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A characterization of skew Hadamard matrices and doubly regular tournaments ☆

Hiroshi Nozaki [a], Sho Suda [b,*]

[a] Department of Mathematics, Aichi University of Education, 1 Hirosawa, Igaya-cho, Kariya, Aichi 448-8542, Japan
[b] Department of Mathematics, International Christian University, 10-2, Osawa 3-chome, Mitaka, Tokyo 181-8585, Japan

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ABSTRACT

We give a new characterization of skew Hadamard matrices of order n in terms of spectral data for tournaments of order $n-2$.

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1. Introduction

An $n \times n$ matrix H with entries ± 1 is a *Hadamard matrix* of order n if $HH^T = nI$. Hadamard matrices of order n exist only if n is two or a multiple of four. A Hadamard matrix H is said to be *skew* if $H + H^T = 2I$, where I denotes the identity matrix. It was conjectured that Hadamard matrices and skew Hadamard matrices of order n exist for $n = 4k$ for any positive integer k. Hadamard matrices appear in theory of combinatorics; finite incomplete block designs, orthogonal arrays and the D-optimal designs [1].

It has been shown that the existence of the following are equivalent:

(1) Skew Hadamard matrices of order n.
(2) Doubly regular tournaments of order $n-1$ [9].

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* Corresponding author.

E-mail addresses: hnozaki@auecc.aichi-edu.ac.jp (H. Nozaki), p001883e@nt.icu.ac.jp (S. Suda).

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(3) Irreducible tournaments of order n having 4 distinct eigenvalues, one of which is zero with algebraic multiplicity 1 [7].
(4) Tournaments of order $n-1$ with spectrum $\left\{k^1, \left(\frac{-1+\sqrt{-k}}{2}\right)^k, \left(\frac{-1-\sqrt{-k}}{2}\right)^k\right\}$, where $n = 2k+2$ [12].
(5) Regular tournaments of order $n-1$ with three distinct eigenvalues [10].

For a skew Hadamard matrix H, we normalize H so that the first row of H consists of the all-ones vector. We can construct a $(0, 1)$-matrix A by $A = \frac{1}{2}(J - H)$, where J denotes the all-ones matrix. Then A is the adjacency matrix of a tournament satisfying condition (3). For such a matrix A, we consider the principal submatrix A_1 of order $n-1$ by deleting the first row and column. Then A_1 satisfies conditions (2), (4) and (5),–and vice versa. Thus skew Hadamard matrices of order n are characterized by certain tournaments of order $n-1$.

As described above, the characterization of some property of an oriented graph in terms of the spectrum of its adjacency matrix is important and useful. For example, a tournament is regular if and only if its adjacency matrix has the all-ones vector as an eigenvector.

In algebraic graph theory, the adjacency matrix plays an important role [4,5]. The adjacency matrix of an undirected graph is always diagonalizable. However that of an oriented graph is not necessarily diagonalizable, and hence dealing with the adjacency matrix for an oriented graph is more difficult than the case of an undirected graph. In the area of two-graphs the Seidel matrix, a $(0, \pm 1)$-adjacency matrix, is used [5, Section 11]. The Seidel matrix for an oriented graph is defined naturally, and since it is always Hermitian, it is easy to use the Seidel matrix in the case of an oriented graph.

In the present paper, we give another characterization of a skew Hadamard matrix of order n in terms of the spectrum of the Seidel matrix of a tournament of order $n-2$. Our main theorem is as follows:

Theorem 1.1. *Let $n = 4k+3$, where k is a non-negative integer. Then there exists a doubly regular tournament of order n if and only if there exists a tournament of order $n-1$ with adjacency matrix A_1 such that $S_1 = \sqrt{-1}(A_1 - A_1^T)$ satisfies the following spectral condition:*

$$(\tilde{\theta}_i)_{i=1}^4 = (\sqrt{n}, 1, -1, -\sqrt{n}) \text{ with } \tilde{\beta}_1 = \tilde{\beta}_4 = 0,\ \tilde{\beta}_2 = \tilde{\beta}_3 = \frac{1}{\sqrt{2}}. \tag{1.1}$$

Here $\tilde{\theta}_i$ $(1 \leqslant i \leqslant 4)$ are the distinct eigenvalues of S_1 and $\tilde{\beta}_i$ $(1 \leqslant i \leqslant 4)$ are the corresponding main angles of S_1 as defined in Section 2. See Theorem 2.5 and Remark 2.6 for the spectra of doubly regular tournaments.

