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Large Time Behavior for Convection-Diffusion Equations in $\mathbf{R}^N$

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We describe the large time behavior of solutions of the convection-diffusion equation

$$u_t - \Delta u = a \cdot \nabla(|u|^{q-1} u) \qquad \text{in } (0, \infty) \times \mathbf{R}^N$$

with $a \in \mathbf{R}^N$ and $q \geqslant 1 + 1/N$, $N \geqslant 1$.

When $q = 1 + 1/N$, we prove that the large time behavior of solutions with initial data in $L^1(\mathbf{R}^N)$ is given by a uniparametric family of self-similar solutions. The relevant parameter is the mass of the solution that is conserved for all t. Our result extends to dimensions $N > 1$ well known results on the large time behavior of solutions for viscous Burgers equations in one space dimension. The proof is based on La Salle's Invariance Principle applied to the equation written in its self-similarity variables.

When $q > 1 + 1/N$ the convection term is too weak and the large time behavior is given by the heat kernel. In this case, the result is easily proved applying standard estimates of the heat kernel on the integral equation related to the problem.

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1. Introduction

This paper is devoted to the study of the large time behavior of solutions of the following convection-diffusion equation

$$u_t - \Delta u = a \cdot \nabla(|u|^{q-1} u) \qquad \text{in } (0, \infty) \times \mathbf{R}^N \tag{1.1}$$

$$u(0) = u_0 \in L^1(\mathbf{R}^N). \tag{1.2}$$

Integrating Eq. (1.1) over all of $\mathbf{R}^N$ we obtain (at least formally; this will be made precise below) that the total mass of solutions is conserved for all time, i.e.,

$$\int_{\mathbf{R}^N} u(t, x)\, dx = \int_{\mathbf{R}^N} u_0(x)\, dx, \qquad \forall t \geqslant 0. \tag{1.3}$$

Therefore, the total mass of solutions should play a crucial role when describing their large time behavior.

On the other hand, multiplying (1.1) by any positive power of u and integrating by parts we obtain the following decay estimate (see Proposition 1 below):

$$\forall r \in [1, \infty), \exists C_r > 0: t^{(N/2)(1-1/r)} \|u(t, \cdot)\|_r \leqslant C_r, \qquad \forall t \geqslant 0. \tag{1.4}$$

(In (1.4) and in all that follows $\|\cdot\|_r$ denotes the norm in $L^r(\mathbf{R}^N)$). At this level the very particular form of the nonlinearity in (1.1) is crucial. Indeed, the integral of the right hand side of (1.1) multiplied by any power of u is zero by Green's formula. Therefore, for (1.1) we obtain the same decay estimates (1.4) as for the linear heat equation. Let us mention that estimates (1.4) were proved by M. E. Schonbek [26, 27] for more smooth initial data by using Fourier transform and a suitable decomposition of the frequency domain.

The estimate (1.4) suggests that the natural question to study is the large time behavior of

$$t^{N(1-1/r)/2} u(t, x) \tag{1.5}$$

in $L^r(\mathbf{R}^N)$.

The answer to this problem is by now well known in several cases.

Let us consider first the linear heat equation with $a = 0$, i.e.,

$$u_t - \Delta u = 0 \qquad \text{in } (0, \infty) \times \mathbf{R}^N. \tag{1.6}$$

The solution of (1.6), (1.2) is given by

$$u(t, \cdot) = G(t, \cdot) * u_0, \tag{1.7}$$

where $G = G(t, x)$ is the heat kernel

$$G(t, x) = (4\pi t)^{-N/2} \exp(-|x|^2/4t). \tag{1.8}$$

Let M be the mass of the initial data

$$M = \int_{\mathbf{R}^N} u_0(x)\, dx. \tag{1.9}$$

When $u_0 \in L^1(\mathbf{R}^N)$ it is easy to see that

$$t^{(N/2)(1-1/r)} \|u(t) - u_M(t)\|_r \to 0 \qquad \text{as} \quad t \to +\infty \tag{1.10}$$

for every $r \in [1, \infty]$, where $u_M(t, x) = MG(t, x)$.

