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Bounds relating the weakly connected domination number to the total domination number and the matching number

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ABSTRACT

Let $G = (V, E)$ be a connected graph. A dominating set S of G is a weakly connected dominating set of G if the subgraph $(V, E \cap (S \times V))$ of G with vertex set V that consists of all edges of G incident with at least one vertex of S is connected. The minimum cardinality of a weakly connected dominating set of G is the weakly connected domination number, denoted $\gamma_{wc}(G)$. A set S of vertices in G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the total domination number $\gamma_t(G)$ of G. In this paper, we show that $\frac{1}{2}(\gamma_t(G) + 1) \leq \gamma_{wc}(G) \leq \frac{3}{2}\gamma_t(G) - 1$. Properties of connected graphs that achieve equality in these bounds are presented. We characterize bipartite graphs as well as the family of graphs of large girth that achieve equality in the lower bound, and we characterize the trees achieving equality in the upper bound. The number of edges in a maximum matching of G is called the matching number of G, denoted $\alpha'(G)$. We also establish that $\gamma_{wc}(G) \leq \alpha'(G)$, and show that $\gamma_{wc}(T) = \alpha'(T)$ for every tree T.

1. Introduction

In this paper we continue the study of weakly connected domination in graphs introduced and studied by Dunbar, Grossman, Hattingh, Hedetniemi and McRae [8] and studied further in [4,12,15] and elsewhere.

For notation and graph theory terminology we in general follow [9]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. We denote the degree of v in G by $\deg_G(v)$, or simply by $\deg(v)$ if the graph G is clear from the context. The minimum degree among the vertices of G is denoted by $\delta(G)$. A cycle on n vertices is denoted by C_n. The girth of G, denoted $g(G)$, is the length of a shortest cycle in G. The number of components of G is denoted by $k(G)$.

A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A tree is a *double star* if it contains exactly two vertices that are not leaves; if one of these vertices is adjacent to a leaves and the other to b leaves, then we denote the double star by $S(a, b)$.

For disjoint subsets $A, B \subseteq V$, we define $[A, B]$ to be the set of all edges of G that joins a vertex in A and a vertex in B. Further we define the distance $d_G(A, B)$ between A and B in G to be the minimum distance between a vertex in A and a vertex in B; that is, $d_G(A, B) = \min\{d_G(u, v) \mid u \in A,\ v \in B\}$.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *dominating set*, denoted DS, of G is a set S of vertices of G such that every vertex in $V \setminus S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum

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cardinality of a DS. A *total dominating set*, abbreviated as TDS, of G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. If X and Y are subsets of vertices in G, then the set X dominates Y in G if $Y \subseteq N[X]$, while X totally dominates Y in G if $Y \subseteq N(X)$. A set $S \subseteq V$ is a *connected dominating set* of G if S is a dominating set of G and the graph $G[S]$ is connected. The *connected domination number* $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G. Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [9,10]. For a recent survey article on total domination in graphs see [11].

Let $G = (V, E)$ be a connected graph and let $S \subseteq V$. The *subgraph weakly induced by* S is the graph $G_S = (N[S]), E \cap (S \times V)$. Thus, G_S has vertex set $N[S]$ and consists of all edges of G incident with at least one vertex of S. The set S is a *weakly connected dominating set* of G, abbreviated as WCDS, if S is a dominating set of G and the graph G_S is connected. The *weakly connected domination number* $\gamma_{\mathrm{wc}}(G)$ is the minimum cardinality of a WCDS of G. A WCDS of G of cardinality $\gamma_{\mathrm{wc}}(G)$ is called a γ_{wc} (G)-set. Notice that every connected dominating set if a WCDS, which in turn is a dominating set. Thus for every connected graph G, we have $\gamma(G) \le \gamma_{\mathrm{wc}}(G) \le \gamma_c(G)$.

Mobile Ad Hoc Networks refer to distributed, wireless, multihop networks that function without using any infrastructure such as a base station or access points for communication, and exhibit dynamic changes in their network topology. Clustering introduces a hierarchy that is otherwise absent in these ad hoc networks, facilitating routing of information through the network. Efficient resource management, routing and better throughput performance can be achieved through adaptive clustering of these mobile nodes. Given the connectivity graph (also known as communication graph, network graph, etc.), $G_0 = (V_0, E_0)$, where the vertices represent the nodes in the network and the edges represent the communication links between pairs of nodes in the network, the clustering problem is to find subsets (not necessarily disjoint) $V_0^1, \dots, V_0^k$ of V_0 such that $V_0 = \cup_{i=1}^k V_0^i$. Each subset is a cluster and induces a graph with small diameter. After clustering, we can abstract the connectivity graph to a graph $G_1 = (V_1, E_1)$ as follows: there exists a vertex $v_i^1 \in V_1$ for every subset V_0^i and there exists an edge between v_1^i and v_1^j if and only if there exist $x_0 \in V_0^i$ and $y_0 \in V_0^j$ such that $x_0y_0 \in E_0$. The concept of a WCDS has also been proposed recently for clustering ad hoc networks [2]. Various algorithms on finding a small WCDS in a graph appear, for example, in [1–3,5–7].

A subset $S \subseteq V$ is a *packing* in G if the closed neighborhoods of vertices in S are pairwise disjoint. A subset $U \subseteq V$ is a *vertex cover* of G if every edge of G is incident with a vertex in U. The minimum cardinality among all the vertex covers in G is called the *vertex cover number* of G and is denoted by $\alpha(G)$. The *independence number* $\beta(G)$ is the maximum cardinality of an independent set of vertices in G.

Two edges in G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G which we denote by $\alpha'(G)$. Let M be a specified matching in a graph G. A vertex v of G is an *M-matched vertex* if v is incident with an edge of M; otherwise, v is an *M-unmatched vertex*. An *M-alternating path* of G is a path whose edges are alternately in M and not in M. An *M-augmenting path* is an M-alternating path that begins and ends with M-unmatched vertices. A *perfect matching M* in G is a matching in G such that every vertex of G is incident to an edge of M. Matchings in graphs are extensively studied in the literature (see, for example, the survey articles by Plummer [13] and Pulleyblank [14]).

2. Known results

The following results on the weakly connected domination number of a connected graph can be found in [8].