In Section 2, we prepare the fundamental notation for oriented graphs and characterize the tournament whose adjacency matrix has a certain spectrum in terms of the spectrum of the Seidel matrix. In Section 3, we prove Theorem 1.1.

2. Tournaments and their Seidel matrices

Let $G = (V, E)$ be an oriented graph of order n; thus the vertex set V consists of n elements and the edge set $E \subset V \times V$ satisfies $E \cap E^T = \emptyset$, where $E^T := \{(x, y) \mid (y, x) \in E\}$. The *adjacency matrix* A of G is indexed by the vertex set V, and its entries are defined as follows:

$$A_{xy} = \begin{cases} 1 & \text{if } (x, y) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Since E satisfies $E \cap E^T = \emptyset$, A satisfies $A \circ A^T = \mathbf{0}$, where $\circ$ is the entrywise product of matrices and $\mathbf{0}$ denotes the zero matrix. The *Seidel matrix* S of G is defined by $S = \sqrt{-1}(A - A^T)$. An oriented graph G is said to be a *tournament* if its adjacency matrix satisfies $A + A^T = J - I$. The vector $A\mathbf{1}$ is called the score vector of the tournament, where $\mathbf{1}$ is the all-ones column vector. Denote the score vector of A by $\boldsymbol{s}$ and the i-th entry of $\boldsymbol{s}$ by s_i. A tournament G of order n is *regular* if all entries of the score vector are equal to $(n-1)/2$, which implies that n must be odd. A regular tournament G is *doubly regular* if the number of common neighbors of a pair of distinct vertices does not depend on the choice of the pair. A tournament G of even order n is *almost regular* if the entries of the column vector $A\mathbf{1}$ are $n/2$ and $(n-2)/2$, each appearing $n/2$ times.

A square matrix M is said to be *normal* if $MM^* = M^*M$, where M^* denotes the transpose conjugate of M. It is known that a normal matrix M can be diagonalized by a unitary matrix, equivalently the eigenspaces corresponding to different eigenvalues are orthogonal. Let τ_i $(1 \leqslant i \leqslant s)$ be the distinct eigenvalues of M. Let m_i be the multiplicity of τ_i, and $\mathcal{E}_i$ the eigenspace of τ_i. Let P_i be the orthogonal projection matrix onto $\mathcal{E}_i$. Then $P_i^* = P_i$, $\sum_{i=1}^{s} P_i = I$ and $P_iP_j = \delta_{ij}P_i$, where δ_{ij} denotes the Kronecker delta. The notion of main angles for the adjacency matrix of a simple undirected graph was introduced in [2]; see [11] for the recent progress. Here we consider the same concept for the $n \times n$ normal matrix M. Define β_i by

$$\beta_i := \frac{1}{\sqrt{n}}\sqrt{(P_i \cdot \mathbf{1})^*(P_i \cdot \mathbf{1})}.$$

We call β_i the *main angle* of τ_i. By the definition of main angles, we have

$$\sum_{i=1}^{s} \beta_i^2 = 1. \tag{2.1}$$

Let G be a tournament of order n with adjacency matrix A and Seidel matrix S. Let $\{\theta_i\}_{i=1}^{s}$ be the distinct eigenvalues of A. Let m_i be the algebraic multiplicity of θ_i. When A is normal, we denote the main angle of θ_i by β_i.

Since the Seidel matrix S is normal, we may define the main angles of S. Moreover S is Hermitian, and all eigenvalues of S are real. Let $\tilde{\theta}_1 > \cdots > \tilde{\theta}_{\tilde{s}}$ be the distinct eigenvalues of S and let $\tilde{m}_i, \tilde{\beta}_i$ be the multiplicity and the main angle of $\tilde{\theta}_i$ for $1 \leqslant i \leqslant \tilde{s}$. The spectral decomposition of the Seidel matrix is $S = \sum_{i=1}^{\tilde{s}} \tilde{\theta}_i P_i$. The following are fundamental results on S.

