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A Characterization of Transversal Topologies

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K. Steffens ("Injektive Auswahlfunktionen," Schriften aus dem Gebiet der Angewandten Mathematik Nr. 2, Aachen, 1972) has shown that a family of finite sets has a transversal if and only if the collection of all "critical" subfamilies forms a topology. In this paper these "transversal topologies" are characterized, as well as families whose transversal topologies satisfy separation axioms.

1. INTRODUCTION

Let $\mathscr{A} = (A_i \mid i \in I)$ be an arbitrary family of finite nonempty subsets of a ground set $E = \bigcup_{i \in I} A_i$. We write

$$A(J) = \bigcup_{i \in J} A_i \qquad \text{and} \qquad \mathscr{A}_J = (A_i \mid i \in J)$$

for $J \subseteq I$. The symbol $J \Subset I$ shall indicate that J is a finite subset of I; $J \subset I$ shall mean that J is a proper subset of I. A set $T \subseteq E$ is called a *transversal* of $\mathscr{A}$ if there is a bijection Φ: $T \to I$ such that $x \in A_{\Phi(x)}$ for all $x \in T$. The set of all transversals of $\mathscr{A}$ shall be denoted by $\mathrm{TR}(\mathscr{A})$. Finally, if $\mathscr{A} = (A_i \mid i \in I)$ and $\mathscr{B} = (B_j \mid j \in J)$ are two families with disjoint index sets, then the family $\mathscr{A} + \mathscr{B}$ is defined as $(C_k \mid k \in I \cup J)$, where $C_k = A_k$ or B_k depending on whether $k \in I$ or $k \in J$.

In [2] Hall proved the following necessary and sufficient conditions for the existence of a transversal:

THEOREM A. *$\mathrm{TR}(\mathscr{A}) \neq \varnothing$ if and only if $|A(J)| \geqslant |J|$ for all $J \Subset I$.*

Steffens [5] introduced the notion of a "critical" subfamily:

DEFINITION A. A subfamily $\mathscr{A}_J$ is called *critical* iff it has a unique transversal.

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Remark 1. If $\mathscr{A}_J$ is a critical subfamily with transversal T, then $T = A(J)$. Therefore, for a critical subfamily $\mathscr{A}_J$ we have $|A(J)| = |J|$.

We shall make use of the following of Steffens' results [5]:

THEOREM B. *Every critical subfamily is the union of finite critical subfamilies.*

THEOREM C. $\mathrm{TR}(\mathscr{A}) \neq \varnothing$ *if and only if there is no* $J \subseteq I$ *and there is no* $i \in I \backslash J$ *such that* $\mathscr{A}_J$ *is critical and* $A_i \subseteq A(J)$.

THEOREM D. $\mathrm{TR}(\mathscr{A}) \neq \varnothing$ *if and only if* $\mathfrak{T}(\mathscr{A}) = \{\mathscr{A}_J \subseteq \mathscr{A} \mid \mathscr{A}_J \text{ critical}\} \cup \{\varnothing, \mathscr{A}\}$ *is a topology on* $\mathscr{A}$.

DEFINITION 1. If $\mathrm{TR}(\mathscr{A}) \neq \varnothing$ we call $\mathfrak{T}(\mathscr{A})$ the *transversal topology* of the family $\mathscr{A}$ and refer to its elements as the *open subfamilies* of $\mathscr{A}$.

The purpose of this paper is to characterize transversal topologies as well as families whose transversal topologies satisfy separation axioms.

2. DEFINITIONS AND PRELIMINARY RESULTS

For the following we assume that $\mathrm{TR}(\mathscr{A}) \neq \varnothing$ and $|I| \geqslant 2$. In [3] we introduced the following notations:

Let $A \in \mathscr{A}$ and $x \in E$. Then we shall write

$$C(A) = \min_{\substack{J \subseteq I \\ A \in \mathscr{A}_J}} (|A(J)| - |J|) \qquad \text{and} \qquad C(x) = \min_{x \in A \in \mathscr{A}} C(A).$$

In view of Definition A and Theorem B, $C(A) = 0$ means that A is contained in a critical subfamily while $C(x) = 0$ means that x is contained in every transversal of $\mathscr{A}$.

