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Journal of Approximation Theory 164 (2012) 776–790
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

Extrapolation on the cone of decreasing functions

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Received 18 February 2011; received in revised form 21 January 2012; accepted 6 February 2012
Available online 15 February 2012

Communicated by Paul Nevai

Abstract

We develop Yano's extrapolation theory for sublinear operators bounded on the cone of positive decreasing functions in $L^p(0,\infty)$. Applications in the setting of bounded operators on this cone are presented.

Keywords: Extrapolation theory; Decreasing function; Hardy operator; Riemann–Liouville; Lorentz spaces

1. Introduction

In 1951, Yano (see [22]) proved that for every sublinear operator T satisfying that, for every $1 < p \le p_0$ (p_0 fixed),

$$\left(\int_N |Tf(x)|^p \, d\nu(x)\right)^{1/p} \le \frac{C}{p-1}\left(\int_M |f(x)|^p \, d\mu(x)\right)^{1/p},$$

with C independent of p and where (N,ν) and (M,μ) are two finite measure spaces, it holds that

$$T : L\log L(\mu) \longrightarrow L^1(\nu)$$

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2012.02.005

is bounded. If the measures involved are not finite, then an easy modification of the above result shows that $T : L \log L(\mu) \longrightarrow L^1_{\mathrm{loc}}(\nu)$ and, in fact, $T : L \log L(\mu) \longrightarrow L^1(\nu) + L^\infty(\nu)$.

More recently, it has been proved (see [5]) that under a weaker condition on the operator T, namely that

$$T : L^{p,1} \longrightarrow \Gamma^{p,\infty}$$

is bounded for every $1 < p \le p_0$, with constant $C/(p-1)$, where

$$\|f\|_{\Gamma^{p,\infty}} = \sup_{t>0} f^{**}(t) t^{1/p},$$

it holds that

$$T : L \log L(\mu) \longrightarrow M(\varphi),$$

is bounded, where $M(\varphi)$ is the maximal Lorentz space associated with the function $\varphi(t) = t/(1 + \log^+ t)$; that is,

$$\|f\|_{M(\varphi)} = \sup_{t>0} \frac{t f^{**}_\nu(t)}{1 + \log^+ t}.$$

It turns out that this space $M(\varphi)$ is strictly embedded in $L^1(\nu) + L^\infty(\nu)$ and therefore, Yano's theorem was improved.

On the other hand, the theory of bounded operators on the cone of decreasing functions

$$L^p_{dec} = \left\{ f \ge 0;\ f \downarrow \text{ and } f \in L^p \right\}$$

has become very active in the last two decades. A starting point of this theory was the characterization of the boundedness of the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

in weighted Lorentz spaces $\Lambda^p(w)$ defined by the condition

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p w(t) dt \right)^{1/p} < \infty.$$

In fact, it was proved, in [1] (see also [16]), that for every f positive and decreasing and every $p > 1$,

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t f(s) ds \right)^p w(t) dt \right)^{1/p} \le A \left(\int_0^\infty f(t)^p w(t) dt \right)^{1/p} \tag{1}$$

if and only if $w \in B_p$; that is,

$$\|w\|_{B_p} := \sup_{r>0} \frac{\int_0^r w(t) dt + r^p \int_r^\infty \frac{w(t)}{t^p} dt}{\int_0^r w(t) dt} < \infty.$$

Now, since

$$(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) ds = f^{**}(t),$$

it was obtained as a consequence that

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w)$$

is bounded if and only if $w \in B_p$.

Moreover, this result remains true for every $p > 0$ and in this case (see also [11,7]), if $p \leq 1$, the constant A in (1) satisfies that $A = \|w\|_{B_p}^{1/p}$ while if $p > 1$, the best result known up to now (see [17]) is $\|w\|_{B_p}^{1/p} \leq A \leq \|w\|_{B_p}$. Consequently, if $Sf(t) = \frac{1}{t}\int_0^t f(s)ds$ is the Hardy operator, we have that

$$\|S\|_{L^p_{dec}(w)\to L^p(w)} \leq \|w\|_{B_p}^{\max(1,1/p)}. \tag{2}$$

On the other hand, Muckenhoupt proved (see [15]) that inequality (1) holds, for every positive function f and $p \geq 1$, if and only if $w \in M_p$, M_p being defined by the condition

$$\left(\int_r^\infty \frac{w(t)}{t^p}dt\right)\left(\int_0^r w(x)^{-p'/p}dx\right)^{p/p'} < \infty,$$

and hence, there are many weights in B_p which are not in M_p (see [1]). In particular, if d is a decreasing weight and

$$u(x) = \begin{cases} d(x)x^{q-1} & \text{if } x \notin (a,b) \\ 0 & \text{if } x \in (a,b) \end{cases}$$

we have that $u \in B_p \setminus M_p$ for $p > q$. Moreover, it is easy to check that

$$\|u\|_{B_p} \lesssim 1 + \frac{1}{p-q}.$$

Then, for every $q < p < p_0$ (p_0 fixed),

$$S : L^p_{dec}(u) \longrightarrow L^p(u)$$

is bounded with constant less than or equal to $C(p-q)^{-1}$, while the result is false if we consider L^p instead of L^p_{dec}. Moreover, taking d appropriately we may have that $u \notin B_q$ and hence S is not bounded on $L^q_{dec}(u)$. But, can we give some estimate at the end point $p = q$?

