Cardinal invariants of monotonically normal spaces

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Received 6 May 1996

Abstract

The basic cardinal invariants of monotonically normal spaces are determined. The gap between cellularity and density is investigated via calibres. © 1997 Elsevier Science B.V.

Keywords: Cardinal invariants; Monotonically normal; Calibres; Trees

AMS classification: Primary 54E35; 54D65, Secondary 54F05

1. Introduction and notation

In this note we will determine the relationships between the basic cardinal invariants in monotonically normal spaces, and probe the gap between cellularity and density with the aid of calibres.

Recall that a space is said to be monotonically normal if there is an operator $V(\cdot, \cdot)$ which assigns to each point $x$ and each open neighbourhood $U$ of $x$ an open set $V(x, U)$ containing $x$ which satisfies

1. $V(x, U) \subseteq V(x, U')$ if $U \subseteq U'$,
2. $V(x, X\setminus\{y\}) \cap V(y, X\setminus\{y\}) = \emptyset$ if $x \neq y$.

Every metrisable space, and every linearly ordered space is monotonically normal. Arbitrary subspaces, and closed images, of monotonically normal spaces are again monotonically normal. Thus, many of the basic spaces of topology are monotonically normal. Indeed, it can be argued that whenever a space can be explicitly and constructively shown to be normal, then it is probably monotonically normal. For further details about monotonically normal spaces see Gruenhage's survey articles [6,7].

The study of cardinal invariants of topological spaces has been very fruitful, yielding many wonderful examples and some striking theorems. Naturally, considering cardinal invariants of restricted classes of spaces should lead to fewer counter-examples, and more
theorems. However, normality has very little affect, and even in the class of hereditarily collectionwise normal spaces only one new relation (cellularity coincides with hereditary cellularity) arises. Monotone normality, on the other hand has a significant impact on cardinal invariants, as we will subsequently demonstrate.

All the cardinal invariants considered in this paper are mentioned in Hodel's excellent survey article [9], and the interested reader is referred there for further information. The invariants we will principally be concerned are the following ones. Let $X$ be a topological space. The density of $X$, denoted $d(X)$, is the infima of cardinalities of dense subsets. Cellularity, denoted $c(X)$, is the suprema of sizes of pairwise disjoint families of open sets. Lindelöf degree, written $L(X)$, is the infima of all cardinals $\kappa$ so that every open cover of $X$ has a subcover of size no more than $\kappa$. The diagonal degree of $X$, $\Delta(X)$, is the infima of all $\kappa$ for which there are open covers $\mathcal{V}_\alpha$ of $X$, where $\alpha \in \kappa$, such that for each point $x$ of $X$, $\bigcap_{\alpha \in \kappa} \text{st}(x, \mathcal{V}_\alpha) = \{x\}$. Note that $X$ has a $G_\delta$ diagonal if and only $\Delta(X) \leq \aleph_0$. Fix, for the moment, a point $x$ in $X$. Write $\psi(x, X)$ for $\min\{|\mathcal{V}|: \text{each } \mathcal{V} \text{ is a collection of open sets, and } \bigcap \mathcal{V} = \{x\}\}$, and $\tau(x, X)$ for the minimal cardinal $\kappa$ so that, whenever $x \in \overline{Y}$, for some $Y \subseteq X$, then there is a subset $A$ of $Y$, with $|A| \leq \kappa$, and $x \in A$. Now we may define the pseudocharacter of $X$, denoted $\pi(X)$, as $\sup\{\psi(x, X): x \in X\}$, and, similarly, $\tau(X) = \sup\{\tau(x, X): x \in X\}$.

In addition to these key cardinal invariants, their hereditary versions will also be studied. If $f$ is any cardinal function then the hereditary version of $f$, denoted $hf$, is defined by $hf(X) = \sup\{f(Y): Y \subseteq X\}$. It will also be convenient to define $I(X)$, where $X$ is an arbitrary topological space, to be the cardinality of the set of isolated points in $X$. A few other cardinal invariants are mentioned in passing. These are, netweight, denoted $nw$, $\pi$-weight and $\pi$-character, denoted $\pi w$ and $\pi X$, respectively. For their definitions see Hodel's paper [9]. Our results concerning cardinal invariants may be summarised as follows.

**Theorem 1.** Let $X$ be monotonically normal.

1. $d(X) = hd(X)$.
2. $d(X) \leq e(X) \cdot \Delta(X) \cdot I(X)$.
3. $\tau(X) \leq c(X)$.

Other results due to Ostaszewski [13], Moody [12], and Williams and Zhou [15], complete our understanding of the cardinal invariants of monotonically normal spaces.