On the other hand, if $u_0 \in L^1(\mathbf{R}^N; 1 + |x|)$ (i.e., $\int_{\mathbf{R}^N} |u_0(x)| (1 + |x|)\, dx < \infty$) we have a faster decay rate

$$\forall r \geqslant 1, \exists C_r > 0\colon t^{(N/2)(1-1/r)+1/2} \|u(t) - u_M(t)\|_r \leqslant C_r, \qquad \forall t > 0. \tag{1.11}$$

The estimate (1.10) answers the question in the linear case. Indeed, (1.10) means that the difference between $t^{(N/2)(1-1/r)}u(t)$ and $Mt^{(N/2)(1-1/r)}G(t)$ decays to zero in $L^r(\mathbf{R}^N)$ as t goes to infinity. Therefore, we can assert that the general solution of (1.6), for t large, behaves like the heat kernel. On the other hand, (1.11) ensures a decay of order $t^{-1/2}$ when $u_0 \in L^1(\mathbf{R}^N; 1 + |x|)$.

The second case where the large time behavior of solutions of convection-diffusion equations is well known is the viscous Burgers equation in one space dimension, i.e., (1.1) with $N = 1$ and $q = 2$. In this case, by using the Hopf–Cole transformation, the convection-diffusion equation may be transformed into the linear heat equation and one obtains (1.10) (resp. (1.11)) if $u_0 \in L^1(\mathbf{R}^N)$ (resp. if $u_0 \in L^1(\mathbf{R}^N; 1 + |x|)$) with

$$u_M(t, x) = t^{-1/2} \exp\left(\frac{-x^2}{4t}\right)\left\{C_M + a \int_{-\infty}^{x/\sqrt{t}} \exp\left(\frac{-s^2}{4}\right) ds\right\}^{-1}, \tag{1.12}$$

where $C_M \in \mathbf{R}$ is a constant so that

$$\int_{\mathbf{R}} u_M(t, x)\, dx = M, \qquad \forall t > 0. \tag{1.13}$$

This result has been extended to systems of viscous conservation laws in one space variable by T. P. Liu [21] and I. L. Chern and T. P. Liu [6]. The stability of viscous scalar shock fronts has also been studied (cf. A. M. Il'in and O. A. Oleĭnik [15] for the one-dimensional case and J. Goodman [13] for the case of several space dimensions). Let us also mention the work by T. P. Liu and M. Pierre [22] on the large time behavior of solutions for hyperbolic conservation laws in one space dimension.

In both cases, we may assert that the large time behavior of solutions of (1.1) is given by the uniparametric family of particular solutions $\{u_M\}$. These solutions are of self-similar form, i.e.,

$$u_M(t,x) = t^{-N/2} f_M\left(\frac{x}{\sqrt{t}}\right) \tag{1.14}$$

with a profile $f_M = f_M(x)$ such that

$$\int_{\mathbf{R}^N} f_M(x)\,dx = M. \tag{1.15}$$

It is easy to check that every function of the form (1.14) satisfies:

(i) Takes $M\delta$, where δ denotes the Dirac mass at the origin, as initial value, i.e.,

$$u(t,x) \to M\delta \qquad \text{as} \quad t \to 0^+ \tag{1.16}$$

in the sense of measures, namely,

$$\lim_{t\to 0^+} \int_{\mathbf{R}^N} u(t,x)\,\psi(x)\,dx = \psi(0) \tag{1.17}$$

for every continuous and bounded function $\psi\colon \mathbf{R}^N \to \mathbf{R}$.

(ii) It is invariant under the rescaling transformation

$$u_\lambda(t,x) = \lambda^N u(\lambda^2 t, \lambda x), \tag{1.18}$$

i.e.,

$$u_\lambda \equiv u, \qquad \forall \lambda > 0. \tag{1.19}$$

In fact, u is of the form (1.14) with $f(x) = u(1,x)$ if and only if (1.18) holds.

In the two particular cases above, Eq. (1.1) is also invariant under transformation (1.18). Therefore, the large time behavior of the general solution is described in terms of *self-similar solutions* of the problem that are also *source solutions* since they verify (1.17).