This simple example explains easily the motivation of this work in which the goal is to develop an extrapolation theory for operators acting on decreasing functions, that allows us to obtain end-point estimates, as in the classical case. As we shall see in Section 3, there are many other situations where this theory has interesting applications.

As usual, the symbol $f \approx g$ will indicate the existence of a universal positive constant C (independent of all parameters involved) such that $(1/C)f \leq g \leq Cf$, while the symbol $f \lesssim g$ means that $f \leq Cg$.

We shall work in the measure space $(\mathbb{R}_+, v)$ with $\mathbb{R}_+ = (0,\infty)$ and v a positive and locally integrable function in $\mathbb{R}_+$ called the weight. We write $\lambda_g^v(y) = \int_{\{x\in\mathbb{R}_+;\ |g(x)|>y\}} v(x)dx$, the distribution function of g with respect to the weight v, $g_v^*(t) = \inf\{s > 0; \lambda_g^v(s) \leq t\}$ is the decreasing rearrangement and $g_v^{**}(t) = (1/t)\int_0^t g_v^*(s)\,ds$ (see [2]). The cone of decreasing functions on $\mathbb{R}_+$ is the set of positive and decreasing functions on $\mathbb{R}_+$.

Also, for a measurable set E in $\mathbb{R}_+$, we shall denote by $|E|$ the Lebesgue measure of the set, and whenever it is not specified, the underlying measure in $\mathbb{R}_+$ will be this one. Finally, given a weight v in $\mathbb{R}_+$, we shall represent with the capital letter V its primitive $V(t) = \int_0^t v(s)ds$.

Given a decreasing function f, it is clear that

$$f(t) = \int_0^\infty \chi_{\{z;\, f(t)>z\}}(y)dy = \int_0^\infty \chi_{(0,\lambda_f(y))}(t)dy$$

and in this paper, all our examples deal with operators satisfying that

$$|Tf(t)| \le \int_0^\infty |T(\chi_{(0,\lambda_f(y))})(t)|dy \tag{3}$$

and hence, contrary to what happens in the classical case, the extrapolation theory of bounded operators on $L^{p,1}_{dec}$ and with values in a normed space is not really of interest since

$$T : L^{p,1}_{dec} \longrightarrow X, \qquad \|T\| \le K_p \iff \|T\chi_{(0,r)}\|_X \le K_p r^{1/p};$$

that is, it is completely characterized by its behavior on characteristic functions. To illustrate what we refer to, let us give a direct proof of the following extrapolation result without using extrapolation techniques.

Theorem 1.1. *Let $p_0 > 1$ be fixed and let T be an operator satisfying (3) such that, for every $1 < p \le p_0$, $T : L^{p,1}_{dec} \to \Gamma^{p,\infty}$ is bounded with constant less than or equal to $\frac{1}{(p-1)^m}$, where $m > 0$ and $L^{p,1}_{dec}$ is the set of decreasing functions such that*

$$\|f\|_{L^{p,1}_{dec}} = \int_0^\infty f(t)t^{1/p}\frac{dt}{t} < \infty.$$

Then we have that

$$T : L(\log L)^m_{dec} \to \Gamma^{1,\infty}(v_m)$$

is bounded, where

$$\|f\|_{\Gamma^{1,\infty}(v_m)} = \sup_{t>0} V_m(t) f^{**}(t),$$

with

$$V_m(t) = \int_0^t v_m(s)ds \approx \frac{t}{(1+\log^+ t)^m}$$

and

$$L(\log L)^m_{dec} = \left\{ f \downarrow;\ \|f\|_{L(\log L)^m_{dec}} = \int_0^\infty f(t)\left(1+\log^+\frac{1}{t}\right)^m dt < \infty \right\}.$$

Proof. By hypothesis, for every $1 < p \le p_0$,

$$(T\chi_{(0,r)})^{**}(t) \le \frac{1}{(p-1)^m}\left(\frac{r}{t}\right)^{1/p}$$

and hence, taking the infimum in p we get

$$(T\chi_{(0,r)})^{**}(t) \lesssim \frac{r}{t}\left(1+\log^+\frac{t}{r}\right)^m$$

and consequently

$$\|T\chi_{(0,r)}\|_{\Gamma^{1,\infty}(v_m)} \approx \sup_{t>0} \frac{t(T\chi_{(0,r)})^{**}(t)}{(1+\log^+ t)^m} \lesssim D_m(r)$$

where $D_m(t) = t\left(1+\log^+ \frac{1}{t}\right)^m$ and hence, using (3), we have that, for every decreasing function,

$$\|Tf\|_{\Gamma^{1,\infty}(v_m)} \lesssim \int_0^\infty D_m(\lambda_f(y))dy \approx \|f\|_{L(\log L)^m_{dec}}. \quad \square$$

The paper is organized as follows. In Section 2, we present the main results of the theory that deal with the following non-trivial cases:

(A) T is defined on $L^{p,1}_{dec}$ but it takes values in a space which is not a normed space, and
(B) T satisfies a strong boundedness condition of the type $T : L^p_{dec} \longrightarrow L^p$ with the right behavior of the constant.