**Theorem A.** Let $X$ be monotonically normal. Then

(1) $c(X) = hc(X) = hL(X)$;

(2) $d(X) \leq c(X)^+$. 

Williams and Zhou also showed that the $\kappa^+$-Souslin hypothesis is equivalent to:

$X$ is monotonically normal, $c(X) = \kappa$ implies $d(X) = \kappa$.

We give two proofs of the result equating density and hereditary density in monotonically normal spaces. The first is an unusual ‘forcing and absoluteness’ argument,
exploiting the simple logical form of the definition of monotone normality. The second
relies on some additional results about calibres of monotonically normal spaces. Calibres
are properties lying between cellularity and density.

Let $\kappa, \lambda$ and $\mu$ be a nonincreasing sequence of cardinals. A topological space $X$ is
said to have calibre $(\kappa, \lambda, \mu)$ if and only if for every collection $\mathcal{U}$ of nonempty open
sets, which has size $\kappa$, there is a subcollection $\mathcal{V}$ of $\mathcal{U}$ of cardinality $\lambda$, such that for any
subcollection $\mathcal{W}$ of $\mathcal{V}$, with $|\mathcal{W}| = \mu$, we have $\bigcap \mathcal{W} \neq \emptyset$.

The relationships between the various calibres, density and cellularity; and the pro-
ductivity of calibres can be summarised as below. For further information about calibres
see [1].

**Theorem B.**

(i) If $\kappa' \geq \kappa$, $\lambda' \leq \lambda$ and $\mu \leq \mu'$, then a space with calibre $(\kappa, \lambda, \mu)$ has calibre
$(\kappa', \lambda', \mu')$.

(ii) $X$ has $c(X) \leq \kappa$ if and only if $X$ has calibre $(\kappa^+, 2, 2)$.

(iii) If $X$ has $d(X) \leq \kappa$ then $X$ has calibre $(\kappa^+, \kappa^+, \kappa^+)$.

(iv) If regular $\kappa \geq \lambda \geq \mu$, $X$ has calibre $(\kappa, \kappa, \mu)$ and $Y$ has calibre $(\kappa, \lambda, \mu)$, then
$X \times Y$ has calibre $(\kappa, \lambda, \mu)$.

We can now state our principal results concerning calibres of monotonically normal
spaces.

**Theorem 2.** Let $X$ be monotonically normal. Then

1. $X$ has calibre $(\kappa^+, \kappa^+, 2)$ if and only if $d(X) \leq \kappa$;

2. $X$ has $c(X) \leq \kappa$ if and only if $X$ has calibre $(\kappa^+, \kappa, \kappa)$;

3. If $X$ has any calibre then it has the same property hereditarily.

The first two claims demonstrate that the calibres of a monotonically normal space fall
into two groups: those equivalent to cellularity, and those equivalent to density. From the
Williams and Zhou result, cited above, these two groups may, consistently, either coincide
or be disjoint. Other results concerning calibres (and compact-calibres) of monotonically
normal spaces may be found in [10]. McIntyre's results were obtained independently,
and are essentially disjoint from those presented above.

Lying at the heart of many of our results lies a construction, due to Williams and
Zhou, of a tree of open subsets of each monotonically normal space. The two theorems
of Williams and Zhou already mentioned are immediate consequences of the existence
of this tree. Our first results require merely the existence of Williams–Zhou trees in mono-
tonically normal spaces. However, in later results it is necessary to vary the construction
to take into account additional available information.

**Lemma C** (Williams and Zhou). Let $X$ be monotonically normal, with monotone nor-
mality operator $V(\cdot, \cdot)$. Call a tree $T$ of open subsets of $X$, ordered by reverse inclusion,
a Williams–Zhou tree for $X$ provided it satisfies:
(1) $X$ is the least element of $T$,
(2) for all $T \in T$ either $|T| \leq 1$ or $T$ has at least two successors,
(3) for all $T \in T$, $T \neq X$, there is an open $G_T$ in the predecessors of $T$, and $x_T \in G_T$ such that $T = \mathcal{V}(x_T, G_T)$,
(4) if $B$ is a branch of $T$, then $\bigcap B = \emptyset$,
(5) for each $T \in T$, the set $\{S \in T : S, T have the same predecessors\}$ is maximal among open collections which have pairwise disjoint closures and the closure of each element is contained in the closure of each predecessor of $T$.

For every space $X$ with monotone normality operator $\mathcal{V}(\cdot, \cdot)$, there is a Williams-Zhou tree, and $\{x_T\}_{T \in T}$ is dense in $X$. Note that incomparable members of the tree are disjoint.