Theorem 1 (*Dunbar et al. [8]*). *If G is a connected graph of order $n \ge 2$, then the following properties hold:*
(a) $\gamma_{\mathrm{wc}}(G) \le \alpha(G)$.
(b) *If G is a tree, then* $\gamma_{\mathrm{wc}}(G) = \alpha(G)$.
(c) $\gamma(G) \le \gamma_{\mathrm{wc}}(G) \le 2\gamma(G) - 1$.
(d) $\gamma_{\mathrm{wc}}(G) \le \gamma_c(G) \le 2\gamma_{\mathrm{wc}}(G) - 1$.
(e) $\gamma_{\mathrm{wc}}(G) = n - \max\{\beta(T) \mid T \text{ is a spanning tree of } G\}$.
(f) $\gamma_{\mathrm{wc}}(G) \le n/2$.

3. The total domination number

Our aim in this section is to establish a relationship between the weakly connected domination number of a connected graph and its total domination number. If G is a connected graph of order $n \ge 2$ with $\Delta(G) = n - 1$, then $\gamma_{\mathrm{wc}}(G) = 1$ and $\gamma_t(G) = 2$. Hence in what follows we may assume that $\Delta(G) < n - 1$.

3.1. Lower bound

First we establish a lower bound on the weakly connected domination number of a graph in terms of its total domination number.

Theorem 2. *If G is a connected graph of order $n \ge 2$ with $\Delta(G) < n-1$, then*

$$\gamma_{\mathrm{wc}}(G) \ge \frac{1}{2}(\gamma_t(G)+1).$$

Proof. Let S be a $\gamma_{\mathrm{wc}}(G)$-set. On the one hand, suppose that S contains two adjacent vertices u and v. For each vertex $w \in S \setminus \{u, v\}$, let w' be a neighbor of w in G and let

$$S' = \bigcup_{w \in S \setminus \{u,v\}} \{w'\}.$$

Then, $S \cup S'$ is a TDS of G, and so $\gamma_t(G) \le |S| + |S'| \le 2|S| - 2 = 2\gamma_{\mathrm{wc}}(G) - 2$. Thus, $\gamma_{\mathrm{wc}}(G) \ge \frac{1}{2}\gamma_t(G) + 1 > \frac{1}{2}(\gamma_t(G)+1)$, as desired. On the other hand, suppose that S is an independent set in G. Let $v \in S$. Since S is a WCDS and $\Delta(G) < n-1$, there is a vertex $u \in S$ at distance 2 from v in G. Let v' be a common neighbor of u and v in G. For each vertex $w \in S \setminus \{u, v\}$, let w' be a neighbor of w in G and let S' be defined as before. Then, $S \cup (S' \cup \{v'\})$ is a TDS of G, and so $\gamma_t(G) \le |S| + |S'| + 1 \le 2|S| - 1 = 2\gamma_{\mathrm{wc}}(G) - 1$. Thus, $\gamma_{\mathrm{wc}}(G) \ge \frac{1}{2}(\gamma_t(G)+1)$, as desired. □

Next we present properties of connected graphs with weakly connected domination number at least 4 that achieve equality in the lower bound of Theorem 2.

Lemma 3. *Let G be a connected graph such that $\gamma_{\mathrm{wc}}(G) \ge 4$ and $\gamma_{\mathrm{wc}}(G) = \frac{1}{2}(\gamma_t(G)+1)$, and let S be a $\gamma_{\mathrm{wc}}(G)$-set. Then, G has the following properties.*

(a) *S is an independent set in G.*
(b) *If $A \subseteq S$ and A' totally dominates A in G, then $|A'| \ge |A| - 1$.*
(c) *There is a vertex v in S such that $S \setminus \{v\}$ is a packing. Further for every $u \in S \setminus \{v\}$, $d(u, v) = 2$ and $N(u) \not\subseteq N(v)$. Moreover, $N(v) \not\subseteq N(S \setminus \{v\})$.*
(d) *Let v be defined as in (c) above and let $S = \{v, v_1, v_2, \dots, v_k\}$. For $i = 1, 2, \dots, k$, let G_i be the star induced by the edges of G incident with v_i. Then, the graph G_S is obtained from the disjoint union of these k stars by adding the vertex v and joining it to at least one but not to every leaf from each star G_i, $1 \le i \le k$, and then adding at least one pendant edge to v.*

Proof. Let $G = (V, E)$ and let $|V| = n$. For each vertex $w \in S$, let w' be a neighbor of w in G.

(a) If S is not an independent set in G, then as shown in the proof of Theorem 2, $\gamma_{\mathrm{wc}}(G) > \frac{1}{2}(\gamma_t(G)+1)$, a contradiction. Hence, S is an independent set. This establishes part (a).

(b) Let $A \subseteq S$ and let A' totally dominate A in G (and so, $A \subseteq N(A')$). Suppose that $|A'| \le |A| - 2$. Let $B = S \setminus A$ and let $S' = \cup_{w \in B}\{w'\}$. Then, $A' \cup S \cup S'$ is a TDS of G. Since $|S'| \le |B| = |S| - |A|$, we therefore have that $\gamma_t(G) \le |A'| + |S| + |S'| \le 2|S| - 2 = 2\gamma_{\mathrm{wc}}(G) - 2$. Thus, $\gamma_{\mathrm{wc}}(G) > \frac{1}{2}(\gamma_t(G)+1)$, a contradiction. Hence, $|A'| \ge |A| - 1$. This establishes part (b).

(c) Among all vertices of S, let v be chosen so that the number of vertices of S at distance 2 from v is a maximum. Since S is a WCDS and $\Delta(G) < n-1$, there is at least one vertex at distance 2 from v in G. Let v_1 be a vertex of S at distance 2 from v and let v_1' be a common neighbor of v and v_1 in G. Let $A = S \setminus \{v, v_1\}$ and let A' be a minimum set of vertices that totally dominates A. By part (a) above, S is an independent set, implying that $A' \subseteq V \setminus S$. Now, $S \cup (A' \cup \{v_1'\})$ is a TDS of G, and so $\gamma_t(G) \le |S| + |A'| + 1$. If $|A'| \le |A| - 1 = |S| - 3$, then $\gamma_t(G) \le 2|S| - 2 = 2\gamma_{\mathrm{wc}}(G) - 2$, a contradiction. Hence, $|A'| \ge |A|$. However the set $\cup_{w \in A}\{w'\}$ totally dominates A, and so $|A'| \le |A|$. Consequently, $|A'| = |A| = |S| - 2$. This implies that the set A is a packing in G. Let $A = \{v_2, \dots, v_k\}$ and note that $k = |S| - 2 \ge \gamma_{\mathrm{wc}}(G) - 2 \ge 2$.