In Section 3 we present several applications in the setting of bounded operators on the cone of decreasing functions in $L^p(u)$.

2. Preliminaries and main results

Let us start this section by recalling the definition of several weighted Lorentz spaces which will be fundamental for our purposes.

Definition 2.1. Given two weights u and v in $\mathbb{R}_+$, the Lorentz spaces $\Gamma^{p,\infty}_u(v)$, $\Lambda^{p,\infty}_u(v)$ and $\Lambda^p_u(v)$ are defined as the sets of measurable functions f satisfying that

$$\|f\|_{\Gamma^{p,\infty}_u(v)} = \sup_{t>0} V(t)^{1/p} f^{**}_u(t) < \infty,$$

$$\|f\|_{\Lambda^{p,\infty}_u(v)} = \sup_{t>0} V(t)^{1/p} f^{*}_u(t) < \infty,$$

and

$$\|f\|_{\Lambda^p_u(v)} = \left(\int_0^\infty f^*_u(t)^p v(t)dt\right)^{1/p} < \infty,$$

respectively.

It is trivially seen that, for every $1 \le p < \infty$, the spaces $\Gamma^{p,\infty}_u(v)$ are Banach spaces, while the Banach property for the spaces $\Lambda^{p,\infty}(v)$ and $\Lambda^p(v)$ depends on the properties of the weight v. In particular, if $p \ge 1$, $\Lambda^{p,\infty}(v)$ is a Banach space if and only if $v \in B_p$ [20]. This condition is also equivalent to $\Lambda^p(v)$ being Banach if $p > 1$ [19].

The following lemma, proved in [12], will be fundamental for our purposes.

Lemma 2.2. *Let f, g, h be three decreasing functions such that $f \le g + h$. Then, there exists a decomposition $f = f_0 + f_1$ with f_j decreasing functions and such that $f_0 \le g$ and $f_1 \le h$.*

Let us start with the weak type version of our extrapolation results. We want to emphasize here that although the space $\Lambda^{p,\infty}$ is normable and it coincides with the space $\Gamma^{p,\infty}$ appearing in the previous theorem if $p > 1$, the behavior of the constant is very important for these kinds of extrapolation results and, hence, we cannot renorm a space without taking into account whether this renorming affects the behavior of the constant. Namely, if $1 < p \leq p_0$ with $p_0 < \infty$ fixed,

$$\|f\|_{\Lambda^{p,\infty}} \leq \|f\|_{\Gamma^{p,\infty}} \lesssim \frac{1}{p-1}\|f\|_{\Lambda^{p,\infty}}.$$

Therefore, if $T : L^{p,1}_{dec} \to \Lambda^{p,\infty}$ is bounded, for every $1 < p \leq p_0$, with constant less than or equal to $\frac{1}{(p-1)^m}$, then $T : L^{p,1}_{dec} \to \Gamma^{p,\infty}$ will also be bounded, for every $1 < p \leq p_0$, but with constant less than or equal to $\frac{C}{(p-1)^{m+1}}$.

Theorem 2.3. *Let $p_0 > 1$, $m > 0$ and let u, v, w be three weights such that $W(s)/s$ is a decreasing function. If T is a sublinear operator such that*

$$T : L^{p,1}(u)_{dec} \to \Lambda^{p,\infty}_v(w)$$

is bounded for every $1 < p \leq p_0$ with constant less than or equal to $\frac{1}{(p-1)^m}$, then

$$T : \big(L(\log L)^m(\log\log\log L)\big)_{dec}(u) \to \Lambda^{1,\infty}_v(w_m)$$

is bounded with w_m such that

$$W_m(t) = \frac{W(t)}{\big(1+\log^+ W(t)\big)^m},$$

and

$$\|f\|_{\left(L(\log L)^m(\log\log\log L)\right)_{dec}(u)} = \int_0^\infty f(t)\left(1+\log^+\frac{1}{U(t)}\right)^m \times\left(1+\log^+\log^+\log^+\frac{1}{U(t)}\right)u(t)dt.$$