Lemma 2.1. *Let G be a tournament of order n with Seidel matrix S. Then*

1. $\tilde{\theta}_{\tilde{s}+1-i} = -\tilde{\theta}_i$, $\tilde{m}_{\tilde{s}+1-i} = \tilde{m}_i$ *and* $\tilde{\beta}_{\tilde{s}+1-i} = \tilde{\beta}_i$ *for* $1 \leqslant i \leqslant \tilde{s}$,
2. $\sum_{i=1}^{\tilde{s}} \tilde{m}_i = n$ *and* $\sum_{i=1}^{\tilde{s}} \tilde{m}_i\tilde{\theta}_i^2 = n^2 - n$.
3. *G is regular if and only if* $S\mathbf{1} = 0$.

Proof. Let A be the adjacency matrix of G.

(1) Follows from that $\sqrt{-1}S$ is skew-symmetric.
(2) Follows from taking the traces of $\sum_{i=1}^{\tilde{s}} P_i = I$ and $\sum_{i=1}^{\tilde{s}} \tilde{\theta}_i^2 P_i = S^2 = -A^2 - (A^T)^2 + AA^T + A^TA$.
(3) Follows from the fact that $S\mathbf{1} = 0$ is equivalent to $s_i = n - 1 - s_i$ $(1 \leqslant i \leqslant n)$. $\square$

The following lemma characterizes almost regularity of a tournament in terms of spectral data for the Seidel matrix.

Lemma 2.2. *Let n be an even integer at least two and G a tournament of order n. Then the following are equivalent:*

1. *G is almost regular,*
2. $\sum_{i=1}^{\tilde{s}} \tilde{\theta}_i^2 \tilde{\beta}_i^2 = 1$.

Proof. First the following equality holds for any tournament:

$$\sum_{i=1}^{n} s_i = \mathbf{s}^T\mathbf{1} = \mathbf{1}^TA\mathbf{1} = \tfrac{1}{2}\mathbf{1}^T(A + A^T)\mathbf{1} = \tfrac{1}{2}\mathbf{1}^T(J - I)\mathbf{1} = \tfrac{n(n-1)}{2}. \tag{2.2}$$

Since the order of the tournament is even, $\frac{n}{4} \leqslant \sum_{i=1}^{n}(s_i - \frac{n-1}{2})^2$ holds. Using (2.2) we have

$$\boldsymbol{s}^T\boldsymbol{s} \geqslant \frac{n(n^2-2n+2)}{4}, \tag{2.3}$$

with equality if and only if G is almost regular.

We calculate $\boldsymbol{s}^T\boldsymbol{s}$ in terms of the data of the Seidel matrix. From $S = \sqrt{-1}(2A - J + I)$ we have $\boldsymbol{s} = \frac{1}{2}((n-1)\mathbf{1} - \sqrt{-1}S\mathbf{1})$. Since

$$\begin{aligned}\boldsymbol{s}^T\boldsymbol{s} &= \frac{1}{4}((n-1)^2\mathbf{1}^T\mathbf{1} + \mathbf{1}^TS^2\mathbf{1}) \\ &= \frac{1}{4}\left(n(n-1)^2 + n\sum_{i=1}^{\tilde{s}}\tilde{\theta}_i^2\tilde{\beta}_i^2\right),\end{aligned}$$

G is almost regular if and only if $\sum_{i=1}^{\tilde{s}}\tilde{\theta}_i^2\tilde{\beta}_i^2 = 1$. □

For a square matrix A, we denote the characteristic polynomial of A by $P_A(x)$, that is $P_A(x) = \det(A - xI)$. We use the following lemma to prove Theorem 1.1. See [3, p.90] for the proof. Its proof is valid for normal matrices.