We show that the set $S \setminus \{v\} = A \cup \{v_1\}$ is a packing in G. Assume, to the contrary, that $S \setminus \{v\}$ is not a packing in G. Since A is a packing, we have that $d(v_1, A) = 2$. Renaming vertices if necessary, we may assume that $d(v_1, v_2) = 2$. Let v_2' be a common neighbor of v_1 and v_2 in G. By part (b) above, every vertex in $V \setminus S$ is adjacent to at most two vertices of S. Thus, $v_1' \ne v_2'$. Since v_1 is at distance 2 from at least two vertices of S, namely, v and v_2, our choice of the vertex v implies that v is at distance 2 from two vertices of S. Suppose v is at distance 2 from a vertex in $S \setminus \{v_1, v_2\}$. Renaming vertices if necessary, we may assume that $d(v, v_3) = 2$. Let v_3' be a common neighbor of v and v_3 in G. Since A is a packing, we note that $v_2' \ne v_3'$. But then the set $\{v_2', v_3'\}$ totally dominates the set $\{v, v_1, v_2, v_3\} \subseteq S$, contradicting part (b) above. Hence the set $S \setminus \{v\}$ is a packing in G. However S is a WCDS, and so G_S is connected. The vertex v is therefore at distance 2 from every vertex of $S \setminus \{v\}$. Thus, $d(u, v) = 2$ for every $u \in S \setminus \{v\}$.

If $u \in S \setminus \{v\}$ and $N(u) \subseteq N(v)$, then replacing u with a neighbor of u in S produces a $\gamma_{\mathrm{wc}}(G)$-set that is not independent set, contradicting part (a) above. Hence, $N(u) \not\subseteq N(v)$ for every $u \in S \setminus \{v\}$.

Finally suppose $N(v) \subseteq N(S \setminus \{v\})$. For $i = 1, 2, \dots, k$, let u_i be a common neighbor of v and v_i. Since the set $S \setminus \{v\}$ is a packing in G, $u_i \ne u_j$ for $1 \le i < j \le k$. Let $S' = \{u_1, u_2, \dots, u_k\}$, and let $D = (S \cup S') \setminus \{v\}$. Then, D is a TDS of G, and so $\gamma_t(G) \le |D| = 2(|S| - 1) = 2\gamma_{\mathrm{wc}}(G) - 2$, a contradiction. Hence, $N(v) \not\subseteq N(S \setminus \{v\})$. This establishes part (c).

(d) By part (c) above, there is a vertex v in S such that $S \setminus \{v\}$ is a packing. Let $S_v = \{v_1, v_2, \dots, v_k\}$, and so $S_v = S \setminus \{v\}$. For $i = 1, 2, \dots, k$, let G_i be the star induced by the edges of G incident with v_i, and let $A_i = N(v) \cap N(v_i)$ and $B_i = N(v_i) \setminus A_i$. Let $N_v = N(v) \setminus N(S_v)$. By part (c) above, we have that $|N_v| \ge 1$. Further for $i = 1, 2, \dots, k$, $|A_i| \ge 1$ and $|B_i| \ge 1$. Thus the graph G_S is the spanning subgraph of G obtained from the disjoint union of the k stars $G_1, G_2, \dots, G_k$ by adding the set of vertices $N_v \cup \{v\}$ and joining v to every vertex of A_i for each i, $1 \le i \le k$, and joining v to every vertex of N_v. The graph G_S is illustrated in Fig. 1. □

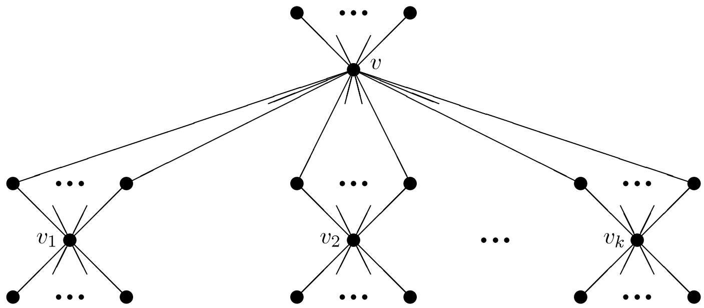

Fig. 1. The graph G_S.

As a consequence of Lemma 3, we have a characterization of special families of graphs, including the family of bipartite graphs and the family of graphs of large girth, with weakly connected domination number at least 4 that achieve equality in the lower bound of Theorem 2. For this purpose, let $\mathcal{G}$ be the family of all trees that can be obtained from a star $K_{1,k}$ for some integer $k \geq 3$, by adding at least one pendant edge to each vertex of the star, and then subdividing every edge of the star exactly once. Let $\mathcal{B}$ be the family of bipartite graphs that can be obtained from $b \geq 3$ disjoint stars each of order at least 3 by adding a new vertex v and joining it to at least one but not to every leaf from each of the b stars, and then adding at least one pendant edge to v.

Corollary 4. *Let G be a connected graph such that* $\gamma_{\mathrm{wc}}(G) \geq 4$ *and* $\gamma_{\mathrm{wc}}(G) = \frac{1}{2}(\gamma_t(G) + 1)$.
(a) *If G is bipartite, then* $G \in \mathcal{B}$.
(b) *If G has girth at least 8, then* $G \in \mathcal{G}$.

Proof. We shall follow the notation introduced in the proof of Lemma 3(d). Let S be a $\gamma_{\mathrm{wc}}(G)$-set. Then the graph G_S is as defined in Lemma 3(d). The only possible edges of G not in G_S are edges joining vertices in $V(G) \setminus S$. However adding any such edge would create an odd cycle of length 3, 5 or 7. Thus if G is bipartite, then no edge can be added to G_S to construct G, whence $G = G_S$ and $G \in \mathcal{B}$. This establishes part (a). If $|A_i| \geq 2$ for some i, $1 \leq i \leq k$, then v and v_i are contained in a common 4-cycle. Thus if G has girth at least 8, then $|A_i| = 1$ for all $i = 1, 2, \ldots, k$ and no edge can be added to G_S to construct G, whence $G = G_S$ and $G \in \mathcal{G}$. This establishes part (b). □

3.2. Upper bound

Next we establish an upper bound on the weakly connected domination number of a graph in terms of its total domination number.