Proof. Let us take first f decreasing with $\|f\|_\infty \leq 2$. Then we have that

$$\begin{aligned}(Tf)^*_v(t)W(t)^{\frac{1}{p}} &\leq \frac{1}{(p-1)^m}\|f\|_{L^{p,1}(u)} = \frac{p}{(p-1)^m}\int_0^2 \lambda^u_f(y)^{\frac{1}{p}}dy \\ &\lesssim \frac{1}{(p-1)^m}\left(\int_0^\infty \lambda^u_f(y)dy\right)^{\frac{1}{p}} = \frac{1}{(p-1)^m}\|f\|^{\frac{1}{p}}_{L^1(u)},\end{aligned} \tag{4}$$

and thus

$$(Tf)^*_v(t) \lesssim \frac{1}{(p-1)^m}\left(\frac{\|f\|_{L^1(u)}}{W(t)}\right)^{\frac{1}{p}} \quad \text{for all } 1 < p \leq p_0.$$

Taking the infimum over all $1 < p \leq p_0$ we obtain that

$$(Tf)^*_v(t) \lesssim \frac{\|f\|_{L^1(u)}}{W(t)}\left(1+\log^+\frac{1}{\|f\|_{L^1(u)}}\right)^m (1+\log^+ W(t))^m.$$

It follows that

$$\|Tf\|_{\Lambda^{1,\infty}_v(w_m)} = \sup_{t>0}\frac{W(t)(Tf)^*_v(t)}{(1+\log^+ W(t))^m} \lesssim D_m(\|f\|_{L^1(u)}), \tag{5}$$

where $D_m(x) = x\left(1+\log+\frac{1}{x}\right)^m$, $x > 0$.

Now, for a decreasing and bounded function with $|f| \geq 2$ whenever $f \neq 0$, we can decompose $f = \sum_{n\in\mathbb{N}\cup\{0\}} f\chi_{E_n}$, where $E_n = \{2^{2^n} < f \leq 2^{2^{n+1}}\}$. Hence, if we define

$$\tilde{f}_n = \begin{cases} 2^{2^{n+1}}, & \bigcup_{i\geq n+1} E_i \\ f, & E_n \\ 0, & \bigcup_{i=0}^{n-1} E_i, \end{cases}$$

we have that $f \leq \sum_n \tilde{f}_n$ with $\tilde{f}_n$ decreasing for every $n \in \mathbb{N}\cup\{0\}$, and using Lemma 2.2, we obtain that there exist decreasing functions $\{g_n\}_n$ such that $f = \sum_n g_n$ and $g_n \leq \tilde{f}_n$ for all n. Let $\phi_n = 2^{-2^{n+1}} g_n$ and let $(c_n)_n$ be a sequence of positive numbers such that $\sum_n c_n = 1$. Using Theorem 2.1 from [8], we have that

$$\left(T\left(\sum_n g_n\right)\right)_v^*(3t) \lesssim \sum_n \left[(Tg_n)_v^*(t) + \frac{1}{t}\int_{c_nt}^t (Tg_n)_v^*(s)ds\right]$$

and since $W(3t) \approx W(t)$, we obtain that

$$\begin{aligned} \|Tf\|_{\Lambda^{1,\infty}_{v(w_m)}} &= \sup_{t>0} \frac{W(t)(Tf)_v^*(t)}{(1+\log^+ W(t))^m} = \sup_{t>0} \frac{W(t)\left(T\left(\sum_n g_n\right)\right)_v^*(t)}{(1+\log^+ W(t))^m} \\ &\lesssim \sup_{t>0} \frac{W(t)\sum_n \left[(Tg_n)_v^*(t) + \frac{1}{t}\int_{c_nt}^t (Tg_n)_v^*(s)ds\right]}{(1+\log^+ W(t))^m} \\ &\approx \sup_{t>0} \left(\frac{W(t)\sum_n 2^{2^{n+1}}(T\phi_n)_v^*(t)}{(1+\log^+ W(t))^m} + \frac{W(t)}{t}\frac{\sum_n 2^{2^{n+1}}\int_{c_nt}^t (T\phi_n)_v^*(s)ds}{(1+\log^+ W(t))^m}\right). \end{aligned}$$

Since ϕ_n is decreasing and $\|\phi_n\|_\infty \leq 1$, we have that

$$(T\phi_n)_v^*(s) \lesssim \frac{(1+\log^+ W(s))^m}{W(s)} D_m(\|\phi_n\|_{L^1(u)}),$$

and taking $c_n \approx 1/n^2$,

$$\begin{aligned} \int_{c_nt}^t (T\phi_n)_v^*(s)ds &\lesssim D_m(\|\phi_n\|_{L^1(u)})(1+\log^+ W(t))^m \int_{c_nt}^t \frac{s}{W(s)}\frac{ds}{s} \\ &\lesssim D_m(\|\phi_n\|_{L^1(u)}) \frac{t(1+\log^+ W(t))^m}{W(t)} \log(n+2) \end{aligned}$$

and thus

$$\|Tf\|_{\Lambda_v^{1,\infty}(w_m)} \lesssim \sum_{n\geq 0} 2^{2^{n+1}} \log(n+2)\cdot D_m(\|\phi_n\|_{L^1(u)}).$$