Lemma 2.3. *Let M be a normal matrix, τ_i the distinct eigenvalues of M, and β_i the main angle of τ_i. Let c be a complex number. Then*

$$P_{M+cJ}(x) = P_M(x)\left(1 + c\sum_{i=1}^{s}\frac{n\beta_i^2}{\tau_i - x}\right).$$

Applying Lemma 2.3 to the Seidel matrix of a tournament, we have the following corollary:

Corollary 2.4. *Let G be a tournament of order n with adjacency matrix A and Seidel matrix S. Then the following holds:*

$$P_A(x) = \left(\frac{-\sqrt{-1}}{2}\right)^n P_S(\sqrt{-1}(2x+1))\left(1 + \sqrt{-1}\sum_{i=1}^{\tilde{s}}\frac{n\tilde{\beta}_i^2}{\tilde{\theta}_i - \sqrt{-1}(2x+1)}\right). \tag{2.4}$$

Proof. Since $A = \frac{1}{2}(-\sqrt{-1}S - I + J)$ holds, applying Lemma 2.3 yields the following equations;

$$\begin{aligned}P_A(x) &= \det(A - xI) \\ &= \left(\frac{-\sqrt{-1}}{2}\right)^n \det(S + \sqrt{-1}J - \sqrt{-1}(2x+1)I) \\ &= \left(\frac{-\sqrt{-1}}{2}\right)^n P_S(\sqrt{-1}(2x+1))\left(1 + \sqrt{-1}\sum_{i=1}^{\tilde{s}}\frac{n\tilde{\beta}_i^2}{\tilde{\theta}_i - \sqrt{-1}(2x+1)}\right).\end{aligned}$$ □

Theorem 2.5. *Let G be a tournament of order n with adjacency matrix A and Seidel matrix S. Then the following are equivalent:*

(1) *G is doubly regular,*
(2) *A is such that $s = 3$ and $(\theta_i)_{i=1}^3 = \left(\frac{n-1}{2}, \frac{-1+\sqrt{-n}}{2}, \frac{-1-\sqrt{-n}}{2}\right)$,*
(3) *S is such that $\tilde{s} = 3$, $(\tilde{\theta}_i)_{i=1}^3 = (\sqrt{n}, 0, -\sqrt{n})$, and $(\tilde{\beta}_i)_{i=1}^3 = (0, 1, 0)$.*

Proof. (1) ⇔ (2): The equivalence is proven in [12, Theorem 3.2].

(1), (2) $\Rightarrow$ (3): Note that by [12] $P_A(x) = -\left(x - \frac{n-1}{2}\right)\left(x^2 + x + \frac{n+1}{4}\right)^{\frac{n-1}{2}}$ and A is a normal matrix. Since G is regular, the main angles of A are given by $(\beta_i)_{i=1}^3 = (1, 0, 0)$. Applying Lemma 2.3 yields the following equation:

$$P_S(x) = -x(x^2 - n)^{\frac{n-1}{2}}.$$

Since G is regular, $\tilde{\beta}_1$ and $\tilde{\beta}_3$ are zero, and thus $\tilde{\beta}_2$ is one.

(3) $\Rightarrow$ (2): By Lemma 2.1 (1) and (2), $(\tilde{m}_i)_{i=1}^3 = \left(\frac{n-1}{2}, 1, \frac{n-1}{2}\right)$. Then it follows from Corollary 2.4. □

Remark 2.6. The multiplicities of eigenvalues for the adjacency matrix and the Seidel matrix are given by $(m_i)_{i=1}^3 = \left(1, \frac{n-1}{2}, \frac{n-1}{2}\right)$ and $(\tilde{m}_i)_{i=1}^3 = \left(\frac{n-1}{2}, 1, \frac{n-1}{2}\right)$.

