda W_\phi u,$$

and $U = W_\phi u \neq 0$; this proves at the same time that $W_\phi u$ is an eigenvector of $\widehat{A}$ because W_ϕ has kernel $\{0\}$. Assume conversely that $\widetilde{A}U = \lambda U$ for $U \in L^2(\mathbb{R}^{2n})$, $U \neq 0$, and $\lambda \in \mathbb{R}$. For every ϕ we have

$$\widehat{A}W_\phi^* U = W_\phi^* \widetilde{A}U = \lambda W_\phi^* U$$

hence λ is an eigenvalue of $\widehat{A}$ and u an eigenvector if $u = W_\phi^* U \neq 0$. That $\widetilde{A}$ has discrete spectrum $(\lambda_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}|\lambda_j| = \infty$ now follows from Proposition 16. We have $W_\phi u = W_\phi W_\phi^* U = P_\phi U$ where P_ϕ is the orthogonal projection on the range $\mathcal{H}_\phi$ of W_ϕ. Assume that $u = 0$; then $P_\phi U = 0$ for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, and hence $U = 0$ in view of Proposition 15. $\square$

Let us now consider the general case of operators $\widetilde{A}_\omega$.

Proposition 19. *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ be real, and $m_0 > 0$. Then*:

- (i) *The operator $\widetilde{A}_\omega$ has discrete spectrum $(\lambda_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}|\lambda_j| = \infty$.*
- (ii) *The eigenfunctions of $\widetilde{A}_\omega$ are the functions $\Phi_j = W_{f,\phi}\phi_j$ where the ϕ_j are the eigenfunctions of the operator $\widehat{A}'$ with Weyl symbol $a' = f^* a$.*
- (iii) *We have $\Phi_{j,k} = W_{f,\phi_j}\phi_k \in \mathcal{S}(\mathbb{R}^{2n})$ and the $\Phi_{j,k}$ form an orthonormal basis of $\mathcal{S}(\mathbb{R}^{2n})$.*

Proof. Recall that we have shown in Proposition 13 that $\widetilde{A}_\omega W_{f,\phi} = W_{f,\phi}\widehat{A}'$ where $\widehat{A}' \overset{\text{Weyl}}{\longleftrightarrow} a \circ f$. In view of Lemma 17 the Shubin class $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ is preserved by linear changes of variables. The proof of the proposition now follows *mutatis mutandis* from that of Proposition 18 replacing $\widehat{A}$ with the operator $\widehat{A}'$ with Weyl symbol $a \circ f$ and using the intertwining formula $\widetilde{A}_\omega W_{f,\phi} = W_{f,\phi}\widehat{A}'$ together with the fact that $W_{f,\phi} = M_f^{-1} W_{f,\phi}$ where M_f^{-1} is a unitary operator. $\square$

4.3. Gelfand triples and generalized eigenvalues

Eigenvectors of pseudo-differential operators are not always elements of a Hilbert space, but of a distribution space. The notion of Gelfand triple (or rigged Hilbert spaces, as it was called by the physicist Dirac) formalizes this observation, that we briefly recall here since it provides the natural setting for the discussion of the spectral properties of our classes of pseudo-differential operators, e.g. if the symbol is not an element of $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$.

A (Banach) Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ consists of a (Banach) Fréchet space $\mathcal{B}$ which is continuously and densely embedded into a Hilbert space $\mathcal{H}$, which in turn is w^*-continuously and densely embedded into the dual (Banach) Fréchet space $\mathcal{B}'$. In this definition one identifies $\mathcal{H}$ with its dual $\mathcal{H}^*$ and the scalar product on $\mathcal{H}$ thus extends in a natural way into a pairing between $\mathcal{B} \subset \mathcal{H}$ and $\mathcal{B}' \supset \mathcal{H}$.

The standard example of a Gelfand triple is $(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ but there are many other examples; one of them is $(M_0^1(\mathbb{R}^n), L^2(\mathbb{R}^n), M_0^1(\mathbb{R}^n)')$ where $M_0^1(\mathbb{R}^n)$ is the Feichtinger algebra which is a particular modulation space (see Section 5.1 below). The use of this Gelfand triple not only offers a better description of self-adjoint operators but it also allows a simplification of many proofs.