2. An unusual proof

The key to our first proof equating density and hereditary density in monotonically normal spaces lies in the observation that the logical sentence in the language of set theory defining monotone normality is $\Sigma_1$. Thus, if $\mathcal{V}(\cdot, \cdot)$ is a monotone normality operator for a space $(X, T)$ in a model of set theory $\mathcal{V}$, then, in any extension $\mathcal{V}'$ of $\mathcal{V}$, $\mathcal{T}$ is a base for a topology on $X$, and $\mathcal{V}(\cdot, \cdot)$ can be extended to a monotone normality operator for this enlarged topology. Additionally, it is trivial to see that density is preserved in any extension of the set theoretic universe. In contrast, a normal space (even a hereditarily collectionwise normal space) may cease to be normal in a forcing extension.

**Theorem 3.** \( ZFC \vdash \text{"In monotonically normal spaces density equals hereditary density"} \)

**Proof.** Let $\mathcal{V}$ be any model for set theory.

Working in $\mathcal{V}$. Suppose $(X, \tau)$ is a monotonically normal space, $d(X) = \kappa$ and $Y \subseteq X$. Construct in the subspace $Y$ of $X$ a tree $T$ of $Y$-open sets and pick points $\{y_T\}_{T \in \mathcal{T}}$ as in Lemma C. It is sufficient to show that $|T| \leq \kappa$.

Let $\mathcal{P}$ be $T^{-1}$ ($T$ turned upside-down) and $G$ be a $\mathcal{P}$-generic filter. By Theorem A, $c(Y) \leq \kappa$, so $\mathcal{P}$ has the $\kappa^+$-chain condition. Force with $\mathcal{P}$.

In $\mathcal{V}[G]$. Here $\bigcup G$ is a chain meeting each level of $\mathcal{P}$. As pointed out above, $X$ is still monotonically normal and $d(X) \leq \kappa$. Hence, again by Theorem A, $c(Y) \leq \kappa$ and chains of $\mathcal{P}$ are of cardinality $\leq \kappa$. In particular $\bigcup G$ has size $\leq \kappa$, so $\mathcal{P}$ has size $\leq \kappa$.

Back in $\mathcal{V}$. Since $\mathcal{P}$ has the $\kappa^+$-chain condition and $\mathcal{P}$ has cardinality $\leq \kappa$ in $\mathcal{V}[G]$, we must have that $|T| = |\mathcal{P}| \leq \kappa$. \( \Box \)

3. Strong calibres

The author would vigorously defend both the validity and merit of the ‘forcing and absoluteness’ proof relating density and hereditary density is monotonically normal
spaces—it says something important about the set theoretic status of monotone normality as opposed to the status of, say, normality or paracompactness. Nonetheless, we now present an alternative proof of our result which capitalises on the connections between the strong calibres and density, and the natural manner in which monotone normality makes these calibres hereditary.

**Theorem 4.** Let \( X \) be monotonically normal. If \( X \) has caliber \((\kappa^+, \kappa^+, \mu)\), where \( \mu \geq 2 \), then \( d(X) \leq \kappa \).

**Proof.** It suffices to assume that \( \mu \) equals 2. So, let \( X \) be monotonically normal with caliber \((\kappa^+, \kappa^+, 2)\). Construct a tree \( T \) as in Lemma C. It is sufficient to show \(|T| \leq \kappa\).

Since \( X \) has caliber \((\kappa^+, \kappa^+, 2)\) the cellularity of \( X \) is \( \leq \kappa \), and chains of \( T \) are of size \( \leq \kappa \). Suppose, for a contradiction, \(|T| \geq \kappa^+\). Then as \( X \) has caliber \((\kappa^+, \kappa^+, 2)\) we can find \( T' \subset T \) of size \( \geq \kappa^+ \) and \( T, T' \in T \Rightarrow T \cap T' \neq \emptyset \). But this means \( T' \) is a chain of cardinality \( \geq \kappa^+ \).

To give our alternative proof of Theorem 3 we recall the definition of a \( K_1 \)-space (due to Van Douwen [2]). (Here, and for the remainder of this section, we write \( \tau Z \) for the topology on a space \( Z \).) A space \( X \) is a \( K_1 \)-space if and only if for every subset \( Y \) of \( X \) there is a \( k : \tau Y \to \tau X \) such that

\[
(1) \quad k(U) \cap Y = U,
(2) \quad k(U_0) \subseteq k(U_1) \text{ if } U_0 \subseteq U_1,
(3) \quad U_0 \cap U_1 \neq \emptyset \text{ if } k(U_0) \cap k(U_1) \neq \emptyset.
\]

The following lemma is immediate from the definitions of a \( K_1 \)-space and caliber \((\kappa, \lambda, 2)\).