Theorem 5. *If G is a connected graph of order at least 2, then* $\gamma_{\mathrm{wc}}(G) \leq \frac{3}{2}\gamma_t(G) - 1$.

Proof. To establish the desired upper bound, we construct a WCDS D that satisfies $|D| \leq \frac{3}{2}\gamma_t(G) - 1$, as follows. Initially, we let D_0 be a $\gamma_t(G)$-set and we let $G_0 = G_{D_0}$. (Recall that G_{D_0} denotes the graph weakly induced by D_0.) If G_0 is connected, then $D = D_0$ is a WCDS, and so $\gamma_{\mathrm{wc}}(G) \leq |D| = \gamma_t(G) \leq \frac{3}{2}\gamma_t(G) - 1$, and we are done. Hence we may assume that G_0 is disconnected. Among all components of G_0, let G_{X_0} and G_{Y_0} be chosen so that the distance between their vertex sets is a minimum. Since G is connected and D_0 is a dominating set, $d_G(V(G_{X_0}), V(G_{Y_0})) = 1$. Hence there exists an edge $x_0y_0 \in E(G)$ where $x_0 \in V(G_{X_0})$ and $y_0 \in V(G_{Y_0})$. Since G_{X_0} and G_{Y_0} are distinct components of G_0, we note that neither x_0 nor y_0 is in D. Hence, x_0 is adjacent to a vertex $u \in D \cap X_0$ and y_0 is adjacent to a vertex $v \in D \cap Y_0$. Notice that u and v are neither adjacent nor do they have a common neighbor. We now let $D_1 = D_0 \cup \{x_0\}$ and we let $G_1 = G_{D_1}$ be the subgraph of G weakly induced by D_1. Since the components G_{X_0} and G_{Y_0} of G_0 are contained in one new component of G_1, we have that $k(G_1) \leq k(G_0) - 1$.

In general, if G_i is connected for some $i \geq 0$, then we let $D = D_i$. Otherwise, if G_i is disconnected, then among all components of G_i, let G_{X_i} and G_{Y_i} be chosen so that the distance between their vertex sets is a minimum. Since G is connected and D_i is a dominating set, $d_G(V(G_{X_i}), V(G_{Y_i})) = 1$. Hence there exists an edge $x_iy_i \in E(G)$ where $x_i \in V(G_{X_i})$ and $y_i \in V(G_{Y_i})$. We now let $D_{i+1} = D_i \cup \{x_i\}$ and we let $G_{i+1} = G_{D_{i+1}}$ denote the subgraph of G weakly induced by D_{i+1}. Since the components G_{X_i} and G_{Y_i} of G_i are contained in one new component of G_{i+1}, we have that $k(G_{i+1}) \leq k(G_i) - 1$.

Since the subgraph $G[D_0]$ induced by the set D_0 contains no isolated vertex, the subgraph G_0 contains at most $|D_0|/2$ components. Hence by adding at most $\frac{1}{2}|D_0| - 1$ vertices to D_0, we can extend D_0 to a WCDS. Thus for some i with $i \leq \frac{1}{2}|D_0| - 1$, the graph G_i is connected, whence $D = D_i$ is a WCDS of G. Therefore, $\gamma_{\mathrm{wc}}(G) \leq |D_i| = |D_0| + i \leq |D_0| + \frac{1}{2}|D_0| - 1 = \frac{3}{2}|D_0| - 1 = \frac{3}{2}\gamma_t(G) - 1$, which completes the proof of the upper bound. □

Next we present properties of connected graphs that achieve equality in the upper bound of Theorem 5.

Lemma 6. *Let G be a connected graph of order at least 2 satisfying* $\gamma_{\mathrm{wc}}(G) = \frac{3}{2}\gamma_t(G) - 1$, *and let D be a* $\gamma_t(G)$*-set. Then, G has the following properties.*
(a) $G[D] = kK_2$ *for some integer* $k \geq 1$.

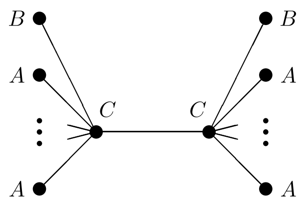

Fig. 2. A labeled double star.

Let $E(G[D]) = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$. For $i = 1, \ldots, k$, let $V_i = \{u_i, v_i\}$ and let G_i be the subgraph induced by the edges of G incident at least one vertex in V_i.

(b) *$G_1, \ldots, G_k$ are the k components of G_D.*
(c) *$d(V_i, V_j) \geq 3$ for $1 \leq i < j \leq k$ and $(N_G[V_1], \ldots, N_G[V_k])$ is a partition of $V(G)$.*
(d) *For $i = 1, \ldots, k$, each G_i contains a leaf adjacent to u_i and a leaf adjacent to v_i.*
(e) *Each vertex is adjacent in G to vertices from at most one component of G_D different from the component of G_D to which it belongs.*

Proof. (a) Since $\gamma_{wc}(G) = \frac{3}{2}\gamma_t(G) - 1$, we have that $3\gamma_t(G) = 2(\gamma_{wc}(G) + 1)$, and so $\gamma_t(G)$ is even. Let D be a $\gamma_t(G)$-set. Each inequality in the proof of Theorem 5 becomes an equality, and so G_D has exactly $|D|/2$ components and each component of $G[D]$ is a K_2-component consisting of two adjacent vertices. Thus, $G[D] = kK_2$ for some integer $k \geq 1$.

(b) Let $E(G[D]) = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$. For $i = 1, \ldots, k$, let $V_i = \{u_i, v_i\}$. Since G_D has exactly $|D|/2$ components, the subgraphs $G_1, \ldots, G_k$ are the k components of G_D.

(c) Thus, $V_i = D \cap V(G_i)$ for $i = 1, \ldots, k$. Since $G_1, \ldots, G_k$ are distinct components of G_D, we have that $d(V_i, V_j) \geq 3$ for $1 \leq i < j \leq k$. By definition of the subgraph G_i, $V(G_i) = N_G[V_i]$ for each $i = 1, \ldots, k$. Thus since D is a dominating set of G, $(N_G[V_1], \ldots, N_G[V_k])$ is a partition of $V(G)$.