Given a Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ one proves that every self-adjoint operator $A : \mathcal{B} \longrightarrow \mathcal{B}$ has a complete family of generalized eigenvectors $(\psi_\alpha)_\alpha = \{\psi_\alpha \in \mathcal{B}' \colon \alpha \in \mathbb{A}\}$ ($\mathbb{A}$ an index set), defined as follows: for every $\alpha \in \mathbb{A}$ there exists $\lambda_\alpha \in \mathbb{C}$ such that

$$(\psi_\alpha, A\phi) = \lambda_\alpha(\psi_\alpha, \phi) \quad \text{for every } \phi \in \mathcal{B}.$$

Completeness of the family $(\psi_\alpha)_\alpha$ means that there exists at least one ψ_α such that $(\psi_\alpha, \phi) \neq 0$ for every $\phi \in \mathcal{B}\backslash\{0\}$. The scalars λ_α are called generalized eigenvectors. For more see [13,18,20].

Proposition 20. *Let a be a real-valued symbol in $\mathcal{S}'\mathbb{R}^n$ and choose $(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ as Gelfand triple.*

(i) *The generalized eigenvalues of $\widetilde{A}_\omega$ and those of the Weyl operator $\widehat{A'} \overset{\text{Weyl}}{\longleftrightarrow} a \circ f$ are the same.*
(ii) *Let u be a generalized eigenvector of $\widehat{A'}$: $\widehat{A'}u = \lambda u$. Then $U = W_{f,\phi}u$ satisfies $\widetilde{A}_\omega U = \lambda U$.*
(iii) *Conversely, if U is a generalized eigenvector of $\widetilde{A}_\omega$ then $u = W_\phi^* U$ is a generalized eigenvector of $\widehat{A'}$ corresponding to the same generalized eigenvalue.*

Proof. Since $\mathcal{S}(\mathbb{R}^n)$ is weak∗-dense in $\mathcal{S}'(\mathbb{R}^n)$, one can extend $\widehat{A'}$ to $\mathcal{S}'(\mathbb{R}^n)$. By the assumption on the symbol a yields bounded self-adjoint operators $\widehat{A}$ and $\widetilde{A}_\omega$ from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, respectively $\mathcal{S}'(\mathbb{R}^{2n})$ to $\mathcal{S}(\mathbb{R}^{2n})$. Therefore, the above mentioned result gives the existence of generalized eigenvectors and eigenvalues for $\widehat{A}$ and $\widetilde{A}_\omega$. Finally, the arguments of the preceding section establishing the final two propositions extend to this setting if we interpret them in a weak sense, which implies the statements (i)–(iii). □

5. Regularity in modulation spaces

The modulation spaces $M_v^{p,q}(\mathbb{R}^n)$ introduced in the 80s by Feichtinger [16,17,19] and developed by Feichtinger and Gröchenig [29] are a tool of choice for relating the regularity properties of the phase space operator $\widetilde{A}_\omega$ to those of the corresponding operator $\widehat{A}$. In addition, the modulation spaces $M_v^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ (which contain as a particular case the so-called Sjöstrand class) will supply us with symbol classes defined without any reference to differentiability properties. We define the modulation spaces in terms of the cross-Wigner transform; in the standard literature on the topic (especially in time-frequency analysis) they are defined using a closely related object, the "windowed short-time Fourier transform". Because of the particular form of the weighting functions we use, it is easy to see that both definitions coincide.

5.1. The spaces M_s^q

Let $s \geqslant 0$ and set $v_s(z) = (1 + |z|^2)^{s/2}$. We note that for every $f \in GL(2n, \mathbb{R})$ there exists a constant $C_{s,f}$ such that

$$v_s(fz) \leqslant C_{s,f} v_s(z). \tag{61}$$

The modulation space $M_s^q(\mathbb{R}^n)$ ($q \geqslant 1$) consists of all distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $W(u,\phi) \in L_s^q(\mathbb{R}^{2n})$ for some window $\phi \in \mathcal{S}(\mathbb{R}^n)$; here $L_s^q(\mathbb{R}^{2n})$ consists of all functions U on $\mathbb{R}^{2n}$ such that $v_s U \in L^q(\mathbb{R}^{2n})$. One shows that this definition is independent of the choice of window ϕ and that if it holds for one ϕ in $\mathcal{S}(\mathbb{R}^n)$ then it holds for all. Moreover the formula