**Lemma 5.** If \( X \) is a \( K_1 \)-space and has caliber \((\kappa, \lambda, 2)\) then \( X \) has caliber \((\kappa, \lambda, 2)\) hereditarily.

The relevance of this to the current situation lies in the fact that every monotonically normal space is a \( K_1 \)-space [2]. Indeed if we say that a space is ‘monotonically \( K_1 \)’ provided for every \( Y \subseteq X \) there is a \( k_Y : \tau Y \to \tau X \) such that \( k_Y \) satisfies (1) to (3) above and

\[
(4) \quad \text{if } U_0 \subseteq U_1, \ Y_0 \cap Y_0 \supseteq Y_1 \cap U_1 \text{ and } U_i \in \tau Y_i, \text{ then } k_{Y_0}(U_0) \subseteq k_{Y_1}(U_1)
\]

then we can characterise monotone normality as follows.

**Lemma 6.** A space is monotonically normal if and only if it is monotonically \( K_1 \).

**Proof (Sketch).** Suppose first that \( X \) has monotone normality operator \( V(\cdot, \cdot) \), and take any \( Y \subseteq X \). Define \( k_Y : \tau Y \to \tau X \) by

\[
k_Y(U) = \bigcup \{ V(x, X \setminus \{Y \setminus U\}) : x \in U \}.
\]

It is routine to check that \( k_Y \) satisfies (1)–(4) above.
Conversely, suppose for every \( Y \subseteq X \) there is a \( k_Y : \tau Y \to \tau X \) satisfying (1)–(4) above. Define for each point \( x \) in an open \( U \), \( V(x, U) = k(x) \cup (X \setminus U)\{\{x\}\} \). Then it is clear that \( V(x, U) \) is an open neighbourhood of \( x \), and it is straightforward to check that \( V(\cdot, \cdot) \) satisfies (1) and (2) in the definition of monotone normality. \( \square \)

**Alternative proof of Theorem 3.** Let \( X \) be monotonically normal with \( d(X) = \kappa \). Then \( X \) has caliber \((\kappa^+, \kappa^+, 2)\) and is a \( K_1 \)-space by the preceding lemma. Hence \( X \) has caliber \((\kappa^+, \kappa^+, 2)\) hereditarily. Therefore by Theorem 4, \( hd(X) \leq \kappa \). \( \square \)

4. Weak calibres

We now turn our attention to the weaker calibres. It will be seen that all the calibres not taken care of by Theorem 4 are all equivalent to cellularity. This is in fact the case for any space such that the cellularity and hereditary cellularity are equal. The following lemma is probably folklore, but a proof is included for completeness.

**Lemma 7.** If \( X \) has \( hc(X) \leq \kappa \) and \( \mathcal{U} \) is a collection of open sets which is point-(\( \leq \kappa \)), then \( \mathcal{U} \) has a subcover of size \( \leq \kappa \).

**Proof.** Suppose \( X \) has \( hc(X) \leq \kappa \), \( \mathcal{U} \) is a point-(\( \leq \kappa \)) collection of open sets. We may assume without loss of generality that \( \mathcal{U} \) covers \( X \).

By transfinite induction find a minimal ordinal \( \alpha \), a discrete set of points \( \{x_\beta\}_{\beta<\alpha} \) and \( \{U_\beta\}_{\beta<\alpha} \) a subcollection of \( \mathcal{U} \) such that

\[
X = \bigcup_{\beta<\alpha} U_\beta.
\]

Since the set of points is discrete, \( \alpha \in \kappa^+ \).

For \( \beta < \alpha \) let \( \mathcal{V}_\beta = \{U \in \mathcal{U} : x_\beta \notin U\} \), \( \mathcal{V}_\alpha = \{U_\beta\}_{\beta<\alpha} \), and \( \mathcal{V} = \bigcup_{\beta<\alpha} \mathcal{V}_\beta \). Clearly \( \mathcal{V} \) is a subcollection of \( \mathcal{U} \) covering \( X \) and from \( \mathcal{U} \) being point-(\( \leq \kappa \)) we see that each \( \mathcal{V}_\beta \) has size \( \leq \kappa \). Thus \( \mathcal{V} \) has size \( \leq \kappa \). \( \square \)

**Theorem 8.** A space \( X \) has \( hc(X) \leq \kappa \) if and only if \( X \) has caliber \((\kappa^+, \kappa, \kappa)\) hereditarily.