DEFINITION 2. An open subfamily $\mathscr{A}_J \neq \varnothing$ is called *minimal* iff $\mathscr{A}_K \notin \mathfrak{T}(\mathscr{A})$ for all $\varnothing \subset K \subset J$.

We write $\mathfrak{T}_m = \{\mathscr{A}_J \in \mathfrak{T}(\mathscr{A}) \mid \mathscr{A}_J \text{ minimal}\}$.

Remark 2. On account of Theorem B every $\mathscr{A}_J \in \mathfrak{T}_m \backslash (\mathscr{A})$ is a finite subfamily of $\mathscr{A}$.

LEMMA 1. *Let* $\mathscr{A}_J \in \mathfrak{T}_m$, $\mathscr{A}_K \in \mathfrak{T}(\mathscr{A})$. *Then either* $J \subseteq K$ *or* $J \cap K = \varnothing$.

Proof. Assume $\varnothing \subset J \cap K \subset J$. By Theorem D we have $\mathscr{A}_{J \cap K} \in \mathfrak{T}(\mathscr{A})$ contradicting the minimality of $\mathscr{A}_J$.

LEMMA 2. *Let $A \in \mathscr{A}$. Then there exists a smallest open subfamily $\mathscr{A}_A$ containing A.*

Proof. If $C(A) = 0$ then, by Theorem B, we know that $\mathscr{A}_A$ is finite. The uniqueness follows from Theorem D. If $C(A) \neq 0$ then $\mathscr{A}$ is the only open family containing A.

DEFINITION 3. $\mathfrak{C}(\mathscr{A}) = \{\mathscr{A} \backslash \mathscr{A}_I \mid \mathscr{A}_I \in \mathfrak{T}(\mathscr{A})\}$ shall denote the set of all *closed subfamilies* of $\mathscr{A}$.

Finally, we briefly recall the separation axioms we shall deal with. More detailed information about them can be found in [1].

Let $\mathfrak{S}$ be a topology on a set X and $x \in X$. Then we write

$$\{x\}' = \{y \neq x \mid \text{for all } G \in \mathfrak{S}\ (y \in G \text{ implies } x \in G)\}$$

and

$$\{x\}^{\sim} = \bigcap_{x \in G \in \mathfrak{S}} G \setminus \{x\}.$$

(A1) $\mathfrak{S}$ is a T_0-topology if and only if for all $x, y \in X$ with $x \neq y$ there exists a set $G \in \mathfrak{S}$ such that $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$.

(A2) $\mathfrak{S}$ is a T_D-topology if and only if for all $x \in X$ there exist an open set G and a closed set C such that $\{x\} = G \cap C$.

(A3) $\mathfrak{S}$ is a T_F-topology if and only if for all $x \in X$ ($y \in \{x\}^{\sim}$ implies $\{y\}^{\sim} = \varnothing$).

(A4) $\mathfrak{S}$ is a T_Y-topology if and only if $\mathfrak{S}$ is a T_F-topology and for all $x, y \in X$ ($x \neq y$ implies $|\{x\}^{\sim} \cap \{y\}^{\sim}| \leqslant 1$).

(A5) $\mathfrak{S}$ is a T_{DD}-topology if and only if $\mathfrak{S}$ is a T_D-topology and $|\{x\}^{\sim}| \leqslant 1$ for all $x \in X$.

(A6) $\mathfrak{S}$ is a T_{FF}-topology if and only if either of the following cases holds:

(i) $\{x\}^{\sim} = \varnothing$ for all but at most one $x \in X$,

(ii) $\{x\}' = \varnothing$ for all but at most one $x \in X$.

(A7) $\mathfrak{S}$ is a T_1-topology if and only if $\{x\}^{\sim} = \varnothing$ for all $x \in X$.

Remark 3. All these topologies satisfy the T_0-property.

(A8) $\mathfrak{S}$ is regular if and only if for all $x \in X$ and for all closed sets C with $x \notin C$ there exist disjoint open sets G_x and G_C with $x \in G_x$ and $C \subseteq G_C$.