$$\|u\|_{\phi, M_s^q} = \| W_\phi u \|_{L_s^q} = \left(\int_{\mathbb{R}^{2n}} |W_\phi u(z)|^q v_s^q(z)\, dz \right)^{\frac{1}{q}}$$

defines a norm on $M_s^q(\mathbb{R}^n)$ and different ϕ lead to equivalent norms. The topology defined by any of these norm endows $M_s^q(\mathbb{R}^n)$ with a Banach space structure. The spaces M_s^q increase with the parameter q: if $q \leqslant q'$ then $M_s^q(\mathbb{R}^n) \subset M_s^{q'}(\mathbb{R}^n)$. Following result summarizes the main algebraic properties of $M_s^q(\mathbb{R}^n)$:

Proposition 21.

(i) *The modulation spaces $M_s^q(\mathbb{R}^n)$ are invariant under the action of the metaplectic group* $\mathrm{Mp}(2n,\sigma)$*: $u \in M_s^q(\mathbb{R}^n)$ if and only $\widehat{S}u \in M_s^q(\mathbb{R}^n)$ for every $\widehat{S} \in \mathrm{Mp}(2n,\sigma)$;*

(ii) *There exists a constant $C > 0$ such that for every $z \in \mathbb{R}^{2n}$ we have*

$$\left\| \widehat{T}(z)u \right\|_{\phi, M_s^q} \leqslant C v_s(z) \|u\|_{\phi, M_s^q};$$

in particular $M_s^q(\mathbb{R}^n)$ is invariant under the action of the Heisenberg–Weyl operators;

(iii) *Let $f \in GL(n,\mathbb{R})$. We have $u \in M_s^q(\mathbb{R}^n)$ if and only if $f^*u = f \circ u \in M_s^q(\mathbb{R}^n)$.*

The properties (i)–(ii) above can be stated in more concise form by saying that the modulation spaces $M_s^q(\mathbb{R}^n)$ are invariant under the action of the inhomogeneous metaplectic group $\mathrm{IMp}(2n,\sigma)$ (it is the group of unitary operators generated by the elements of $\mathrm{Mp}(2n,\sigma)$ together with the Heisenberg–Weyl operators).

In the particular case $s = 0$, $q = 1$ one obtains the Feichtinger algebra $S_0(\mathbb{R}^n) = M^1(\mathbb{R}^n)$. It is an algebra for both pointwise multiplication and convolution. It is the smallest Banach algebra containing $\mathcal{S}(\mathbb{R}^n)$ and invariant under the action of the Heisenberg–Weyl operators (and hence of $\mathrm{IMp}(2n,\sigma)$), and we have

$$M^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap F\big(L^1(\mathbb{R}^n)\big),$$

using the Riemann–Lebesgue theorem it follows in particular that

$$M^1(\mathbb{R}^n) \subset C^0(\mathbb{R}^n).$$

The following easy observation will be used in the forthcoming sections:

Lemma 22. *We have $u \in M_s^q(\mathbb{R}^n)$ if and only if $W_{f,\phi}u \in L_s^q(\mathbb{R}^n)$.*

Proof. Since $W_{f,\phi} = M_f^{-1} W_\phi$ and $W_\phi u$ is proportional to $W(u,\phi)$ it suffices to show that if $U \in L_s^q(\mathbb{R}^{2n})$ then $M_f^{-1}U \in L_s^q(\mathbb{R}^{2n})$. In view of definition (45) of $M_f U$ we have, using the inequality (61),

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} \left| M_f^{-1} U(z) \right|^q v_s^q(z)\, dz &= |\det f|^{-1/2} \int_{\mathbb{R}^{2n}} \left| U\big(f^{-1}z\big) \right|^q v_s^q(z)\, dz \\
&= |\det f|^{1/2} \int_{\mathbb{R}^{2n}} \left| U(z) \right|^q v_s^q(fz)\, dz \\
&\leqslant C \int_{\mathbb{R}^{2n}} \left| U(z) \right|^q v_s^q(z)\, dz
\end{aligned}$$