**Proof.** It is sufficient to show that if \( X \) has \( hc(X) \leq \kappa \) then every point-(\( < \kappa \)) collection, \( \mathcal{U} \) say, of nonempty open sets is of size \( \leq \kappa \).

Define \( \mathcal{R}_\alpha = \mathcal{U} \) and inductively \( \mathcal{L}_\alpha \) to be a subcollection from \( \mathcal{R}_\alpha \) of size \( \leq \kappa \) covering \( \bigcup \mathcal{R}_\alpha \) (existence of \( \mathcal{L}_\alpha \) is assured by the preceding lemma) and

\[
\mathcal{R}_\alpha = \mathcal{U} \setminus \bigcup_{\beta<\alpha} \mathcal{L}_\beta.
\]

For some minimal ordinal \( \alpha \), \( \bigcup \mathcal{L}_\alpha = \emptyset \).

Note that for \( \gamma < \beta < \alpha \) we have:

(1) \( |\mathcal{L}_\beta| \leq \kappa \),
Now suppose, for a contradiction, that \(|U| \geq \kappa^+\). From (1) \(\alpha \geq \kappa^+\), so certainly we can pick \(x \in \bigcup L_{k+1} \neq \emptyset\). But from (2) and (3) \(x\) is in at least \(\kappa\) many members of \(U\), and this contradicts \(U\) point-(\(< \kappa\)).

**Corollary 9.** Let \(X\) be monotonically normal. If \(X\) has \(c(X) \leq \kappa\), and \(\lambda \leq \kappa\), then \(X\) has caliber \((\kappa^+, \lambda, \mu)\).

### 5. Tightness vs. pseudocharacter

In this section we consider the interaction in monotonically normal spaces of tightness and pseudocharacter. From Ostaszewski’s Theorem A, every monotonically normal space with countable cellularity is hereditarily Lindelöf, and so has countable pseudocharacter. We now show that the analogous result holds for tightness.

**Theorem 10.** Let \(X\) be a monotonically normal space. Then \(\tau(X) \leq c(X)\).

**Proof.** Since cellularity and hereditary cellularity coincide in monotonically normal spaces it is sufficient to show:

- if \(X\) is monotonically normal and \(x \in X\) nonisolated
- then there is an \(A \subseteq X \setminus \{x\}\) with \(|A| \leq c(X)\) and \(x \in \overline{A}\).

So take any \(X\) and \(x\) as above. Define \(U_0 = X \setminus \{x\}\) and pick \(a_0 \in U_0\). By transfinite induction, given points \(\{a_{\beta}\}_{\beta < \alpha}\) and open sets \(\{U_{\beta}\}_{\beta < \alpha}\) define

\[
U_\alpha = X \setminus \bigcup_{\beta < \alpha} V(a_\beta, U_\beta) \cup \{x\}
\]

and if \(U_\alpha \neq \emptyset\), pick \(a_\alpha \in U_\alpha\). Let \(\alpha\) be minimal such that \(U_\alpha = \emptyset\).

Consider \(\{V(a_\beta, U_\beta)\}_{\beta < \alpha}\) and \(A\) which we define to be \(\{a_\beta\}_{\beta < \alpha}\). By construction the \(V(a_\beta, U_\beta)\)'s are pairwise disjoint, so \(\alpha < c(X)^+\). Further, as \(x\) is nonisolated,

\[
\bigcup_{\beta < \alpha} V(a_\beta, U_\beta) = X.
\]

Now \(A \subseteq X \setminus \{x\}\), \(|A| \leq c(X)\), and, given any open \(U\) containing \(x\),

\[
V(x, U) \cap V(a_\beta, U_\beta) \neq \emptyset
\]

for some \(\beta < \alpha\). Hence \(a_\beta \in U\), and \(x \in \overline{A}\). □

Note that the one point compactification of an uncountable discrete space is monotonically normal, compact, has countable tightness but uncountable pseudocharacter. And while every compact space with countable pseudocharacter has countable tightness, the following example shows that we can not weaken ‘compact’ here, to ‘Lindelöf’.
Example 11. Let $X$ be $\mathbb{R} \times \{0, 1\}$. Isolate points of $\mathbb{R} \times \{1\}$. Let

$$B(n, x) = (x - 1/n, x + 1/n),$$

and give each point $(x, 0)$ basic neighbourhoods of the form

$$(B(n, x) \times \{0\}) \cup \left(\left[B(n, x) \setminus \{x\} \cup A\right] \times \{1\}\right),$$

where $n \geq 1$ and $A$ is a countable subset of $\mathbb{R}$.