Since $\|\phi_n\|_{L^1(u)} \leq \lambda_f^u(2^{2^n})$ and the function D_m is equivalent to a concave function, it follows, using the same computation than in [9, Theorem 2.2, 3], that

$$\|Tf\|_{\Lambda_v^{1,\infty}(w_m)} \lesssim \int_1^\infty D_m\left(\lambda_f^u(y)\right)(1+\log^+\log^+\log^+ y)dy$$

$$\lesssim \int_0^\infty f_u^*(t) \left(1 + \log^+ \frac{1}{t}\right)^m \left(1 + \log^+ \log^+ \log^+ \frac{1}{t}\right) dt.$$

Now, for a general decreasing function g and a general weight u,

$$\lambda_g^u(y) = \int_{\{x; |g(x)|>y\}} u(s)ds = \int_0^{\lambda_g(y)} u(s)ds = U(\lambda_g(y)),$$

and hence, for every weight h in $\mathbb{R}_+$ and every f decreasing,

$$\begin{aligned}\int_0^\infty f_u^*(t)h(t)dt &= \int_0^\infty H(\lambda_g^u(y))dy = \int_0^\infty H(U(\lambda_g(y)))dy \\ &= \int_0^\infty f(t)h(U(t))u(t)dt,\end{aligned}$$

and consequently

$$\begin{aligned}&\int_0^\infty f_u^*(t) \left(1 + \log^+ \frac{1}{t}\right)^m \left(1 + \log^+ \log^+ \log^+ \frac{1}{t}\right) dt \\ &= \int_0^\infty f(t) \left(1 + \log^+ \frac{1}{U(t)}\right)^m \left(1 + \log^+ \log^+ \log^+ \frac{1}{U(t)}\right) u(t)dt.\end{aligned}$$

as we wanted to see. □

If the operator T satisfies a stronger condition than in the previous theorem such as that $T : L^p_{dec}(u) \to L^p(v)$ is bounded with the same behavior of the constant, then we can say more, but to this end we need to adapt the technique on [10] to the cone of decreasing functions.

Let us start with a technical lemma (if the reader is not familiar with the theory of interpolation and the K-functional, we recommend just reading the statement and skipping the proof for the present).

Lemma 2.4. *Let $q \geq 1$, u a weight in $\mathbb{R}_+$ and g a positive decreasing function. Let G be a concave function such that*

$$G(t) \approx \left(\int_0^{t^q} g_u^*(s)^q ds\right)^{1/q}.$$

Set, for every $i \in \mathbb{Z}$,

$$E_i = \{s \in (0, \infty);\ G'(s) > 2^i\}.$$

Then there exist $(g_i)_{i\in\mathbb{Z}}$ positive and decreasing functions such that $g = \sum_i 2^i g_i$ and

$$\left(\int_0^{t^q} (g_i)_u^*(s)^q ds\right)^{1/q} \lesssim \min(t, |E_i|).$$

Proof. Let us first mention that since the function

$$H(t) = \left(\int_0^{t^q} g_u^*(s)^q ds\right)^{1/q}$$

is increasing and $H(s)/s$ is decreasing, H is quasi-concave and hence equivalent to a concave function; so the existence of G is clear. Since G' is a decreasing function we have that

$G'(s) \lesssim \sum_i 2^i \chi_{E_i}(s)$ and $G(t) \lesssim \sum_i 2^i \min(t, |E_i|)$. On the other hand, it is known that $H(t) \approx K(g, t; L^q(u), L^\infty)$, but since g is decreasing and the cone of decreasing functions is a Marcinkiewicz cone for the couple $(L^q(u), L^\infty)$ [6], we have that

$$H(t) \approx K^{dec}(g, t; L^q(u), L^\infty),$$

where

$$K^{dec}(g, t; L^q(u), L^\infty) = \inf\{\|g_0\|_{L^q(u)} + t\|g_1\|_{L^\infty};\ g = g_0 + g_1, g_j \text{ is decreasing}\}.$$

Now, it was proved in [6] that

$$K^{dec}(g, t; L^q(u), L^\infty) = K(t, g; L^q(u)_D, L^\infty_D),$$

where for a Banach space X, $X_D = \{f;\ Df \in X\}$ with $\|f\|_{X_D} = \|Df\|_X$, with Df being the least decreasing majorant of f. Therefore,

$$G(t) \approx K(t, g; L^q(u)_D, L^\infty_D)$$

and thus we can use the K-divisibility theorem of interpolation theory [4, p. 325] to prove that there exists $(f_i)_i$ such that $g = \sum_i 2^i f_i$ and

$$K(t, f_i; L^q(u)_D, L^\infty_D) \lesssim \min(t, |E_i|).$$

Now, since $g \le \sum_i 2^i Df_i$ and Df_i is decreasing we can use Lemma 2.2 to conclude that there exist decreasing functions g_i such that $g = \sum_i 2^i g_i$ and $g_i \le Df_i$. Then

$$\begin{aligned} K(t, g_i; L^q(u), L^\infty) &\le K(t, Df_i; L^q(u), L^\infty) \approx K(t, f_i; L^q(u)_D, L^\infty_D) \\ &\lesssim \min(t, |E_i|) \end{aligned}$$

and the result follows. $\square$

Theorem 2.5. *Let $p_0 > 1$, $q \ge 1$ and let T be a sublinear operator such that, for every $q < p \le p_0$, $T : L^p_{dec}(u) \to L^p(v)$ is bounded with constant less than or equal to $\frac{1}{(p-q)^m}$; then for every decreasing function f,*

$$\sup_{t>0} \frac{\left(\int_0^t [(Tf)^*_v(s)]^q ds\right)^{1/q}}{(1+\log^+ t)^m} \lesssim \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^t f(s)^q u(s) ds\right)^{1/q}}{U(t)} \times \left(\log^+ \frac{1}{U(t)}\right)^{m-1} u(t) dt.$$