which proves the assertion. □

The dual Banach space $M_0^1(\mathbb{R}^n)'$ consists of all $u \in S'(\mathbb{R}^n)$ such that $W(u,\phi) \in L^\infty(\mathbb{R}^{2n})$ for some (and hence every) window $\phi \in M_0^1(\mathbb{R}^n)$; the duality bracket is given by the pairing

$$(u, u') = \int_{\mathbb{R}^{2n}} W(u,\phi)(z)\overline{W(u',\phi)(z)}\, dz, \tag{62}$$

and the formula

$$\|\psi\|^{\hbar}_{\phi,(M_0^1)'} = \sup_{z\in\mathbb{R}^{2n}} \left|W(\psi,\phi)(z)\right| \tag{63}$$

defines a norm on $M_0^1(\mathbb{R}^n)'$ for which this space is complete.

5.2. The symbol class $M_s^{\infty,1}$

Let us now introduce a different class of modulation spaces, which contains as a particular case the Sjöstrand classes, defined by other methods in Sjöstrand [39]; also see the paper [10] by Boulkhemair. It is interesting to view these modulation spaces as symbol classes: in contrast to the cases traditionally considered in the literature, membership of a symbol a in $M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ does not require any smoothness properties of a. It turns out that this point of view allows to recover many classical and difficult regularity results (for instance then Calderón–Vaillancourt theorem) in a rather simple way; see for instance Gröchenig [30,31]. In a recent paper [27] two of us pointed out the relevance of Sjöstrand classes for deformation quantization.

As before we set $v_s(z) = (1+|z|^2)^{s/2}$ for $z \in \mathbb{R}^{2n}$. The modulation space $M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ consists of all distributions in $\mathcal{S}'(\mathbb{R}^{2n})$ (viewed as pseudo-differential symbols, and hence denoted $a, b, \ldots$) such that

$$\sup_{z\in\mathbb{R}^{2n}} \left|W(a,\Phi)(z,\zeta)v_s(z)\right| \in L^1(\mathbb{R}^n \oplus \mathbb{R}^n) \tag{64}$$

for every $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$. Here $W(a,\Phi)$ is the cross-Wigner transform of functions (or distributions) defined on $\mathbb{R}^n \oplus \mathbb{R}^n$. When $s = 0$ the space $M_0^{\infty,1}(\mathbb{R}^{2n}) = M^{\infty,1}(\mathbb{R}^{2n})$ is called the Sjöstrand class. It thus consists of all symbols $a \in \mathcal{S}'(\mathbb{R}^n \oplus \mathbb{R}^n)$ such that

$$\sup_{z\in\mathbb{R}^{2n}} \left|W(a,\Phi)(z,\zeta)\right| \in L^1(\mathbb{R}^n \oplus \mathbb{R}^n)$$

for every $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$, and we have

$$S^0_{0,0}(\mathbb{R}^{2n}) \subset C_b^{2n+1}(\mathbb{R}^{2n}) \subset M^{\infty,1}(\mathbb{R}^{2n}) \tag{65}$$

where $C_b^{2n+1}(\mathbb{R}^{2n})$ is the vector space of all bounded complex functions on $\mathbb{R}^{2n}$ with continuous and bounded derivatives up to order $2n+1$ and the symbol class $S^0_{0,0}(\mathbb{R}^{2n})$ consists of all infinitely differentiable complex functions a on $\mathbb{R}^n \oplus \mathbb{R}^n$ such that $\partial_z^\alpha a$ is bounded for all multi-indices $\alpha \in \mathbb{N}^{2n}$.

It is clear that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a complex vector space for the usual operations. In fact:

Proposition 23. *We have $\Psi \in M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ if and only if (64) holds for one $\Phi \in \mathcal{S}(\mathbb{R}^n \oplus \mathbb{R}^n)$, and*

(i) *The equalities*

$$\|a\|^{\Phi}_{M_s^{\infty,1}} = \int_{\mathbb{R}^{2n}} \sup_{z\in\mathbb{R}^{2n}} \left|W(a,\Phi)(z,\zeta)v_s(z)\right| d\zeta$$

define a family of equivalent norms on $M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ for different $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$.