Then $X$ is monotonically normal, Lindelöf, and has countable pseudocharacter, but $\tau(X) = \aleph_1$.

Question 1. For $X$ monotonically normal, do we have $|X| \leq 2^{L(X) \cdot \psi(X)}$?

Question 2. For $X$ monotonically normal, do we have $\tau(X) \leq (L(X) \cdot \psi(X))^+$?

Recently Williams and Zhou introduced the class of extremely normal spaces [15]. A space $X$ is extremely normal (EN) if it has a monotone normality operator $V(\cdot, \cdot)$ satisfying:

\begin{align*}
\text{(EN)} \quad & \text{if } x \neq x' \text{ and } V(x, U) \cap V(x', U') \neq \emptyset \\
& \text{then either } V(x, U) \subseteq U' \text{ or } V(x', U') \subseteq U.
\end{align*}

For these spaces we can give positive answers to both questions.

Theorem 12. Let $X$ be EN. Then $\tau(X) \leq L(X) \cdot \psi(X)$. Consequently,

$$|X| \leq 2^{L(X) \cdot \psi(X)}.$$

Proof. Let $\kappa = L(X) \cdot \psi(X)$. It is sufficient to show:

- if $x$ in $X$ is nonisolated, and $Y$ is a dense subspace of $X$,
- then there is an $A \in [Y]^{< \kappa}$ so that $x \in A$.

Let $\mathcal{U}$ be a pseudo-base at $x$, with $|\mathcal{U}| \leq \kappa$, and, for each $U \in \mathcal{U}$, let $U^*$ be an open neighbourhood of $x$ so that $\overline{U^*} \subseteq U$. For each $U \in \mathcal{U}$, $X \setminus U$ is closed, hence $I(X \setminus U) \leq \kappa$. Thus the open cover $\{V(y, X \setminus \overline{U^*}) : y \in X \setminus U\}$ has a subcover of size $\leq \kappa$, $\{V(y_{U, \alpha}, X \setminus \overline{U^*}) : \alpha < \kappa\}$ say. As $Y$ is dense in $X$ we may pick, for each $U \in \mathcal{U}$ and $\alpha < \kappa$, a point $z_{U, \alpha}$ in $V(y_{U, \alpha}, X \setminus \overline{U^*}) \cap Y$.

Let $A = \{z_{U, \alpha} : U \in \mathcal{U} \text{ and } \alpha < \kappa\}$. Note that $|A| \leq \kappa$ and

$$X \setminus \{x\} = \bigcup \{V(y_{U, \alpha}, X \setminus \overline{U^*}) : U \in \mathcal{U} \text{ and } \alpha < \kappa\}.$$

Take any open neighbourhood $T$ of $x$. As $x$ is nonisolated,

$$V(x, T) \cap V(y_{U, \alpha}, X \setminus \overline{U^*}) \neq \emptyset$$

for some $U \in \mathcal{U}$ and $\alpha < \kappa$. From $x \notin X \setminus \overline{U^*}$ and $X$ EN, it follows that

$$z_{U, \alpha} \in V(y_{U, \alpha}, X \setminus \overline{U^*}) \subseteq T,$$

so $T \cap A \neq \emptyset$, and $x \in \overline{A}$, as required.
It is well known that for any space $X$ we have $|X| \leq 2^{\mu(X) \cdot \psi(X) \cdot \tau(X)}$ (see [9], for example), so the second part of the theorem follows immediately from the first. 

In [15], Williams and Zhou show that every EN space is hereditarily paracompact. It is not difficult to check that in paracompact spaces, Lindelöf degree and extent coincide. Thus we may replace Lindelöf degree in the statement of Theorem 12 by extent.

6. Restrictions on density

In this section we give a bound on density by cardinal invariants which individually place no such restriction. It will be seen that for every monotonically normal space, $X$ say, a Williams–Zhou tree (as in Lemma C) can be constructed whose height is bounded by the diagonal degree, $\Delta(X)$, alone. But to bound the width of a Williams–Zhou tree of a monotonically normal space requires the maximum of all three of diagonal degree, extent and the size of the set of isolated points. To see that no two together bound density consider: the Michael–Bernstein line, a disjoint sum of uncountably many copies of the real line, and the double arrow space. Recall that $e(X)$, the extent of a space $X$, is the infima of all cardinalities of closed, discrete subsets of $X$.