Proof. By the previous lemma, we can proceed exactly as in the proof of Theorem 2.4 in [10] to obtain that

$$\sup_{t>0} \frac{\left(\int_0^t [(Tf)^*_v(s)]^q ds\right)^{1/q}}{(1+\log^+ t)^m} \lesssim \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^t f^*_u(s)^q ds\right)^{1/q}}{t} \left(\log \frac{1}{t}\right)^{m-1} dt := \mathrm{I}.$$

Now, without loss of generality we can assume that U is strictly increasing and since f is decreasing, we have that, if $0 < t < \|u\|_1$,

$$\int_0^t f^*_u(s)^q ds = \int_0^t f(U^{-1}(s))^q ds = \int_0^{U^{-1}(t)} f(y)^q u(y) dy$$

and hence,

$$\mathrm{I} = \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^{U^{-1}(t)} f(s)^q u(s)ds\right)^{1/q}}{t} \left(\log \frac{1}{t}\right)^{m-1} dt$$
$$\lesssim \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^t f(s)^q u(s)ds\right)^{1/q}}{U(t)} \left(\log \frac{1}{U(t)}\right)^{m-1} u(t)dt$$

and the result follows. □

3. Examples and applications

I. Let us start by solving the motivation example given in the introduction.

Corollary 3.1. *Let $p_0 > 1$, $q \geq 1$ and let $u \in B_p \setminus B_q$ for every $p > q$ and such that, if $q < p \leq p_0$,*

$$\|u\|_{B_p} \lesssim \frac{1}{p-q}.$$

Then, for every decreasing function,

$$\sup_{0<t} \frac{\left(\int_0^t [(Sf)(s)]^q u(s)ds\right)^{1/q}}{(1+\log^+ U(t))} \lesssim \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^t f(s)^q u(s)ds\right)^{1/q}}{U(t)} u(t)dt.$$

Proof. By hypothesis, we already know that, for every $q < p \leq p_0$,

$$S : L^p_{dec}(u) \longrightarrow L^p(u)$$

is bounded with constant less than or equal to $C/(p-q)$. Then, by Theorem 2.5,

$$\sup_{t>0} \frac{\left(\int_0^t (Sf)^*_u(s)^q ds\right)^{1/q}}{(1+\log^+ t)} \lesssim \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^t f(s)^q u(s)ds\right)^{1/q}}{U(t)} u(t)dt.$$

Now, since Sf is decreasing, we can rewrite the term on the left hand side as

$$\sup_{0<t\leq\|u\|_1} \frac{\left(\int_0^{U^{-1}(t)} [(Sf)(s)]^q u(s)ds\right)^{1/q}}{(1+\log^+ t)}$$

and the result follows. □

II. Integral operators

An important class of operators in this setting of the cone of decreasing functions are the integral operators with positive kernels k; that is

$$Tf(x) = \int_0^\infty k(x,t) f(t)dt,$$

since this class includes many interesting operators such as generalized Hardy operators

$$S_{u,v} f(x) = \frac{1}{u(x)} \int_0^x f(s)v(s)ds,$$

generalized conjugate Hardy operators

$$\tilde{S}_{u,v} f(x) = v(x) \int_x^\infty \frac{f(s)}{u(s)} ds,$$

the fractional Riemann–Liouville operator

$$R_\lambda f(x) = x^{-\lambda} \int_0^x (x-t)^{\lambda-1} f(t) dt,$$

with $\lambda > 0$, the Laplace operator, Hardy operator with variable limits, etc.

In many cases, these operators satisfy that Tf is decreasing if f is decreasing.

In order to give examples of operators satisfying the hypothesis of Theorem 2.5 we need to start by computing the norm of several integral operators

$$T : L^p_{dec}(w) \to L^p(w),$$

and this is the aim of the following three results. We shall present the results for $p > 0$ since the proof is the same as for the case $p \geq 1$.