The third case where the large time behavior for (1.1) is well known, is when q is large enough. When q is large, since $\|u(t)\|_{L^\infty(\mathbf{R}^N)} \to 0$ as $t \to +\infty$ (with a decay rate that does not depend on q), one should expect a *weakly nonlinear behavior*, i.e., solutions of (1.1) should behave as the solutions of the linear heat equation. This indeed happens when $N=1$ and $q>2$ (cf. [6, 21]) and when $N>1$ and $q\geqslant 2$ (cf. [20]).

Therefore, once again, the large time behavior is given by self-similar functions but now they are not solutions of the full Eq. (1.1).

This type of result is also well known for Navier–Stokes equations (cf. [16, 25]) and for two-dimensional vorticity equations (cf. [11]).

In order to understand when being of self-similar form is compatible with being a solution of (1.1), let us observe that, if u solves (1.1) then u_λ solves

$$u_{\lambda,t}-\Delta u_\lambda=\lambda^{N(1-q)+1}a\cdot\nabla(|u_\lambda|^{q-1}u_\lambda) \qquad \text{in } (0,\infty)\times\mathbf{R}^N. \tag{1.20}$$

Observe that when

$$q=1+\frac{1}{N} \tag{1.21}$$

(or $a=0$) Eq. (1.1) is invariant under the scaling transformation (1.18). Note that when $N=1$, then $q=2$ and we obtain the one-dimensional viscous Burgers equation.

For $q\neq 1+1/N$, Eq. (1.1) is not invariant under the scaling transformation and we must distinguish the cases $q>1+1/N$ and $1<q<1+1/N$. We shall answer the question for $q>1+1/N$. The problem remains open for $1<q<1+1/N$.

In the examples above, since the functions u_M describing the large time behavior are of self-similar form, (1.10) and (1.11) are respectively equivalent to

$$u_\lambda(1)\to f_M \qquad \text{in } L^r(\mathbf{R}^N) \text{ as } t\to\infty \tag{1.22}$$

and

$$\lambda^{1/2}\,\|u_\lambda(1,x)-f_M(x)\|_r\leqslant C_r, \qquad \forall\lambda\geqslant 1. \tag{1.23}$$

Therefore, the large time behavior of u may be understood in terms of the behavior of u_λ as $\lambda\to\infty$.

When $q>1+1/N$, the power $N(1-q)-1$ of λ on the right hand side of (1.20) is negative. Therefore, formally (this will be made precise below), the convection term should vanish as $\lambda\to+\infty$.

As a consequence of these remarks we should expect the two following results:

(a) If $q=1+1/N$, $N\geqslant 1$, the large time behavior of the general solution of (1.1) with initial data in $L^1(\mathbf{R}^N)$ should be given by a uniparametric family of self-similar solutions of the full equation. (This would extend to dimensions $N > 1$ the well known results on viscous Burgers equation mentioned above.)

(b) If $q > 1 + 1/N$, $N \geqslant 1$, the large time behavior of solutions should be given by the heat kernel.

The main aim of this paper is to state more precisely and to prove these two results that were announced in [9].

In order to state the main result for the case $q = 1 + 1/N$ we must recall the results by J. Aguirre, M. Escobedo, and E. Zuazua [2, 3, 4] on the existence and uniqueness of self-similar solutions for (1.1): "If $q = 1 + 1/N$, for every $M \in \mathbf{R}$ there exists a unique self-similar solution of (1.1) with a smooth profile $f_M = f_M(y)$ verifying

$$-\Delta f_M - \frac{1}{2} y \cdot \nabla f_M = \frac{N}{2} f_M + a \cdot \nabla(|f_M|^{q-1} f_M) \qquad \text{in } \mathbf{R}^N$$

such that $\int_{\mathbf{R}^N} f_M(y)\, dy = M$, f_M is of constant sign and decays exponentially to zero as $|y| \to \infty$."

Concerning the large time behavior we have the following result.