(d) Suppose that there is a vertex $v \in D$ that does not have a leaf-neighbor in the component of G_D to which it belongs. We may assume that $v = u_1$. Thus, G_1 does not contain a leaf adjacent to u_1. Hence, v_1 dominates $V(G_1)$. Since G is connected and D is a dominating set of G, there exists an edge x_1y_1 that joins a vertex of G_1 to a vertex of G_i for some $i \geq 2$. But then $D' = (D \setminus \{u_1\}) \cup \{x_1\}$ is a $\gamma_t(G)$-set of G with $G_{D'}$ having at most $|D|/2 - 1$ components, and so letting $D_0 = D'$ in the proof of Theorem 5, at least one inequality in the proof becomes a strict inequality, whence $\gamma_{wc}(G) < \frac{3}{2}\gamma_t(G) - 1$, a contradiction.

(e) Suppose there is a vertex v that is adjacent in G to vertices from two or more components of G_D different from the component of G_D containing v. By part (c) above, $v \notin V_1$. We now let $D_0 = D$ and $x_0 = v$. As in the proof of Theorem 5, we let $D_1 = D_0 \cup \{x_0\}$ and $G_1 = G_{D_1}$. Since at least three components of G_0 are all contained in one new component in G_1, we have that $k(G_1) \leq k(G_0) - 2$. Hence at least one inequality in the proof of Theorem 5 becomes a strict inequality, whence $\gamma_{wc}(G) < \frac{3}{2}\gamma_t(G) - 1$, a contradiction. $\square$

Next we provide a constructive characterization of trees that achieve the upper bound of Theorem 5. For this purpose, we shall need the following definitions.

Definition 1. We define a *labeling* of a tree T as a function $S: V(T) \to \{A, B, C\}$. The label of a vertex v is also called its *status*, denoted sta(v). A labeled tree is denoted by a pair (T, S).

Definition 2. Let $L(a, b)$ be the labeled double star obtained from the double star $S(a, b)$, where $a \geq 1$ and $b \geq 1$, by labeling the two central vertices of $S(a, b)$ with status C, labeling exactly one leaf adjacent to each central vertex of $S(a, b)$ with status B, and labeling all other leaves of $S(a, b)$ with status A. A labeled double star is shown in Fig. 2.

Definition 3. Let operation $\mathcal{O}_1$ extend a labeled tree (T, S) by adding a labeled double star $L(a, b)$, where $a \geq 1$ and $b \geq 2$, and joining a vertex u of status A in (T, S) to a vertex v of status A in $L(a, b)$, and then relabeling both u and v with status B in the resulting labeled tree while keeping all other labels unchanged.

We now describe a procedure to build labeled trees.

Definition 4. Let $\mathcal{L}$ be the family of labeled trees that contain all labeled double stars and is closed under the operation $\mathcal{O}_1$.

We remark that if $(T, S) \in \mathcal{L}$ for some labeling S, then, by construction, T is a tree and every vertex of status A in (T, S) is a leaf whose neighbor has status C, while every vertex of status C in (T, S) is a support vertex with exactly one leaf-neighbor of status B. Furthermore, every support vertex in (T, S) has status C while every leaf in (T, S) has status A or B.

Definition 5. Let $\mathcal{T}$ be the family of all unlabeled trees T such that $(T, S) \in \mathcal{L}$ for some labeling S. We remark that every graph in the family $\mathcal{T}$ is a tree.

We are now in a position to present our constructive characterization of trees that achieve the upper bound of Theorem 5.

Theorem 7. *Let T be a tree of order at least 2. Then, $\gamma_{wc}(T) = \frac{3}{2}\gamma_t(T) - 1$ if and only if $T \in \mathcal{T}$.*

Proof. Suppose $T \in \mathcal{T}$. We show that $\gamma_{wc}(T) = \frac{3}{2}\gamma_t(T) - 1$, by using induction on $\text{num}(T)$, the number of operations required to construct the tree $(T, S) \in \mathcal{L}$ for some labeling S. If $\text{num}(T) = 0$, then $T = L(a, b)$ for some $a \geq 1$ and $b \geq 1$, and so $\gamma_{wc}(T) = \gamma_t(T) = 2$, whence $\gamma_{wc}(T) = \frac{3}{2}\gamma_t(T) - 1$. Assume, then, that for all trees $T' \in \mathcal{T}$ with $\text{num}(T') < k$, where $k \geq 1$ is an integer, that $\gamma_{wc}(T') = \frac{3}{2}\gamma_t(T') - 1$. Let $T \in \mathcal{T}$ be a tree with $\text{num}(T) = k$. Then, $(T, S) \in \mathcal{L}$ is obtained from some labeled tree $(T', S') \in \mathcal{L}$ by Operation $\mathcal{O}_1$. But then $T' \in \mathcal{T}$ and $\text{num}(T') < k$. Applying the inductive hypothesis to T', $\gamma_{wc}(T') = \frac{3}{2}\gamma_t(T') - 1$.

Suppose (T, S) is obtained from (T', S') by joining the leaf ℓ_1 in T' to the leaf ℓ_2 of the added labeled double star L. Suppose u_1 is the central vertex of L adjacent to ℓ_2, and let u_2 be the other central vertex of L. Note that both u_1 and u_2 are support vertices in T. Let v denote the neighbor of the leaf ℓ_1 in the tree T'.

We first show that $\gamma_t(T) = \gamma_t(T') + 2$. Let D be a $\gamma_t(T)$-set. As every support vertex of T must be in D, we immediately see that D consists of exactly the vertices labeled C. But then $D \setminus \{u_1, u_2\}$ is a TDS of T', whence $\gamma_t(T') \leq \gamma_t(T) - 2$. Moreover, every $\gamma_t(T')$-set can be extended to a TDS of T by adding the two vertices u_1 and u_2, and so $\gamma_t(T) \leq \gamma_t(T') + 2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$.