3. Main Theorems

THEOREM 1. *A topology $\mathfrak{S}$ on X is homeomorphic to a transversal topology $\mathfrak{T}(\mathscr{A})$ if and only if $\mathfrak{B}(\mathfrak{S}) = \{G \in \mathfrak{S} \mid |G| < \aleph_0\} \cup \{X\}$ is a base for $\mathfrak{S}$.*

Proof. The necessity follows from Theorem B.

Sufficiency: We write $X^* = \bigcup_{G \in \mathfrak{S} \setminus \{X\}} G$. If $x \in X^*$ let G_x denote the smallest $G \in \mathfrak{B}(\mathfrak{S})$ such that $x \in G$. If $x \in X \setminus X^*$ let $G_x = \{a_x, b_x\}_{\neq}$ such that $G_x \cap X^* = \varnothing$ and $G_{x_1} \cap G_{x_2} = \varnothing$ for all $x_1, x_2 \in X \setminus X^*$ with $x_1 \neq x_2$. Then

$$\mathscr{A} = (G_x \mid x \in X)$$

is a family of finite sets with $\mathrm{TR}(\mathscr{A}) \neq \varnothing$. Define

$$f: X \to \mathscr{A} \qquad \text{by} \quad f(x) = G_x .$$

f is a bijection since if $G_x = G_y$ for $x \neq y$ we distinguish between the two sets according to their indices. To show that f is open and continuous it suffices to consider the elements of the bases.

(a) Let $G \in \mathfrak{B}(\mathfrak{S})$. If $G = \varnothing$ or $G = X$, then $\mathscr{A}_G = \varnothing$ or $\mathscr{A}_G = \mathscr{A}$ which are both elements of $\mathfrak{T}(\mathscr{A})$. Now let $\varnothing \subset G \subset X$. Then G is finite, say $G = \{x_1, ..., x_k\}$ and we have

$$k = |\{x_1, ..., x_k\}| \leqslant |G_{x_1} \cup \cdots \cup G_{x_k}| \leqslant |G| = k,$$

which means that $\mathscr{A}_G$ is critical.

(b) Let $Y \subseteq X$ with $\mathscr{A}_Y \in \{\mathscr{A}_J \subseteq \mathscr{A} \mid \mathscr{A}_J \text{ critical, finite}\} \cup \{\varnothing, \mathscr{A}\}$. If $Y \in \{\varnothing, X\}$ then $Y \in \mathfrak{S}$. Assume $\varnothing \subset Y \subset X$. Then $Y \subseteq \bigcup_{y \in Y} G_y = A(Y)$. Since $\mathscr{A}_Y$ is critical,

$$|A(Y)| = |Y| \leqslant \left| \bigcup_{y \in Y} G_y \right| = |A(Y)|,$$

and therefore $Y = \bigcup_{y \in Y} G_y \in \mathfrak{S}$, which completes the proof.

LEMMA 3. *If $\mathfrak{T}(\mathscr{A})$ is a T_0-topology, then $|J| = 1$ for all $\mathscr{A}_J \in \mathfrak{T}_m$.*

Proof. $|J| \geqslant 2$ for $\mathscr{A}_J \in \mathfrak{T}_m$ would imply that no two sets in $\mathscr{A}_J$ could be separated by open subfamilies.

LEMMA 4. *If $\mathfrak{T}(\mathscr{A})$ is a T_0-topology, then there is at most one $A_i \in \mathscr{A}$ with $C(A_i) \neq 0$.*

Proof. Assume $C(A_i) \neq 0$, $C(A_j) \neq 0$ for $i \neq j$. By Lemma 2, the only open subfamily containing A_i is $\mathscr{A}$ itself, which also contains A_j, and vice versa.

Let $\mathfrak{T}(\mathscr{A})$ be a T_0-topology. Then we write $I^* = I \backslash \{i \mid C(A_i) \neq 0\}$.