(ii) *The space $M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ is a Banach space for the topology defined by any of the norms $\|\cdot\|^{\Phi}_{M_s^{\infty,1}}$ and $\mathcal{S}(\mathbb{R}^{2n})$ is a dense subspace of $M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$.*

The interest of $M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ comes from the following property of the twisted product (Gröchenig [31]):

Proposition 24. *Let $a, b \in M_s^{\infty,1}(\mathbb{R}^{2n})$. Then $a\#b \in M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$. In particular, for every window Φ there exists a constant $C_\Phi > 0$ such that*

$$\|a\#b\|^{\Phi}_{M_s^{\infty,1}} \leqslant C_\Phi \|a\|^{\Phi}_{M_s^{\infty,1}} \|b\|^{\Phi}_{M_s^{\infty,1}}.$$

Recall that the twisted product $a\#b$ is the Weyl symbol of the product $\widehat{A}\widehat{B}$ of the operators $\widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a$ and $\widehat{B} \overset{\text{Weyl}}{\longleftrightarrow} b$. Since obviously $\bar{a} \in M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ if and only and $a \in M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ the property above can be restated by saying that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a Banach $*$-algebra with respect to the twisted product $\#$ and the involution $a \longmapsto \bar{a}$.

The following property follows from Theorem 4.1 and its Corollary 4.2 in [31] (also see [29], Theorem 14.5.6); it is a particular case of more general results in Toft [41].

In the case of the Sjöstrand class $M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ one has the following more precise results:

Proposition 25. *Let $\widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a$. We have*:

(i) *If $a \in M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ then $\widehat{A}$ is bounded on $L^2(\mathbb{R}^n)$ and on all $M^q(\mathbb{R}^n) = M_0^q(\mathbb{R}^n)$*;
(ii) *If $a \in M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ then $\widehat{A}$ is bounded on every modulation space $M_s^q(\mathbb{R}^n)$*;
(iii) *If $\widehat{A}$ with $a \in M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$ is invertible with inverse $\widehat{B} \overset{\text{Weyl}}{\longleftrightarrow} b$ then $b \in M^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$.*

Property (i) thus extends the L^2-boundedness property of operators with symbols in $S^0_{0,0}(\mathbb{R}^n \oplus \mathbb{R}^n)$. Property (iii) is called the *Wiener property* of $M^{\infty,1}(\mathbb{R}^{2n})$.

5.3. Regularity results

Before we prove our main result, Proposition 27, let us show that the symbol spaces $M_s^{\infty,1}(\mathbb{R}^{2n})$ are invariant under linear changes of variables:

Lemma 26. *Let $f \in GL(2n, \mathbb{R})$ and set $f^*a = a \circ f$. There exists a constant $C_A > 0$ such that*

$$\|f^*a\|_{\Phi, M_s^{\infty,1}} \leqslant C_s \|a\|_{(f^{-1})^*\Phi, M_s^{\infty,1}} \tag{66}$$

for every $\Phi \in \mathcal{S}(\mathbb{R}^n \oplus \mathbb{R}^n)$. In particular $a \in M_s^{\infty,1}(\mathbb{R}^{2n})$ if and only $f^*a \in M_s^{\infty,1}(\mathbb{R}^n \oplus \mathbb{R}^n)$.

Proof. Let us set $b = f^*a$. We have, by definition of the cross-Wigner transform,

$$W(b, \Phi)(z, \zeta) = \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^{2n}} e^{-i\zeta\cdot\eta} a\left(fz + \frac{1}{2} f\eta\right) \overline{\Phi\left(z - \frac{1}{2}\eta\right)}\, d\eta$$

thus, performing the change of variables $\xi = f\eta$,

$$W(b, \Phi)\left(f^{-1}z, f^T\zeta\right) = \left(\frac{1}{2\pi}\right)^{2n} |\det f|^{-1} \int_{\mathbb{R}^{2n}} e^{-i\zeta\cdot\xi} a\left(z + \frac{1}{2}\xi\right) \overline{\left(f^{-1}\right)^* \Phi\left(z - \frac{1}{2}\xi\right)}\, d\xi,$$

and hence

$$W(b, \Phi)(z, \zeta) = |\det f|^{-1} W\left(a, \left(f^{-1}\right)^* \Phi\right)\left(fz, \left(f^T\right)^{-1}\zeta\right), \tag{67}$$

taking the suprema of both sides of this equality and integrating we get

$$\|f^*a\|^{\Phi}_{M_s^{\infty,1}} = \int_{\mathbb{R}^{2n}} \sup_{z\in\mathbb{R}^{2n}} \left|W\left(a, \left(f^{-1}\right)^*\Phi\right)(z, \zeta) v_s\left(f^{-1}z\right)\right| d\zeta.$$