**Theorem 13.** Let $X$ be monotonically normal. Then $d(X) \leq \Delta(X) \cdot e(X) \cdot I(X)$.

**Proof.** Fix $V(\cdot, \cdot)$ a monotone normality operator for our space $X$. Set $\kappa$ equal to the max of $\Delta(X), e(X)$ and $|\{x \in X : x \text{ isolated}\}|$. Since $\Delta(X) \leq \kappa$, we may select a collection $\{V_\alpha\}_{\alpha \in \kappa}$ of open covers of $X$ so that, for each $x$ in $X$, $\bigcap_{\alpha \in \kappa} \text{st}(x, V_\alpha) = \{x\}$. For each $x$ in $X$, and $\alpha$ in $\kappa$, fix $V_{\alpha, x} \in V_\alpha$ containing $x$.

Repeat the standard construction of the Williams–Zhou tree, as in [15], but at level $\alpha$, where $\alpha < \kappa$, use the monotone normality operator $V^\alpha(\cdot, \cdot)$ defined by $V^\alpha(x, U) = V(x, U \cap V_{\alpha, x})$ (otherwise, use $V(\cdot, \cdot)$). Denote this tree $T$.

**Claim.** If $x$ and $y$ are distinct points of $X$, then there is an $\alpha$ in $\kappa$ such that, if $y$ is in $T$, $T$ in $T_\alpha$, then $x \notin T$. Hence, all elements of $T_\kappa$ consist of one point, and $T_{\kappa+1}$ is empty.

**Proof.** Pick $\alpha$ so that $x \notin \text{st}(y, V_\alpha)$. Suppose $y$ in $T$, $T$ in $T_\alpha$. Then $T \subseteq V_{\alpha, z}$, for some $z$ in $X$. In particular, $y$ in $V_{\alpha, z}$. If $x$ in $T$, then $x$ is also in $V_{\alpha, z}$. But now $x \in V_{\alpha, z} \subseteq \text{st}(y, V_\alpha)$, contradicting the choice of $\alpha$. 

Thus $T$ has only $\kappa$ many levels. It now suffices to show that, for each $\alpha \in \kappa$, level $T_\alpha$ has size less than or equal to $\kappa$. To this end, fix $\alpha$ and write $T_\alpha = \{T_\lambda\}_{\lambda \in A}$ where $T_\lambda \subseteq V(z_\lambda, U_\lambda)$. By the definition of $\kappa$, at most $\kappa$-many $z_\lambda$'s are isolated. So we may focus on those $z_\lambda$'s which are not isolated.

Let $R = \{z_\lambda\}_{\lambda \in A}$. For each $\lambda$ in $A$, pick $s_\lambda$ in $V(z_\lambda, T_\lambda)$ distinct from $z_\lambda$. Let $S$ be $\{s_\lambda\}_{\lambda \in A}$, and let $T = \bigcup T_\alpha$. Since $e(X) \leq \kappa$, $S$, and $T_\alpha$, will have size no more than $\kappa$, provided $S$ is the union of $\kappa$ many closed discrete subsets. This follows from the final claim.
Claim. For each $\beta \in \kappa$, define
\[ W_\beta = \bigcup \{ V(x, V_{\beta,x}) : x \in X \setminus T \} \cup \bigcup \{ T_\lambda \setminus \{ s_\lambda \} : \lambda \in \Lambda \}. \]
Then $X \setminus S = \bigcap_{\beta \in \kappa} W_\beta$.

Proof. Evidently, $X \setminus S \subseteq \bigcap_{\beta \in \kappa} W_\beta$. Fix $\lambda \in \Lambda$. Then $s_\lambda \in V(z_\lambda, T_\lambda)$ but $s_\lambda \not= z_\lambda$. Pick $\beta$ so that $z_\lambda \not\in \text{st}(s_\lambda, V_\beta)$. Suppose $s_\lambda \in W_\beta$; so $s_\lambda \in V(x, V_{\beta,x})$ for some $x \in X \setminus T$. Since $x \not\in T_\lambda$, we must have $z_\lambda \in V_{\beta,x}$. But this means $z_\lambda \in V_{\beta,x} \subseteq \text{st}(s_\lambda, V_\beta)$—a contradiction. \(\square\)

7. Some consequences

We conclude with some applications of the results established above. These include applications to topological groups and compact spaces.

Corollary 14 (Souslin line type spaces). Call a nonseparable monotonically normal space, with the countable chain condition, a Souslin line type space. There are no Souslin line type spaces with a $G_\delta$ diagonal.