THEOREM 2. *The following statements are equivalent*:

(a) $\mathfrak{T}(\mathscr{A})$ *is a* T_0-*topology*,

(b) $\mathfrak{T}(\mathscr{A})$ *is a* T_D-*topology*,

(c) *there exist an ordinal* α *and a bijection* $\psi\colon \alpha \to I^*$ *such that*

$$\Big| A_{\psi(\beta)} \Big\backslash \bigcup_{\gamma<\beta} A_{\psi(\gamma)} \Big| = 1 \qquad \textit{for all} \quad \beta < \alpha, \tag{1}$$

(d) *there exist an ordinal* α *and a bijection* $\psi\colon \alpha \to I^*$ *such that*

$$\mathscr{A}_{\psi(\beta)} = (A_{\psi(\gamma)} \mid \gamma \leqslant \beta) \qquad \textit{is critical for all } \beta < \alpha,$$

(e) *there is exactly one injective choice function of* $\mathscr{A}_{I^*}$.

Proof. (c) $\Rightarrow$ (d). We shall use transfinite induction.

(1) $\mathscr{A}_{\psi(0)} = (A_{\psi(0)})$ is critical since $| A_{\psi(0)} | = 1$.

(2) If β has an immediate predecessor $\beta - 1$, then $\mathscr{A}_{\psi(\beta)} = \mathscr{A}_{\psi(\beta-1)} + (A_{\psi(\beta)})$. By induction hypothesis $\mathscr{A}_{\psi(\beta-1)}$ is critical and therefore by (1) also $\mathscr{A}_{\psi(\beta)}$.

(3) If β is a limit ordinal then $\mathscr{A}_{\psi(\beta)} = (A_{\psi(\gamma)} \mid \gamma < \beta) + (A_{\psi(\beta)})$, where

$$(A_{\psi(\gamma)} \mid \gamma < \beta) = \bigcup_{\gamma<\beta} \mathscr{A}_{\psi(\gamma)}$$

is critical. Again using (1) we obtain that $\mathscr{A}_{\psi(\beta)}$ is critical.

(d) $\Rightarrow$ (b). Take $A_i \in \mathscr{A}$. If $C(A_i) \neq 0$, then $(A_i) \in \mathfrak{C}(\mathscr{A})$ and $(A_i) = \mathscr{A} \cap (A_i)$. If $C(A_i) = 0$ there exists an ordinal β such that $\psi(\beta) = i$ and $\mathscr{A}_{\psi(\beta)} \in \mathfrak{T}(\mathscr{A})$. Furthermore, $\mathscr{A} \backslash \bigcup_{\gamma<\beta} \mathscr{A}_{\psi(\gamma)} \in \mathfrak{C}(\mathscr{A})$ and $(A_i) = \mathscr{A}_{\psi(\beta)} \cap (\mathscr{A} \backslash \bigcup_{\gamma<\beta} \mathscr{A}_{\psi(\gamma)})$. Therefore $\mathfrak{T}(\mathscr{A})$ is a T_D-topology.

(b) $\Rightarrow$ (a). Remark 3.

(a) $\Rightarrow$ (c). By Lemma 3 we know that there is at least one set (say A_0) in $\mathscr{A}$ with $| A_0 | = 1$. To establish the bijection ψ we define $\psi(0) = 0$, and if $\psi(\gamma)$ has already been defined for $\gamma < \beta$, set $\psi(\beta) = i_0$ where i_0 satisfies $| A_{i_0} \backslash \bigcup_{\gamma<\beta} A_{\psi(\gamma)} | = 1$.

It remains to show that as long as $\{\psi(\gamma) \mid \gamma < \beta\} \subset I^*$, a suitable i_0 can be found. Assume the contrary: there is a minimal β_0 such that

$$\Big| A_i \Big\backslash \bigcup_{\gamma<\beta_0} A_{\psi(\gamma)} \Big| \neq 1 \qquad \text{for all} \quad i \in I^* \backslash \{\psi(\gamma) \mid \gamma < \beta_0\}.$$

By the same argument as in (c) $\Rightarrow$ (d) we can deduce that $(A_{\psi(\gamma)} \mid \gamma < \beta_0)$

is critical, therefore $|A_i \backslash \bigcup_{\gamma<\beta_0} A_{\psi(\gamma)}| = 0$ would contradict Theorem C. On the other hand, if

$$\left| A_i \Big\backslash \bigcup_{\gamma<\beta_0} A_{\psi(\gamma)} \right| \geqslant 2 \qquad \text{for all} \quad i \in I^* \backslash \{\psi(\gamma) \mid \gamma < \beta_0\} \tag{2}$$