Theorem 3.2. *Let T be an integral operator with positive kernel and such that Tf is decreasing for every f decreasing and let $p > 0$. If*

$$K = \sup_{r,s>0} \frac{\int_0^r \int_0^s k(x,t) dt dx}{\min(r,s)} < \infty, \tag{6}$$

then

$$\|T\|_{L^p_{dec}(w) \to L^p(w)} \leq K \|w\|_{B_p}^{\max(1/p,1)}.$$

Proof. Let f be a positive decreasing function. Then,

$$\begin{aligned}
\int_0^u Tf(x) dx &= \int_0^u \int_0^\infty k(x,t) f(t) dt dx = \int_0^u \int_0^\infty \int_0^{\lambda_f(y)} k(x,t) dt dy dx \\
&= \int_0^\infty \int_0^u \int_0^{\lambda_f(y)} k(x,t) dt dx dy \leq K \int_0^\infty \min(u, \lambda_f(y)) dy \\
&= K \int_0^u f(t) dt
\end{aligned}$$

and hence

$$S(Tf)(t) \leq K Sf(t), \quad \forall\, t > 0.$$

Consequently, since $Tf \leq S(Tf)$, we obtain by (2) that

$$\|Tf\|_{L^p(w)} \leq \|S(Tf)\|_{L^p(w)} \leq K \|Sf\|_{L^p(w)} \leq K \|w\|_{B_p}^{\max(1/p,1)} \|f\|_{L^p(w)}. \quad \square$$

Using the same idea, we can also prove the following result.

Theorem 3.3. *Let T be an integral operator with positive kernel and such that Tf is decreasing for every f decreasing. If, for some $\alpha > -1$,*

$$K_\alpha = \sup_{r,s>0} \frac{\int_0^r \int_0^s k(x,t) x^\alpha dt dx}{\min(r,s)^{\alpha+1}} < \infty, \tag{7}$$

then

$$\|T\|_{L^p_{dec}(w)\to L^p(w)} \le K_\alpha \|w\|_{B_{p(1+\alpha)}}^{\max(1/p,1)}.$$

Proof. Let f be a positive decreasing function. Then, as in the previous theorem, it is easy to see that

$$\begin{aligned} S_\alpha(Tf)(u) &:= \frac{1+\alpha}{u^{1+\alpha}} \int_0^u Tf(x)x^\alpha dx \le K_\alpha \frac{1+\alpha}{u^{1+\alpha}} \int_0^\infty \min(u, \lambda_f(y))^{1+\alpha} dy \\ &= K_\alpha(1+\alpha) S_\alpha f(u) \end{aligned}$$

and hence since $Tf \le S_\alpha(Tf)$, we obtain by (2) that

$$\begin{aligned} \|Tf\|_{L^p(w)} &\le \|S_\alpha(Tf)\|_{L^p(w)} \le K_\alpha(1+\alpha)\|S_\alpha f\|_{L^p(w)} \\ &\le K_\alpha(1+\alpha)\|w\|_{B_{p(1+\alpha)}}^{\max(1/p,1)} \|f\|_{L^p(w)}, \end{aligned}$$

where the last inequality follows on changing variables and applying (2). $\square$

Corollary 3.4. *Let T and K_α be as in the previous theorem and let us assume that for every $-1 < \alpha < 0$, $K_\alpha < \infty$. Then, for every $w \in B_p$,*

$$T : L^p_{dec}(w) \longrightarrow L^p(w)$$

is bounded and

$$\|T\|_{L^p_{dec}(w)\to L^p(w)} \le \inf_{-1<\alpha<0} K_\alpha \|w\|_{B_{p(1+\alpha)}}^{\max(1/p,1)}.$$

Proof. This is an immediate consequence of the previous theorem and the fact that if $w \in B_p$, there exists $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$ and hence

$$\inf_{-1<\alpha<0} K_\alpha \|w\|_{B_{p(1+\alpha)}}^{\max(1/p,1)} < \infty. \quad \square$$

Examples:

I. A generalized Hardy operator

There are many operators in the literature under the name of generalized Hardy operator. The one that we refer to here is given by

$$S_v f(t) = \frac{1}{V(t)} \int_0^t f(s)v(s)ds,$$

with v an arbitrary weight. Boundedness properties of these operators have been considered in several papers which can be found in the book of Kufner and Persson [13].

Using Theorem 3.2, we immediately see that the following result holds:

Proposition 3.5. *If a weight v satisfies that*

$$\int_r^\infty \frac{1}{V(x)} dx \lesssim \frac{r}{V(r)},$$

then, for every $w \in B_p$ with $p > 0$,

$$\|S_v\|_{L^p_{dec}(w)\to L^p(w)} \lesssim \|w\|_{B_p}^{\max(1/p,1)}.$$

Remark 3.6. As a consequence, S_v satisfies the same inequality as S in Corollary 3.1.