Since $v_s(f^{-1}z) \leqslant C_{s,f} v_s(z)$ for some constant $C_{s,f} > 0$ (cf. the inequality (61)) the estimate (66) follows. □

Let us now introduce the following notation: for an arbitrary window ϕ set

$$\mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n}) = W_{f,\phi}\big(M^q_s(\mathbb{R}^n)\big) \subset L^q_s(\mathbb{R}^{2n}). \tag{68}$$

Clearly $\mathcal{L}_{f,\phi}(\mathbb{R}^{2n})$ is a closed linear subspace of $L^q_s(\mathbb{R}^{2n})$.

Proposition 27. *Let $\widetilde{A}_\omega$ be associated to the Weyl operator $\widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a$. If $a \in M^{\infty,1}_s(\mathbb{R}^{2n})$ then*

$$\widetilde{A}_\omega : \mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n}) \longrightarrow \mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$$

(*continuously*) *for every window $\phi \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. Let $U \in \mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$; by definition there exists $u \in M^q_s(\mathbb{R}^n)$ such that $U = W_{f,\phi}u$. In view of the first intertwining relation (58) we have

$$\widetilde{A}_\omega W_{f,\phi}u = W_{f,\phi}\widehat{A'}u$$

where $\widehat{A'} \overset{\text{Weyl}}{\longleftrightarrow} a'$ with $a'(z) = a(fz)$. In view of Lemma 26 above we have $a' \in M^{\infty,1}_s(\mathbb{R}^{2n})$ and hence $\widehat{A'}u \in M^q_s(\mathbb{R}^n)$ and is bounded in view of Proposition 25(ii). It follows that $W_{f,\phi}\widehat{A'}u \in \mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$. □

It is worthwhile (and important, in a quantum mechanical context) to note that the spaces $\mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$ cannot contain functions which are "too concentrated" around a point; this is reminiscent of the uncertainty principle. In particular the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ is not contained in any of the $\mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$. This observation is based on the following result, proved in de Gosson and Luef [24,25] using Hardy's uncertainty principle for a function and its Fourier transform: assume that $u \in \mathcal{S}(\mathbb{R}^n)$ is such that $Wu \leqslant Ce^{-Mz\cdot z}$ for some $C > 0$ and a real matrix $M = M^T > 0$. Consider now the eigenvalues of JM; these are of the form $\pm i\lambda_j$ with $\lambda_j > 0$. Then we must have $\lambda_j \leqslant 1$ for all $j = 1, \ldots, n$. Equivalently, the symplectic capacity $c(\mathcal{W}_M)$ of the "Wigner ellipsoid" $\mathcal{W}_M$: $Mz \cdot z \leqslant 1$ satisfies $c(\mathcal{W}) \geqslant \pi$. [Recall [33,37] that the symplectic capacity of an ellipsoid $\mathcal{W}$ in $\mathbb{R}^{2n}$ is the number πR^2 where R is the supremum of the radii of all balls $B^{2n}(r)$ that can be sent into $\mathcal{W}_M$ using symplectomorphisms of $(\mathbb{R}^{2n}, \sigma)$.] This result in fact also holds true for the cross-Wigner transform [32]: if $|W(u,\phi)(z)| \leqslant Ce^{-Mz\cdot z}$ for some $\phi \in \mathcal{S}(\mathbb{R}^n)$ then $c(\mathcal{W}) \geqslant \pi$. Assume now that $U \in \mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$ satisfies the sub-Gaussian estimate $|U(z)| \leqslant Ce^{-Mz\cdot z}$; by definition of $\mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$ this is equivalent to

$$\big|W(u,\phi)(fz)\big| \leqslant Ce^{-(f^{-1})^T Mf^{-1}z\cdot z}$$

hence the ellipsoid $f(\mathcal{W}_M)$ must have symplectic capacity at least equal to π. We remark that a complete characterization of the spaces $M^q_s(\mathbb{R}^n)$ and $\mathcal{L}^q_{f,\phi}(\mathbb{R}^{2n})$ in terms of the uncertainty principle is still lacking; we hope to come back to this important question in a near future.