Proof. Since every monotonically normal space with countable cellularity has the countable chain condition hereditarily, and so certainly has countable extent, by Theorem 13, every such space with a $G_\delta$ diagonal is separable. The significance of this is as follows. Every linearly ordered space with a $G_\delta$ diagonal is metrisable, so no Souslin line has a $G_\delta$ diagonal. But subspaces of linearly ordered spaces may have a $G_\delta$ diagonal, and yet not be metrisable—for example, the Sorgenfrey line. Thus the possibility of a dense subspace of a Souslin line possessing a $G_\delta$ diagonal has, up to now, remained open. \(\square\)

Corollary 15 (Function spaces). If $C_p(X)$ contains a dense monotonically normal subspace then $C_p(X)$ is hereditarily separable. (This is a useful first step in showing that under these hypotheses $C_p(X)$ is in fact metrisable, see [4].)

Proof. The space $C_p(X)$ is dense in $\mathbb{R}^X$, which has caliber $(\omega_1, \omega_1, 2)$. Hence the dense monotonically normal subspace of $C_p(X)$ also has this caliber. By monotone normality and Theorems 3, 4, the dense subspace is hereditarily separable. \(\square\)

Corollary 16 (Topological groups). Let $G$ be a monotonically normal topological group. Then $e(G), \psi(G) = nw(G)$. (See [5] for details.)

Corollary 17 (Acyclic monotone normality). There is no way to distinguish acyclic monotone normality and monotone normality via calibres. In particular, there are no monotonically normal $\lambda$-Gower spaces. (See Moody [12] for the definitions of acyclic monotone normality and $\lambda$-Gower spaces.)
Proof. Just as monotonically normal spaces 'naturally' make caliber \((\kappa, 2, 2)\) hereditary, so acyclic monotonically normal spaces 'naturally' make caliber \((\kappa, n, n)\) hereditary for each integer \(n \geq 2\). Thus, a natural approach to showing that not every monotonically normal space is acyclic monotonically normal is to find a monotonically normal space with caliber \((\kappa, 2, 2)\) which does not hereditarily have caliber \((\kappa, 3, 3)\). However, Theorem 4 shows that such a strategy will always fail. \(\square\)

It should be noted that Rudin \([14]\) has recently constructed an example of a separable monotonically normal space which is not acyclically monotonically normal.

**Corollary 18** (Compactifications 1). Let \(X\) monotonically normal. Then

\[ d(\beta X) = d(X). \]

**Proof.** Van Douwen showed in [3] that \(d(\beta X) \leq \kappa\) if and only if \(X\) has a base which is the union of \(\kappa\) many subcollections each with the finite intersection property. He remarked that it is possible to have \(d(\beta X) < d(X)\), and asked for a large class of spaces where equality holds. We present the class of all monotonically normal spaces as a suitable candidate.

To see this let \(X\) be monotonically normal, and suppose \(d(\beta X) \leq \kappa\). We need to show that \(d(X) \leq \kappa\). But a space with the base property mentioned above can easily be seen to have caliber \((\kappa^+, \kappa^+, 2)\). The claim now follows from Theorem 4. \(\square\)

**Corollary 19** (Compactifications 2). For those spaces \(X\) with a monotonically normal compactification: \(\pi w(X) = d(X) = \text{hd}(X) = h\pi w(X)\).

**Proof.** Suppose a space \(X\) has monotonically normal compactification \(\gamma X\). Then, as \(X\) is dense in \(\gamma X\), from Theorem 3 we see that \(d(X) = \text{hd}(\gamma X)\). For any compact space \(K\), \(\text{hd}(K) = h\pi(K)\). The claim now easily follows. \(\square\)

However, unlike density and in contrast to the preceding result, \(\pi\)-weight and hereditary \(\pi\)-weight may differ in monotonically normal spaces.

**Example.** The subspace \(H\) consisting of eventually-zero rational sequences taken from a countable box product of the rationals is a countable, stratifiable but nonmetrisable, topological group (see Heath [8]). The \(\pi\)-character of \(H\) must be uncountable because \(\pi\)-character and character are equal in topological groups, and first countable topological groups are metrisable.

Let \(X\) be the topological space with underlying set \(X \times (\omega + 1)\), and topology obtained by first taking the product topology then isolating points of \(H \times \omega\). \(X\) is countable and stratifiable. The isolated points of \(X\) form a countable \(\pi\)-base but the hereditary \(\pi\)-character of \(X\) is uncountable.
Acknowledgements

My thanks to Phil Moody and Robin Knight for many useful discussions. And also to Dave McIntyre whose doctoral thesis on calibres in linearly ordered spaces and first countable spaces [11] prompted my own investigations.

References