fix i and let $\mathscr{A}_{A_i}$ be the smallest open subfamily containing A_i. Since $i \in I^*$, $\mathscr{A}_{A_i}$ is finite. Now $\mathscr{A}_{A_i} \backslash (A_i) \not\subseteq (A_{\psi(\gamma)} \mid \gamma < \beta_0)$ since the contrary would imply that $\mathscr{A}_{A_i} \cup (A_{\psi(\gamma)} \mid \gamma < \beta_0)$ is critical, contradicting (2). Therefore, there exists $j \in I^* \backslash \{\psi(\gamma) \mid \gamma < \beta_0\}$, $j \neq i$ with $A_j \in \mathscr{A}_{A_i}$. Now $\mathscr{A}_{A_j} = \mathscr{A}_{A_i}$ would be a contradiction to the T_0-property. So $\mathscr{A}_{A_j} \subset \mathscr{A}_{A_i}$ and we can apply the same argument as before to $\mathscr{A}_{A_j}$. Continuing this process we arrive after finitely many steps either at a contradiction to the T_0-property or we obtain a critical subfamily $\mathscr{A}_{A_k}$ with $|\mathscr{A}_{A_k}| = 1$ which contradicts (2).

(c) $\Leftrightarrow$ (e). See [6].

Before dealing with the other separation axioms we observe that

$$(A_i)' = (A_j \mid j \neq i, A_i \in \mathscr{A}_{A_j}),$$

while

$$(A_i)^{\sim} = (A_j \mid j \neq i, A_j \in \mathscr{A}_{A_i}).$$

THEOREM 3. *$\mathfrak{T}(\mathscr{A})$ is a T_F-topology if and only if one of the following conditions holds*:

(i) *If $I^* \subset I$ then $|I \backslash I^*| = 1$ and $|A_i| = 1$ for all $i \in I^*$,*

(ii) *If $I^* = I$ then*

$$\left| A_i \Big\backslash \bigcup_{\substack{|A_j|=1 \\ j\neq i}} A_j \right| = 1 \qquad \textit{for all} \quad i \in I.$$

Proof. (i) Necessity: From Remark 3 and Lemma 4 we deduce that $|I \backslash I^*| = 1$. Let A_i be the set with $C(A_i) \neq 0$ then $(A_i)^{\sim} = (A_j \mid j \in I^*)$. The T_F-property now implies that $|A_j| = 1$ for all $j \in I^*$.

The sufficiency is obvious.

(ii) Necessity: Let $|A_i| > 1$ and $\mathscr{A}_{A_i} = (A_{j_1}, \ldots, A_{j_k})$. Then, by the T_F-property, we have $|A_{j_h}| = 1$ for all $j_h \neq i$ and the assertion follows.

Sufficiency: Let $A_j \in (A_i)^{\sim}$. From

$$\left| A_i \Big\backslash \bigcup_{\substack{|A_l|=1 \\ l\neq i}} A_l \right| = 1$$

it follows that all but one element of $A_i = \{x_1, \ldots, x_k\}$ are contained in sets $\{x_h\} \in \mathscr{A}$ $(1 \leqslant h \leqslant k-1$, say). Therefore

$$\mathscr{A}_{A_i} = (\{x_1\}, \ldots, \{x_{k-1}\}, A_i),$$

hence

$$(A_j)^{\sim} = \varnothing .$$

THEOREM 4. *$\mathfrak{T}(\mathscr{A})$ is a T_Y-topology if and only if $\mathfrak{T}(\mathscr{A})$ is a T_F-topology and $|A_i \cap A_j| \leqslant 1$ for all $i \neq j$.*

Proof. We may assume $I^* = I$.

Necessity: Assume $\{x, y\}_{\neq} \subseteq A_i \cap A_j$. The T_F-property implies $\{x\} \in \mathscr{A}$ or $\{y\} \in \mathscr{A}$. But if $\{y\} \notin \mathscr{A}$, then y would be the representative of both A_i and A_j, which is, of course, a contradiction. Theorem C implies $(\{x\}, \{y\}) \subseteq \mathscr{A}_{A_i} \cap \mathscr{A}_{A_j}$ and so $|(A_i)^{\sim} \cap (A_j)^{\sim}| \geqslant 2$.