THEOREM 1. *Assume* $q = 1 + 1/N$, $N \geqslant 1$. *Let be* $u_0 = u_0(x) \in L^1(\mathbf{R}^N)$ *with*

$$M = \int_{\mathbf{R}^N} u_0(x)\, dx. \tag{1.24}$$

Then the solution $u = u(t, x)$ *of* (1.1)–(1.2) *satisfies*

$$t^{(N/2)(1-1/r)} \|u(t) - t^{-N/2} f_M(x/\sqrt{t})\|_r \to 0 \qquad \text{as} \quad t \to +\infty \tag{1.25}$$

for every $1 \leqslant r \leqslant \infty$.

Moreover, if $u_0 \in L^2(\mathbf{R}^N; \exp(|x|^2/4)) \cap L^\infty(\mathbf{R}^N)$ *and* $M = 0$ *then*

$$t^{(N/2)(1-1/r) + 1/2 - \varepsilon} \|u(t)\|_r \leqslant C_{r,\varepsilon}, \qquad \forall t \geqslant 1 \tag{1.26}$$

for every $r \in [1, \infty]$ *and* $\varepsilon > 0$.

The first statement (1.25) asserts that the general solution $u = u(t, x)$ of (1.1) behaves like the corresponding self-similar one as $t \to +\infty$. We only obtain the almost sharp decay rate (1.26) when the mass $M = 0$. Note that, when $N = 1$, (1.11) holds for every $M \in \mathbf{R}$.

Writing Eq. (1.1) in its similarity variables, the profile of a self-similar solution becomes a stationary solution of the new equation. On the other hand, (1.25) is equivalent to asserting that the transformed trajectories converge to the equilibrium f_M as $t \to \infty$. Therefore, when proving (1.25), we shall work in the similarity variables and apply La Salle's Invariance

Principle. The precompactness of trajectories will be proved by working on some weighted Sobolev spaces that are naturally related to the equation in the similarity variables and that were introduced by M. Escobedo and O. Kavian in [7, 8] (cf. also [1, 19]).

Concerning the large time behavior of solutions for $q > 1 + 1/N$ we have the following result.

THEOREM 2. *Assume* $q > 1 + 1/N$ *and* $u_0 \in L^1(\mathbf{R}^N)$ *and let be*

$$M = \int_{\mathbf{R}^N} u_0(x)\,dx.$$

Then, the unique solution $u = u(t, x)$ *of* (1.1)–(1.2) *verifies*

$$t^{(N/2)(1-1/r)} \|u(t) - MG(t)\|_r \to 0 \qquad as \quad t \to +\infty \tag{1.27}$$

for every $r \in [1, \infty]$.

Moreover, if $u_0 \in L^1(\mathbf{R}^N; 1 + |x|) \cap L^q(\mathbf{R}^N)$ *then*

$$\forall r \in [1, \infty],\ \exists C_r > 0\colon t^{\alpha(r,q)} \|u(t) - MG(t)\|_r \leqslant C_r \tag{1.28}$$

with

$$\alpha(r, q) = \begin{cases} (N/2)(1 - 1/r) + \frac{1}{2} & \text{when } q > 1 + 2/N \\ (N/2)(1 - 1/r) + \frac{1}{2} - \varepsilon & \text{when } q = 1 + 2/N \text{ for every } \varepsilon > 0 \\ \frac{1}{2} - (N/2)(q - 1/r) & \text{when } 1 + 1/N < q < 1 + 2/N. \end{cases} \tag{1.29}$$

We prove Theorem 2 directly applying standard estimates for the heat kernel and decay estimates (1.4) in the integral equation associated with (1.1)–(1.2).

Observe that $\alpha(r, q) > (N/2)(1 - 1/r)$ for every $r \in [1, \infty]$ and $q > 1 + 1/N$. Therefore Theorem 2 makes more precise the large time behavior of u than the first estimate (1.4) does. Therefore, roughly, Theorem 2 asserts that as $t \to \infty$, solutions of (1.1)–(1.2) behave like the heat kernel.

Note that when $q > 1 + 2/N$ we obtain (1.10), namely, the same behavior as for the linear heat equation. When $1 + 1/N < q \leqslant 1 + 2/N$ we obtain a lower decay rate $\alpha(r, q) < (N/2)(1 - r) + \frac{1}{2}$ and we do not know whether it is sharp.

The assumption $u_0 \in L^q(\mathbf{R}^N)$ is probably unnecessary but we need it for technical reasons.