II. The Riemann–Liouville fractional operator

The boundedness properties of the Riemann–Liouville operator

$$R_\lambda f(x) = x^{-\lambda} \int_0^x (x-t)^{\lambda-1} f(t)dt,$$

with $\lambda > 0$ have been studied in several papers such as [3,14,18,21], among many others. Also in [7] it was proved that if $0 < \lambda \le 1$,

$$R_\lambda : L^p_{dec}(w) \longrightarrow L^p(w)$$

is bounded for every $w \in B_p$, but the norm of this operator was not explicitly computed there. Let us see now that the kernel satisfies the condition of Corollary 3.4. In this case

$$k(x,t) = \frac{(x-t)^{\lambda-1}}{x^\lambda} \chi_{(0,x)(t)}$$

and hence

$$\int_0^s k(x,t)dt = \frac{1}{\lambda}\frac{1}{x^\lambda}\big(x^\lambda - (x - \min(s,x))^\lambda\big)$$

and

$$\begin{aligned}\int_0^r \int_0^s k(x,t)dt x^\alpha dx &= \int_0^r \frac{1}{\lambda}\frac{1}{x^\lambda}\big(x^\lambda - (x-\min(s,x))^\lambda\big)x^\alpha dx \\ &= \frac{1}{\lambda(1+\alpha)}\min(r,s)^{1+\alpha} + \frac{1}{\lambda}\left[\int_s^r \Big(1 - \big(1 - \frac{s}{x}\big)^\lambda\Big) x^\alpha dx\right]_+ \\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

To estimate II, we proceed by a change of variables assuming that $s < r$ (the contrary case trivially gives 0):

$$\begin{aligned}\mathrm{II} &= \frac{1}{\lambda}\left(\frac{r^{1+\alpha}}{1+\alpha} - \frac{s^{1+\alpha}}{1+\alpha}\right) - \frac{1}{\lambda}\left[\int_s^r \left(1-\frac{s}{x}\right)^\lambda x^\alpha dx\right] \\ &= \frac{1}{\lambda}\left(\frac{r^{1+\alpha}}{1+\alpha} - \frac{s^{1+\alpha}}{1+\alpha}\right) - \frac{s^{1+\alpha}}{\lambda}\int_{s/r}^1 (1-u)^\lambda \frac{du}{u^{2+\alpha}}.\end{aligned}$$

Therefore to have the above term controlled by a constant $K_\alpha s^{1+\alpha}$, we need only to see that

$$\sup_{0<s<r<\infty}\left(\frac{1}{\lambda}\left(\frac{r^{1+\alpha}}{s^{1+\alpha}(1+\alpha)}\right) - \frac{1}{\lambda}\int_{s/r}^1 (1-u)^\lambda \frac{du}{u^{2+\alpha}}\right) = K_\alpha < \infty.$$

That is,

$$\sup_{0<\mu<1}\left(\frac{1}{\lambda}\left(\frac{1}{\mu^{1+\alpha}(1+\alpha)}\right) - \frac{1}{\lambda}\int_{\mu}^1 (1-u)^\lambda \frac{du}{u^{2+\alpha}}\right) = K_\alpha < \infty.$$

Now, it is easy to see that

$$\begin{aligned}K_\alpha &= \frac{1}{\lambda(1+\alpha)} + \frac{1}{\lambda}\int_0^1 \frac{(1-(1-u)^\lambda)}{u}\frac{du}{u^{1+\alpha}} \\ &\lesssim \frac{1}{\lambda(1+\alpha)} + \frac{1}{\lambda}\int_0^1 \frac{du}{u^{1+\alpha}} = \frac{-1}{\lambda\alpha(1+\alpha)}\infty.\end{aligned}$$

By Corollary 3.4, we obtain the following result.

Corollary 3.7. *If* $0 < \lambda < 1$,

$$\|R_\lambda\|_{L^p_{dec}(w)\to L^p(w)} \lesssim \inf_{0<\mu<1} \frac{1}{\mu(1-\mu)} \|w\|_{B_{p\mu}}^{\max(1/p,1)}.$$

And, as a consequence, we can prove our last corollary.

Corollary 3.8. *Let* $p_0 > 1$, $q \geq 1$ *and let* $u \in B_p \setminus B_q$ *for every* $p > q$ *and such that if* $q < p \leq p_0$,

$$\|u\|_{B_p} \lesssim \left(\frac{1}{p-q}\right)^m.$$

Then, for every decreasing function f,

$$\sup_{0<t} \frac{\left(\int_0^t [(R_\lambda f)(s)]^q u(s)ds\right)^{1/q}}{(1+\log^+ U(t))^{1+m}} \lesssim \|f\|_{L^q(u)} + \int_0^1 \frac{\left(\int_0^t f(s)^q u(s)ds\right)^{1/q}}{U(t)} \times \left(1+\log^+ \frac{1}{U(t)}\right)^m u(t)dt.$$

Proof. By hypothesis and the previous corollary, we have to compute

$$K = \inf_{\frac{q}{p}<\mu<1} \frac{1}{\mu(1-\mu)} \left(\frac{1}{p\mu-q}\right)^m,$$

for which simple computations lead us to $K \lesssim \left(\frac{1}{p-q}\right)^{1+m}$ and the result follows by Theorem 2.5 and the fact that $R_\lambda f$ is decreasing. □

Acknowledgments

The first author was supported by MTM2010-14946 and the last two authors were partially supported by CNCSIS-UEFISCSU project number 538/2009 PNII-IDEI code 1905/2008. We finally want to thank the referee for his/her comments and suggestions.

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