We finally notice that Lieb [35] has studied integral bounds for ambiguity and Wigner distributions; how are his results related to ours? This is certainly worth being explored, especially since he obtains an interesting characterization for Gaussians in terms of L^2 norms. In [8] Bonami et al. extend Beurling's uncertainty principle into a characterization of Hermite functions. They obtain sharp results for estimates of the Wigner distribution; it would perhaps be useful to study their results in our context; we hope to come back to these possibilities in a near future.

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References

[1] C. Bastos, O. Bertolami, N.C. Dias, J.N. Prata, Weyl–Wigner formulation of noncommutative quantum mechanics, J. Math. Phys. 49 (2008) 072101 (24 pages).

[2] C. Bastos, N.C. Dias, J.N. Prata, Wigner measures in noncommutative quantum mechanics, Communications in Mathematical Physics 299 (3) (2010) 709–740, doi:10.1007/s00220-010-1109-5.

[3] C. Bastos, O. Bertolami, N.C. Dias, J.N. Prata, Phase-space noncommutative quantum cosmology, Phys. Rev. D 78 (2008) 023516 (10 pages).

[4] C. Bastos, O. Bertolami, N.C. Dias, J.N. Prata, Black holes and phase-space noncommutativity, Phys. Rev. D 80 (2009) 124038 (7 pages).

[5] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantization. I. Deformation of symplectic structures, Annals of Physics 111 (1978) 6–110.

[6] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantization. II Physical applications, Annals of Physics 110 (1978) 111–151.

[7] O. Bertolami, J.G. Rosa, C.M.L. de Aragão, P.P. Castorina, D. Zappalà, Noncommutative gravitational quantum well, Phys. Rev. D (3) 72 (2) (2005) 025010–025018.

[8] A. Bonami, B. Demange, P. Jaming, Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms, Rev. Mat. Iberoamericana 19 (2003) 23–55.

[9] F. Bopp, La mécanique quantique est-elle une mécanique statistique particulière?, Ann. Inst. H. Poincaré 15 (1956) 81–112.

[10] A. Boulkhemair, Remarks on a Wiener type pseudodifferential algebra and Fourier integral operators, Mat. Res. Lett. 4 (1997) 53–67.

[11] E. Buzano, F. Nicola, L. Rodino, Global Pseudo-Differential Calculus on Euclidean Spaces, Springer-Verlag, 2010.

[12] S.M. Carroll, J.A. Harvey, V.A. Kostelecký, C.D. Lane, T. Okamoto, Noncommutative field theory and Lorentz violation, Phys. Rev. Lett. 87 (14) (2001) 141601–141605.

[13] E. Cordero, H.G. Feichtinger, F. Luef, Banach Gelfand triples for Gabor analysis, in: Pseudo-Differential Operators, in: Lecture Notes in Mathematics, vol. 1949, Springer, Berlin, 2008, pp. 1–33.

[14] N.C. Dias, M. de Gosson, F. Luef, J.N. Prata, A deformation quantization theory for noncommutative quantum mechanics, J. Math. Phys. 51 (2010) 072101, doi:10.1063/1.3436581 (12 pages).

[15] M.R. Douglas, N.A. Nekrasov, Noncommutative field theory, Rev. Mod. Phys. 73 (2001) 977–1029.

[16] H.G. Feichtinger, On a new Segal algebra, Monatsh. Math. 92 (4) (1981) 269–289.

[17] H.G. Feichtinger, Modulation spaces: Looking back and ahead, Sampl. Theory Signal Image Process 5 (2) (2006) 109–140.

[18] H.G. Feichtinger, W. Kozek, Quantization of TF lattice-invariant operators on elementary LCA groups, in: H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms. Theory and Applications, in: Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Boston, MA, 1998, pp. 233–266, 452–488.

[19] H.G. Feichtinger, Modulation spaces of locally compact Abelian groups, in: R. Radha, M. Krishna, S. Thangavelu (Eds.), Proc. Internat. Conf. on Wavelets and Applications, Chennai, January 2002, New Delhi Allied Publishers, 2003, pp. 1–56.