Theorem 2.7. *Let G be a tournament of order $n-1$ with adjacency matrix A and Seidel matrix S. Then the following are equivalent:*

(1) *A is such that $s = 4$, $(\theta_i)_{i=1}^4 = \left(\frac{-1+\sqrt{-n}}{2}, \frac{-1-\sqrt{-n}}{2}, \frac{n-3+\sqrt{(n-3)(n+1)}}{4}, \frac{n-3-\sqrt{(n-3)(n+1)}}{4}\right)$,*

(2) *S is such that $\tilde{s} = 4$, $(\tilde{\theta}_i)_{i=1}^4 = (\sqrt{n}, 1, -1, -\sqrt{n})$, $(\tilde{\beta}_i)_{i=1}^4 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.*

Proof. (1) $\Rightarrow$ (2): The algebraic multiplicities of θ_1 and θ_2 (θ_3 and θ_4) are equal since they are algebraically conjugate. Define m_1 (resp. m_3) as the algebraic multiplicity of θ_1 (resp. θ_3). Since the size of the matrix A is $n-1$ and the trace of A is 0, we have

$$2m_1 + 2m_3 = n - 1,$$
$$-m_1 + \frac{n-3}{2}m_3 = 0.$$

These equations yield $m_1 = \frac{n-3}{2}$, $m_3 = 1$. By [7, Lemma 1(i)], all eigenvectors of A for eigenvalue θ_i for $i = 1, 2$ are also eigenvectors of S with eigenvalue $-2\mathrm{Im}\theta_i$. Moreover by [7, Lemma 1(i)] the corresponding main angles are zero. Since the dimension of the subspace of $\mathbb{C}^{n-1}$ spanned by those eigenvectors is $2m_1 = n-3$ and S is skew-symmetric, we set the remaining eigenvalues of S as $\tau, -\tau$, where τ is a non-negative real number. By Lemma 2.1 (2), we obtain $n(n-3) + 2\tau^2 = (n-1)(n-2)$. Thus $\tau = 1$ and $(\tilde{\theta}_i)_{i=1}^4 = (\sqrt{n}, 1, -1, -\sqrt{n})$. By (2.1) and Lemma 2.1 (1), $(\tilde{\beta}_i)_{i=1}^4 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

(2) $\Rightarrow$ (1): By Lemma 2.1 (1) and (2), $(\tilde{m}_i)_{i=1}^4 = \left(\frac{n-3}{2}, 1, 1, \frac{n-3}{2}\right)$. Then (1) follows from Corollary 2.4. □

Remark 2.8

(1) When G is a tournament satisfying the conditions in Theorem 2.7, the algebraic multiplicities of the eigenvalues for the adjacency matrix and the Seidel matrix are given by

$$(m_i)_{i=1}^4 = \left(\frac{n-3}{2}, \frac{n-3}{2}, 1, 1\right), (\tilde{m}_i)_{i=1}^4 = \left(\frac{n-3}{2}, 1, 1, \frac{n-3}{2}\right) \tag{2.5}$$

(2) As will be shown in the next section, the tournament of order $n-1$ considered in Theorem 2.7 is obtained from a doubly regular tournament of order n.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let G be a doubly regular tournament of order $n = 4k + 3$ with adjacency matrix A. Take a vertex x in G. Let A_1 be the adjacency matrix of the graph obtained by deleting the vertex x from G, namely after reordering of the vertices of the tournament G we have

$$A = \begin{pmatrix} 0 & \boldsymbol{v}^T \\ \mathbf{1} - \boldsymbol{v} & A_1 \end{pmatrix}, \tag{3.1}$$

where $\boldsymbol{v}$ is a $(0, 1)$-column vector. We calculate the spectrum of A_1. Let $\tau_1 \geqslant \cdots \geqslant \tau_{n-1}$ be all the eigenvalues of the Seidel matrix $S_1 = \sqrt{-1}(A_1 - A_1^T)$. The interlacing eigenvalues theorem for bordered matrices [6, Theorem 4.3.8] shows that

$$\begin{aligned} \tau_1 = \cdots = \tau_{2k} &= \sqrt{n}, \\ \tau_{2k+3} = \cdots = \tau_{n-1} &= -\sqrt{n}. \end{aligned}$$