Sufficiency: Assume $(A_1, A_2) \subseteq (A_i)^{\sim} \cap (A_j)^{\sim}$. From the T_F-property we derive $A_1 = \{x_1\}$ and $A_2 = \{x_2\}$. But $\{x_1\} \in \mathscr{A}_{A_i}$ implies $x_1 \in A_i$. Analogously we get $x_2 \in A_i$ as well as $\{x_1, x_2\} \subseteq A_j$ and therefore $|A_i \cap A_j| \geqslant 2$.

THEOREM 5. *$\mathfrak{T}(\mathscr{A})$ is a T_{DD}-topology if and only if $|A_i| \leqslant 2$ and*

$$\Big| A_i \Big\backslash \bigcup_{\substack{|A_j|=1 \\ j\neq i}} A_j \Big| = 1 \qquad \textit{for all} \quad i \in I.$$

Proof. Necessity: $|A_i| \geqslant 3$ would imply $|\mathscr{A}_{A_i}| \geqslant 3$ and therefore $|(A_i)^{\sim}| \geqslant 2$. Similarly, $A_i = \{x, y\}_{\neq}$ with $\{x\} \notin \mathscr{A}$ and $\{y\} \notin \mathscr{A}$ would yield $|\mathscr{A}_{A_i}| \geqslant 3$, since $\mathscr{A}_{A_i} = (\{x, y\}_{\neq}, \{x, y\}_{\neq})$ would contradict the T_0-property.

Sufficiency: If $|A_i| = 1$, then $|(A_i)^{\sim}| = 0$. If $|A_i| = 2$, say $A_i = \{x, y\}_{\neq}$, then by hypothesis $\{x\} \in \mathscr{A}$ or $\{y\} \in \mathscr{A}$, hence $|\mathscr{A}_{A_i}| = 2$ and $|(A_i)^{\sim}| = 1$.

THEOREM 6. *$\mathfrak{T}(\mathscr{A})$ is a T_{FF}-topology if and only if either of the following conditions holds:*

(a) *$|A_i| = 1$ for all but at most one $i \in I$,*

(b) *there exists a set $A_i \in \mathscr{A}$ with $|A_i| = 1$ such that $|A_j \backslash A_i| = 1$ for all $j \in I \backslash \{i\}$.*

Proof. Statement (a) is equivalent to part (i) of the definition of a T_{FF}-topology.

Necessity: Assume $(A_i)' \neq \varnothing$ but $(A_j)' = \varnothing$ for all $j \neq i$. Then $\mathscr{A}_{A_j} = (A_j)$ or $\mathscr{A}_{A_j} = (A_j, A_i)$ and since $\mathscr{A}_{A_i} = (A_i)$ the assertion follows.

Sufficiency: If $|A_j| = 2$ then $A_i \in \mathscr{A}_{A_j}$ and therefore

$$(A_i)' = \{A_j \mid |A_j| = 2\},$$

but $(A_j)' = \varnothing$.

Since A_i is the only set in $\mathscr{A}$ which can be subset of others, we also have $(A_j)' = \varnothing$ for all $j \neq i$ with $|A_j| = 1$.

THEOREM 7. *The following statements are equivalent:*

(a) $\mathfrak{T}(\mathscr{A})$ *is a* T_1*-topology,*

(b) $|A_i| = 1$ *for all* $i \in I$,

(c) $\mathfrak{T}(\mathscr{A})$ *is the discrete topology.*

Proof. $(A_i)^{\sim} = \varnothing$ if and only if $|A_i| = 1$.

LEMMA 5. *Let* $C(x) = 0$ *for all* $x \in E$ *and* $K = \bigcup_{\mathscr{A}_J \in \mathfrak{T}_m} J \subset I$. *Then* $A(K) \cap A(I \backslash K) \neq \varnothing$.

Proof. Assume the contrary. We know that all elements of E are needed to represent the sets of $\mathscr{A}$ and since $\mathscr{A}_K$ is critical (Theorem D) the elements of $A(K)$ can only represent the sets of $\mathscr{A}_K$. Therefore all elements of $A(I \backslash K)$ are needed to represent $\mathscr{A}_{I \backslash K}$ which implies that $\mathscr{A}_{I \backslash K} \in \mathfrak{T}(\mathscr{A})$. But then there exists a minimal open subfamily of $\mathscr{A}_{I \backslash K}$ contradicting the definition of K.