Next we show that $\gamma_{wc}(T) = \gamma_{wc}(T') + 3$. Let W be a $\gamma_{wc}(T)$-set. Let $u \in \{u_1, u_2\}$. If $u \notin W$, then all neighbors of u must be in W. Let u' be a leaf-neighbor of u in T. Then, $(W \setminus \{u'\}) \cup \{u\}$ produces a new $\gamma_{wc}(T)$-set. Hence we may assume that $\{u_1, u_2\} \subseteq W$. With this assumption, the minimality of W implies that no leaf-neighbor of u_1 or u_2 in T belongs to W. On the one hand, suppose $\ell_2 \in W$. If $\ell_1 \in W$, then $W \setminus \{\ell_2\}$ is a WCDS of T, contradicting the minimality of W. Hence, $\ell_1 \notin W$. Since T_W is connected, we have that $v \in W$. Thus, $W \setminus \{\ell_2, u_1, u_2\}$ is a WCDS of T', whence $\gamma_{wc}(T') \leq \gamma_{wc}(T) - 3$. On the other hand, suppose $\ell_2 \notin W$. Since T_W is connected, we have that $\ell_1 \in W$. This in turn implies that either $v \in W$ or $v \notin W$ and $N_{T'}(v) \subseteq W$. Since the leaf ℓ_1 has status A in (T', S'), the vertex v has status C in (T', S') and is a support vertex with a leaf-neighbor, say ℓ_3, of status B. But then $W' = (W \setminus \{u_1, u_2, \ell_2, \ell_3\}) \cup \{v\}$ is a WCDS of T', and so $\gamma_{wc}(T') \leq |W'| \leq |W| - 3 = \gamma_{wc}(T) - 3$. However every $\gamma_{wc}(T')$-set contains either the support vertex v or its leaf-neighbor ℓ_1. Thus every $\gamma_{wc}(T')$-set can be extended to a WCDS of T by adding to it the vertices in the set $\{\ell_2, u_1, u_2\}$, and so $\gamma_{wc}(T) \leq \gamma_{wc}(T') + 3$.

Hence we have shown that if $T \in \mathcal{T}$, then $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{wc}(T) = \gamma_{wc}(T') + 3$. This implies that $\gamma_{wc}(T) = \gamma_{wc}(T') + 3 = \frac{3}{2}\gamma_t(T') - 1 + 3 = \frac{3}{2}(\gamma_t(T') + 2) - 1 = \frac{3}{2}\gamma_t(T) - 1$, as desired.

Conversely, suppose $\gamma_{wc}(T) = \frac{3}{2}\gamma_t(T) - 1$. By part (a) of Lemma 6, $\gamma_t(T) = 2k$ for some integer $k \geq 1$. We show that $T \in \mathcal{T}$. We proceed by induction on k. Suppose $k = 1$. If T is a star, then $\gamma_{wc}(T) = 1 < \frac{3}{2}\gamma_t(T) - 1$, a contradiction. Hence, T is a double star. Let S be a labeling of T that produces a labeled double star. Then, $(T, S) \in \mathcal{L}$, whence $T \in \mathcal{T}$. This establishes the base case. Assume, then, that $k \geq 2$ and that for all trees T' with $\gamma_{wc}(T') = \frac{3}{2}\gamma_t(T') - 1$ where $\gamma_t(T') < 2k$ we have $T' \in \mathcal{T}$.

Let T be a tree with $\gamma_{wc}(T) = \frac{3}{2}\gamma_t(T) - 1$ where $\gamma_t(T) = 2k$. Let D be a $\gamma_t(T)$-set. Then, T has the properties listed in Lemma 6. We shall adopt the notation introduced in Lemma 6. Since T is a tree, each of the k components of T_D is a double star. By part (c) and part (d) of Lemma 6, the two central vertices of each such double star belong to D and have degree at least 2 in the double star, while all leaves in the double star belong to $V(T) \setminus D$. By part (c), each vertex of D belongs to exactly one component of T_D. Further the sets $N(u)$, where $u \in D$, form a partition of $V(T)$.

We show that each vertex in $V(T) \setminus D$ has degree at most 2 in T. Let $v \in V(T) \setminus D$. Then, v is a leaf in some component of T_D. Since T is a tree, the vertex v is adjacent in T to at most one vertex from every component of T_D. By part (e) of Lemma 6, the vertex v is adjacent in T to vertices from at most one component of T_D different from the component of T_D that contains v. Hence, $\deg_T(v) \leq 2$ for all $v \in V(T) \setminus D$.

We show next that each vertex of D has a leaf-neighbor in T. Let $u \in D$. Let w be the vertex of D adjacent to u, and let $N_u = N(u) \setminus \{w\}$. Thus, $N_u \subset V(T) \setminus D$ is the set of leaf-neighbors of u in the (double star) component of T_D that contains u. Suppose that u has no leaf-neighbor in T. Then, $\deg_T(v) = 2$ for all $v \in N_u$. Since T is a tree, no two vertices in N_u are adjacent to vertices from the same component of T_D other than from the component of T_D that contains u. Hence letting $D' = (D \setminus \{u\}) \cup N_u$ and letting $T' = T_{D'}$ denote the subgraph of T weakly induced by D', at least $\deg_T(u)$ components of T_D are all contained in one new component in T'. Thus, $|D'| = |D| + d_T(u) - 2$ while $k(T') \leq k(T_D) - d_T(u) + 1 = \frac{1}{2}|D| - d_T(u) + 1$. Hence we need to add at most $\frac{1}{2}|D| - d_T(u)$ vertices to D' to form a WCDS for T, whence $\gamma_{wc}(T) \leq |D'| + \frac{1}{2}|D| - d_T(u) = |D| + d_T(u) - 2 + \frac{1}{2}|D| - d_T(u) = \frac{3}{2}|D| - 2 = \frac{3}{2}\gamma_t(T) - 2$, a contradiction. Therefore, each vertex of D has a leaf-neighbor in T.

We are now ready to show that $T \in \mathcal{T}$. As remarked earlier, we adopt the notation introduced in Lemma 6. Let F be the tree of order k where the vertices of F correspond to the k components of T_D and where two vertices of F are adjacent if the corresponding components of T_D are joined by an edge. Let X be a leaf of F and let Y be the neighbor of X in F. Renaming vertices if necessary, we may assume that X and Y correspond to the components G_1 and G_2, respectively, and that xy is an edge of T where $x \in V(G_1)$ and $y \in V(G_2)$. From our earlier remarks, both x and y are leaves in their respective components G_1 and G_2. We may assume that $x \in N(v_1)$ and that $y \in N(u_2)$. Since xy is the only edge of T with exactly one end in $V(G_1)$, every leaf of the double star G_1 different from x is also a leaf of T, while $\deg_T(x) = \deg_T(y) = 2$. As observed earlier, each vertex of D has a leaf-neighbor in T. In particular, v_1 has a leaf-neighbor in T, say v_1', and u_2 has a leaf-neighbor in T, say z. Let $T' = T - V(G_1)$.