Next we determine τ_{2k+1} and τ_{2k+2}. By Lemma 2.1 (1) $\tau_{2k+1} = -\tau_{2k+2}$. And by Lemma 2.1 (2) $\tau_{2k+1}^2 + \tau_{2k+2}^2 + 2\left(\frac{n-1}{2} - 1\right)n = n^2 - 3n + 2$. Thus $\tau_{2k+1} = -\tau_{2k+2} = 1$ as desired. It follows from $A^2 = kA + (k+1)A^T$ and (3.1) that $A_1\mathbf{1} = 2k\mathbf{1} + \boldsymbol{v}$, $A_1\boldsymbol{v} = k\mathbf{1}$, $A_1^T\mathbf{1} = (2k+1)\mathbf{1} - \boldsymbol{v}$ and $A_1^T\boldsymbol{v} = (k+1)\mathbf{1} - \boldsymbol{v}$. Thus $\boldsymbol{y} = \mathbf{1} + (\sqrt{-1} - 1)\boldsymbol{v}$ is the eigenvector of S_1 with eigenvalue 1, and its conjugate vector is that of S_1 with eigenvalue -1. Now we denote by $\tilde{\theta}_1 > \cdots > \tilde{\theta}_4$ the distinct eigenvalues of S_1, β_i $(i = 1, \ldots, 4)$ the corresponding main angles. Then direct calculation of the norm of $\boldsymbol{y}^T\mathbf{1}$ and $\bar{\boldsymbol{y}}^T\mathbf{1}$, where $\bar{\boldsymbol{y}}$ denotes the complex conjugate vector of $\boldsymbol{y}$, shows that $\tilde{\beta}_2 = \tilde{\beta}_3 = \frac{1}{\sqrt{2}}$. By (2.1) we have $\tilde{\beta}_1 = \tilde{\beta}_4 = 0$.

Conversely let G_1 be a tournament of order $n - 1$ with adjacency matrix A_1 and Seidel matrix S_1 satisfying property (1.1). By Remark 2.8 (1) the multiplicities of S_1 are $(\tilde{m}_i)_{i=1}^4 = \left(\frac{n-3}{2}, 1, 1, \frac{n-3}{2}\right)$. It follows from Lemma 2.2 that G_1 is almost regular.

Hence we can add one more vertex to G_1 so that it becomes a regular tournament G of order n. Let S be the Seidel matrix of G. We may express

$$S = \begin{pmatrix} 0 & \boldsymbol{w}^T \\ -\boldsymbol{w} & S_1 \end{pmatrix}$$

for some $(\pm\sqrt{-1})$-column vector $\boldsymbol{w}$ such that $\boldsymbol{w}^T\mathbf{1} = 0$ and $S_1\mathbf{1} = \boldsymbol{w}$. Then by Lemma 2.3 we have

$$\begin{aligned} P_S(t) &= \det\begin{pmatrix} -t & \boldsymbol{w}^T \\ -\boldsymbol{w} & -tI + S_1 \end{pmatrix} \\ &= \det\begin{pmatrix} -t & \boldsymbol{w}^T \\ -t\mathbf{1} & -tI + S_1 \end{pmatrix} \\ &= \det\begin{pmatrix} -nt & -t\mathbf{1}^T \\ -t\mathbf{1} & -tI + S_1 \end{pmatrix} \\ &= t\det\begin{pmatrix} -n & -\mathbf{1}^T \\ 0 & -tI + S_1 + \frac{t}{n}J \end{pmatrix} \\ &= (-n)tP_{S_1 + \frac{t}{n}J}(t) \end{aligned}$$

$$
\begin{aligned}
&= (-n)tP_{S_1}(t)\left(1+\tfrac{(n-1)t}{n}\sum_{i=1}^{4}\tfrac{\tilde{\beta}_i^2}{\tilde{\theta}_i-t}\right)\\
&= (-n)t(t^2-n)^{\frac{n-3}{2}}(t^2-1)\left(1+\tfrac{(n-1)t}{n}\left(\tfrac{1/2}{-1-t}+\tfrac{1/2}{1-t}\right)\right)\\
&= -t(t^2-n)^{\frac{n-1}{2}}.
\end{aligned}
$$

Since G is regular, the main angle corresponding to the eigenvalue 0 is one and the others are zero. Therefore G is a doubly regular tournament by Theorem 2.5. □

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