[20] H.G. Feichtinger, Banach Gelfand triples for applications in physics and engineering, in: AIP Conf. Proc., vol. 1146, Amer. Inst. Phys., 2009, pp. 189–228.

[21] M. de Gosson, Symplectic Geometry and Quantum Mechanics, Birkhäuser, Basel, 2006.

[22] M. de Gosson, Spectral properties of a class of generalized Landau operators, Communications in Partial Differential Operators 33 (11) (2008) 2096–2104.

[23] M. de Gosson, The symplectic camel and the uncertainty principle: The tip of an iceberg?, Found. Phys. 39 (2) (2009) 194–214.

[24] M. de Gosson, F. Luef, Quantum states and Hardy's formulation of the uncertainty principle: a symplectic approach, Lett. Math. Phys. 80 (2007) 69–82.

[25] M. de Gosson, F. Luef, Principe d'Incertitude et Positivité des Opérateurs à Trace ; Applications aux Opérateurs Densité, Ann. H. Poincaré 9 (2) (2008).

[26] M. de Gosson, F. Luef, A new approach to the ⋆-genvalue equation, Lett. Math. Phys. 85 (2008) 173–183.

[27] M. de Gosson, F. Luef, On the usefulness of modulation spaces in deformation quantization, J. Phys. A: Math. Theor. 42 (31) (2009) 315205, (17pp).

[28] M. de Gosson, F. Luef, Spectral and regularity properties of a pseudo-differential calculus related to Landau quantization, Journal of Pseudo-Differential Operators and Applications 1 (1) (2010).

[29] K. Gröchenig, Foundations of Time-Frequency Analysis, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2001.

[30] K. Gröchenig, Composition and spectral invariance of pseudodifferential operators on modulation spaces, Journal d'analyse mathématique 98 (2006) 65–82.

[31] K. Gröchenig, Time-frequency analysis on Sjöstrand's class, Rev. Mat. Iberoamericana 22 (2) (2006) 703–724.

[32] K. Gröchenig, G. Zimmermann, Hardy's theorem and the short-time Fourier transform of Schwartz functions, J. London Math. Soc. 63 (2001) 205–214.

[33] H. Hofer, E. Zehnder, Symplectic Invariants and Hamiltonian Dynamics, Birkhäuser Advanced Texts, Basler Lehrbücher, Birkhäuser Verlag, 1994.

[34] L. Hörmander, The Weyl calculus of pseudo-differential operators, Comm. Pure Appl. Math. 32 (1979) 359–443.

[35] E.H. Lieb, Integral bounds for radar ambiguity functions and Wigner distributions, J. Math. Phys. 31 (1990) 594–599.

[36] J.M. Maillard, On the twisted convolution product and the Weyl transformation of tempered distributions, J. Geom. Phys. 3 (2) (1986) 232–261.

[37] L. Polterovich, The Geometry of the Group of Symplectic Diffeomorphisms, Lectures in Mathematics, Birkhäuser, 2001.

[38] M.A. Shubin, Pseudodifferential Operators and Spectral Theory, first edition, Springer-Verlag, 1987; second edition 2001; original Russian edition in Nauka, Moskva, 1978.

[39] J. Sjöstrand, An algebra of pseudodifferential operators, Math. Res. Lett. 1 (2) (1994) 185–192.

[40] E.M. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, 1993.

[41] J. Toft, Continuity properties for modulation spaces with applications in pseudo-differential calculus, II, Ann. Global Anal. Geom. 26 (1) (2004) 73–106.

[42] R.J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rep. 378 (2003) 207.

[43] A. Unterberger, Quantization, symmetries and relativity, in: Perspectives on Quantization, South Hadley, MA, 1996, in: Comtemp. Math., vol. 214, Amer. Math. Soc., Providence, RI, 1998, pp. 169–187.

[44] A. Unterberger, H. Upmeier, Pseudodifferential Analysis on Symmetric Cones, Sud. Adv. Math., CRC Press, Boca Raton, FL, 1996.

[45] H. Weyl, Gruppentheorie und Quantenmechanik, transl. by H.P. Robertson, The Theory of Groups and Quantum Mechanics, 1931, reprinted Dover, 1950.

[46] M.W. Wong, Weyl Transforms, Springer-Verlag, 1998.