We first show that $\gamma_t(T) = \gamma_t(T') + 2$. On the one hand, since $D \setminus \{u_1, v_1\}$ is a TDS of T', we have that $\gamma_t(T') \leq |D| - 2 = \gamma_t(T) - 2$. On the other hand, if D' is a $\gamma_t(T')$-set, then $D' \cup \{u_1, v_1\}$ is a TDS of T, and so $\gamma_t(T) \leq |D'| + 2 = \gamma_t(T') + 2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$.

Next we show that $\gamma_{\rm wc}(T) = \gamma_{\rm wc}(T') - 3$. Let W be a $\gamma_{\rm wc}(T)$-set. Since each vertex of D is a support vertex of T, we may choose W so that $D \subseteq W$. In particular, $\{u_1, u_2, v_1\} \subset W$. But then exactly one vertex of $\{x, y\}$, say r, is in W. It follows that $W \setminus \{u_1, v_1, r\}$ is a WCDS of T', and so $\gamma_{\rm wc}(T') \leq \gamma_{\rm wc}(T) - 3$. We now consider a $\gamma_{\rm wc}(T')$-set, say W'. Since u_2 is a support vertex of T', we may choose W' so that $u_2 \in W$. Hence, $W' \cup \{u_1, v_1, x\}$ is a WCDS of T, and so $\gamma_{\rm wc}(T) \leq \gamma_{\rm wc}(T') + 3$. Consequently, $\gamma_{\rm wc}(T) = \gamma_{\rm wc}(T') - 3$.

Thus, $\gamma_{\rm wc}(T') = \gamma_{\rm wc}(T) - 3 = (\frac{3}{2}\gamma_t(T) - 1) - 3 = \frac{3}{2}(\gamma_t(T') + 2) - 4 = \frac{3}{2}\gamma_t(T') - 1$. Applying the inductive hypothesis to T', we have that $T' \in \mathcal{T}$. Hence, $(T', S') \in \mathcal{L}$ for some labeling S'. Since u_2 is a support vertex of (T', S'), exactly one leaf-neighbors of u_2 in (T', S') has status B and all other leaf-neighbors of u_2 in (T', S') have status A. Renaming vertices, if necessary, we may assume that the leaf-neighbor z of u_2 in (T', S') has status B. Hence the leaf-neighbor y of u_2 in (T', S') has status A. Now, let (G_1, S_1) be a labeled double star with labeling S_1 obtained from the double star G_1 by labeling v_1' with status B. Then, T can be obtained from T' by operation $\mathcal{O}_1$ by adding (G_1, S_1) and joining the vertex y of status A in (T', S') to the vertex x of status A in (G_1, S_1), and then relabeling both x and y with status B in the resulting labeled tree while keeping all other labels unchanged. Thus, $T \in \mathcal{T}$. $\square$

4. The matching number

In this section, we show that the weakly connected domination number of a connected graph of order at least two is at most the matching number of the graph.

Theorem 8. *For every connected graph G of order $n \geq 2$, $\gamma_{\rm wc}(G) \leq \alpha'(G)$.*

Proof. Let $G = (V, E)$. For every matching M in G, we let $U_M \subset V$ be the set of vertices not incident with an edge in M. Thus, U_M is the set of M-unmatched vertices in G. Among all matchings M in G and among all subsets S of vertices in G, let M and S be chosen so that

(1) M be a maximum matching in G (and so, $\alpha'(G) = |M|$).
(2) Subject to (1), $|S| = |M|$ and S contains exactly one vertex incident with each edge of M.
(3) Subject to (2), the number of vertices in U_M dominated by S is maximized.
(4) Subject to (3), the component of largest order in G_S is maximized.

By the maximality of M, the set U_M is an independent set in G. For each vertex $v \in S$, let e_v be the edge of M incident with v and let $\overline{v}$ be the other vertex incident with e_v, and so e_v joins v and $\overline{v}$. Let $\overline{S}$ be the set of M-matched vertices not in S; that is, $\overline{S} = V \setminus (S \cup U_M)$.

We show first that S dominates V. By construction, S dominates $V \setminus U_M$. Assume that there is a vertex $u \in U_M$ not dominated by S. Since G is connected, u must then be adjacent to some vertex $\overline{v} \in \overline{S}$. By assumption, u and v are not adjacent. By the maximality of the matching M, the vertex v is not adjacent with any vertex of U_M. Thus, $S' = (S \setminus \{v\}) \cup \{\overline{v}\}$ satisfies condition (2) but S' dominates more vertices in U_M than does S, contradicting our choice of S. Hence, S dominates V.

We show next that S is a WCDS of G. Assume, to the contrary, that S is not a WCDS of G. Then, G_S contains at least two components. Let G_1 be a component in G_S of largest order. Since G is connected, there must be an edge $ab \in E$ where $a \in V(G_1)$ and $b \in V \setminus V(G_1)$. Let G_2 be the component of G_S containing b. Since G_1 and G_2 are distinct components of G_S, we note that $\{a, b\} \subseteq V \setminus S$.

Suppose $a \in U_M$. Since U_M is an independent set in G and $b \in V \setminus S$, we must have that $b \in \overline{S}$. Thus, $b = \overline{v}$ for some vertex $v \in S$. This implies that $v \in V(G_2)$. By the maximality of the matching M, the vertex v is not adjacent with any vertex of $U_M \setminus \{a\}$. Since G_1 and G_2 are distinct components of G_S, the vertices a and v are not adjacent in G. Hence, the vertex v is not adjacent with any vertex of U_M. Let $S' = (S \setminus \{v\}) \cup \{\overline{v}\}$. Then, S' satisfies conditions (2) and (3) but $G_{S'}$ contains a component of order strictly larger than G_1, contrary to our choice of S. Hence, $a \notin U_M$, and so $a \in \overline{S}$.