Let K be as in the previous lemma. Then we write

$$\mathfrak{T}_K = \{\mathscr{A}_J \in \mathfrak{T}_m \mid A(J) \cap A(I \backslash K) \neq \varnothing\}.$$

LEMMA 6. *Let* $L = (I \backslash K) \cup \bigcup_{\mathscr{A}_J \in \mathfrak{T}_K} J$. *Then* $\mathscr{A}_L$ *is the smallest open subfamily containing* $\mathscr{A}_{I \backslash K}$.

Proof. Certainly, $\mathscr{A}_L \in \mathfrak{T}(\mathscr{A})$. Let $\mathscr{A}_{L'} \in \mathfrak{T}(\mathscr{A})$, with $I \backslash K \subseteq L'$. The proof of Lemma 5 has shown that $I \backslash K \subset L'$. Furthermore, we have $L' \cap J \neq \varnothing$ for all J with $\mathscr{A}_J \in \mathfrak{T}_K$, since $A(L')$ contains at least one element of $A(J)$ which must not be used to represent $\mathscr{A}_{L'}$. But then, by Lemma 1, we have $J \subseteq L'$ and hence $L \subseteq L'$.

THEOREM 8. *The following statements are equivalent:*

(a) $\mathfrak{T}(\mathscr{A})$ *is regular,*

(b) $\mathscr{A}$ *is the union of minimal open subfamilies,*

(c) $\mathfrak{T}(\mathscr{A}) = \mathfrak{C}(\mathscr{A})$.

Proof. The theorem is true if $\mathfrak{T}(\mathscr{A}) = \{\varnothing, \mathscr{A}\}$, so assume there is at least one critical subfamily not equal to $\mathscr{A}$.

(a) $\Rightarrow$ (b). The regularity (every set A_i and every closed subfamily $\mathscr{A}_J$ not containing A_i are contained in disjoint open subfamilies) implies that $C(A) = 0$ for all $A \in \mathscr{A}$. For, if $A \notin \mathscr{A}_J$ then $A \in \mathscr{A} \backslash \mathscr{A}_J \in \mathfrak{T}(\mathscr{A})$; if $A \in \mathscr{A}_J$ then there is an open subfamily $\mathscr{A}_{J'}$ with $A \in \mathscr{A}_J \subseteq \mathscr{A}_{J'} \subset \mathscr{A}$. It follows that $C(x) = 0$ for all $x \in E$.

Now assume $K = \bigcup_{\mathscr{A}_J \in \mathfrak{T}_m} J \subset I$. By Lemma 6, the smallest open subfamily containing $\mathscr{A}_{I \backslash K}$ contains at least one $\mathscr{A}_J \in \mathfrak{T}_K$. Take an $A_0 \in \mathscr{A}_J$, then A_0 and $\mathscr{A}_{I \backslash K} \in \mathfrak{C}(\mathscr{A})$ violate the definition of regularity.

(b) $\Rightarrow$ (c). Since $\mathscr{A}$ is the union of minimal open subfamilies we derive from Lemma 1 that every $\mathscr{A}_J \in \mathfrak{T}(\mathscr{A})\backslash\{\varnothing\}$ is also

$$\mathscr{A}_J = \bigcup_{\substack{\mathscr{A}_{J'} \in \mathfrak{T}_m \\ J' \subseteq J}} \mathscr{A}_{J'} .$$

Therefore

$$\mathscr{A}\backslash\mathscr{A}_J = \bigcup_{\substack{\mathscr{A}_{J'} \in \mathfrak{T}_m \\ J' \nsubseteq J}} \mathscr{A}_{J'} \in \mathfrak{T}(\mathscr{A}),$$

which implies $\mathscr{A}_J \in \mathfrak{C}(\mathscr{A})$, hence $\mathfrak{T}(\mathscr{A}) \subseteq \mathfrak{C}(\mathscr{A})$.

(c) $\Rightarrow$ (a). Trivial.

Applications of these results to counting finite topologies can be found in [4].

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