Suppose $b \in U_M$. Since S dominates V, there is a vertex $v \in S$ adjacent to b. This implies that $v \in V(G_2)$. By the maximality of the matching M, the vertex $\overline{v}$ is not adjacent with any vertex of $U_M \setminus \{b\}$. Let $M' = (M \setminus \{v\overline{v}\}) \cup \{vb\}$; that is, we replace the edge $v\overline{v}$ in M with the edge vb. Then, $|M'| = |M|$, and so M' is a maximum matching of G. Therefore, M' satisfies condition (1). Let $S' = S$. Then, S' satisfies condition (2) (with respect to the matching M'). Since S dominates V, the set S' dominates V. Thus, S' satisfies condition (3) (with respect to the matching M'). Further, G_1 is a component of $G_{S'}$, and so S' satisfies condition (4) (with respect to the matching M'). Hence, by replacing $\overline{v}$ with b, we may assume that the matching M was chosen so that $b \in \overline{S}$.

Since $\{a, b\} \subseteq \overline{S}$, we have that $a = \overline{u}$ and $b = \overline{v}$ for some vertices $u \in S$ and $v \in S$. Thus, $u \in V(G_1)$ and $v \in V(G_2)$. Let $S' = (S \setminus \{v\}) \cup \{\overline{v}\}$. If S' is a dominating set of G, then S' satisfies conditions (2) and (3) but $G_{S'}$ contains a component of order strictly larger than G_1, contrary to our choice of S. Hence, S' does not dominate V. Thus, there must exist a vertex $x \in U_M$ such that $N(x) \cap (S \cup \{\overline{v}\}) = \{v\}$. By the maximality of the matching M, the vertex x is not adjacent with any vertex of U_M.

Let Z be the set of all vertices of $S \cap V(G_1)$ that are connected to u by an M-alternating path in $G_1 - \overline{u}$ such that consecutive vertices alternate between $S \cap V(G_1)$ and $\overline{S} \cap V(G_1)$. Then, $u \in Z$. Suppose some vertex $z \in Z$ is adjacent to a vertex $z' \in U_M$. Let P_z be an M-alternating path in $G_1 - \overline{u}$ from u to z such that consecutive vertices alternate between $S \cap V(G_1)$ and $\overline{S} \cap V(G_1)$. Then the x-u path $x, v, \overline{v}, \overline{u}, u$, followed by the u-z path P_z and then the trivial z-z' path along the edge zz' is an M-augmenting x-z' path in G, contradicting the maximality of M. Hence no vertex of Z is adjacent to a vertex of U_M.

Let $V_1 = V(G_1)$, and let $S_1 = S \cap V_1$. Let U_1 be the set of all vertices of U_M dominated by S_1 and let $W = S_1 \setminus Z$. Thus, W is the set of all vertices of $S \cap V(G_1)$ not in Z. Let $\overline{Z} = \{\overline{z} \mid z \in Z\}$ and let $\overline{W} = \{\overline{w} \mid w \in W\}$. Note that $V_1 = W \cup \overline{W} \cup Z \cup \overline{Z} \cup U_1$. If there is an edge $z\overline{w}$ joining a vertex $z \in Z$ and a vertex $\overline{w} \in \overline{W}$, then there would be an M-alternating path from w to u in $G_1 - \overline{u}$, contradicting the fact that $w \in W$. Hence, $[Z, \overline{W}] = \emptyset$; that is, there is no edge joining a vertex in Z and a vertex in $\overline{W}$. Since every edge in G_1 is incident with at least one vertex of V_1, there is no edge in G_1 joining a vertex in $\overline{Z}$ and a vertex in $\overline{W}$. Hence, $[\overline{Z}, \overline{W}] = \emptyset$.

Let $D = (S \setminus Z) \cup \overline{Z}$ and let $D_1 = D \cap V_1$. Thus, $D_1 = W \cup \overline{Z}$. Let F be the component of G_D containing u. Since $\overline{u} \in \overline{Z}$ and $u\overline{u} \in E$, we have that $\overline{u} \in V(F)$. If $z \in Z$ and P_z is an M-alternating path from z to u in $G_1 - \overline{u}$, then the path P_z is also a path in F. Hence, $Z \cup \overline{Z} \subseteq V(F)$. Since no vertex of Z is adjacent to a vertex of U_1, the set U_1 is dominated by the set W. Thus, D dominates V. In particular, we note that the set D satisfies conditions (2) and (3).

We show now that $V_1 \subseteq V(F)$. Let $w \in W \cup \overline{W}$. Since G_1 is connected, there is a path from w to every vertex of $Z \cup \overline{Z}$ in G_1. Let z be a vertex in $Z \cup \overline{Z}$ at minimum distance from w in G_1 and let P_w be a shortest w-z path in G_1. Let t be the vertex immediately preceding z on P_w (possibly, $w = t$). Since there is no edge in G_1 joining a vertex in $Z \cup \overline{Z}$ and a vertex in $\overline{W}$, we have that $t \in W$. Since there is no edge in G_1 with both ends in $\overline{W}$, every edge of P_w is therefore incident with at least one vertex of W. Hence since $Z \cup \overline{Z} \subseteq V(F)$ and since $W \subset D$, the path P_w is also a path in F. Thus, $W \cup \overline{W} \subseteq V(F)$. Further since $W \subset D$ and $W \subseteq V(F)$, we have that $U_1 \subset V(F)$. Hence, $V_1 \subseteq V(F)$.

Since $\overline{u} \in D$ and $\overline{u}\,\overline{v} \in E$, we have that $\overline{v} \in V(F)$. Hence since $v \in D$ and $\{x, \overline{v}\} \subseteq N(v)$, we have that $\{x, v\} \subset V(F)$. Hence, $V_1 \cup \{x, v, \overline{v}\} \subseteq V(F)$. Thus, F is a component of G_D of order exceeding that of G_1, contrary to our choice of S. We deduce, therefore, that S is a WCDS of G. Hence, $\gamma_{\rm wc}(G) \le |S| = \alpha'(G)$. $\square$

Since the matching number α' is at most the vertex covering number α, we have the following immediate consequence of Theorem 8.

Corollary 9. *For every connected graph G of order $n \ge 2$, $\gamma_{\rm wc}(G) \le \alpha'(G) \le \alpha(G)$.*

As a consequence of Theorem 1(b) and Corollary 9 we have that the weakly connected domination of a tree is precisely its matching number.

Corollary 10. *For every tree T, $\gamma_{\rm wc}(T) = \alpha'(T)$.*

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