The nature of the behavior of solutions when $1 < q < 1 + 1/N$ seems to be completely different and none of the parabolic techniques we shall develop here seem to apply. The large time behavior of solutions for this range of q remains open.

Let us also mention that combining the methods of [8, 12, 19] with the techniques of this paper we may also study the large time behavior of solutions of convection-diffusion equations with absorption terms like

$$u_t - \Delta u + |u|^{p-1} u = a \cdot \nabla(|u|^{q-1} u) \qquad \text{in } (0, \infty) \times \mathbf{R}^N$$

with $p > 1$ but this will be done in a future paper.

Let us also mention the works by S. Kamin and L. Peletier [17, 18], where the large time behavior of degenerate parabolic equations with absorption is studied. As far as we know, the large time behavior of degenerate parabolic equations with convection terms is unknown.

The rest of the paper is organized as follows. In Section 2, for the sake of completeness, we give an existence and uniqueness result for the Cauchy problem (1.1)–(1.2) as well as the proof of estimates (1.4) and some other estimates for ∇u. In Section 3 we recall some facts about Eq. (1.1) written in its similarity variables and about the weighted Sobolev spaces where it is well posed. We also prove some a priori estimates, that will ensure, in particular, the precompactness of trajectories. In Section 4 we give the proof of Theorem 1 and in Section 5 we prove Theorem 2. Finally, in Section 6 we extend Theorems 1 and 2 to convection-diffusion equations with more general nonlinearities of the form

$$u_t - \Delta u = a \cdot \nabla(g(u))$$

and we also consider initial data that tend to a constant state as $|x| \to \infty$.

2. The Cauchy Problem: Existence, Uniqueness, and Decay Estimates

In this section we give a global existence result and some decay estimates for solutions of (1.1)–(1.2) with initial data in $L^1(\mathbf{R}^N)$. Let us mention that similar decay rates were proved by M. Schonbek [27] by different techniques and for more smooth initial data.

Proposition 1. *For any $a \in \mathbf{R}^N$, $q > 1$ and initial data $u_0 \in L^1(\mathbf{R}^N)$ there exists a unique classical solution $u \in C([0, \infty); L^1(\mathbf{R}^N))$ of* (1.1)–(1.2) *such that*

$$u \in C((0, \infty); W^{2,p}(\mathbf{R}^N)) \cap C^1((0, \infty); L^p(\mathbf{R}^N))$$

for every $p \in (1, \infty)$.

This solution satisfies the following decay estimates

(i) *For every* $p \in [1, \infty)$ *there exists some constant* $C_p = C(p, \|u_0\|_1)$ *such that*

$$\begin{cases} \|u(t)\|_p \leqslant C_p t^{-(N/2)(1-1/p)}, & \forall t > 0 \\ \|u(t)\|_1 \leqslant \|u_0\|_1, & \forall t > 0. \end{cases} \tag{2.1}$$

(ii) *If* $u_0 \in L^1(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ *with* $p \in (1, \infty)$, *then*

$$\|u(t)\|_p \leqslant (C_p t + \|u_0\|_p^{-2p/N(p-1)})^{-(N/2)(1-1/p)}, \qquad \forall t > 0 \tag{2.2}$$

for some $C_p = C(p, \|u_0\|_1)$.

If $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ *then*

$$\|u(t)\|_\infty \leqslant \|u_0\|_\infty, \qquad \forall t > 0. \tag{2.3}$$

(iii) *If* $q \geqslant 1 + 1/N$, $u_0 \in L^1(\mathbf{R}^N)$ *and* $t_0 > 0$ *there exist some constants* $C_\infty = C(\|u_0\|_1, t_0) > 0$ *and* $C'_p = C(p, \|u_0\|_1, t_0)$ *such that*

$$\|u(t)\|_\infty \leqslant C_\infty t^{-N/2}, \qquad \forall t \geqslant t_0 \tag{2.4}$$

and

$$\|\nabla u(t)\|_p \leqslant C'_p t^{-(N/2)(1-1/p)-1/2}, \qquad \forall t \geqslant t_0 \tag{2.5}$$

for every $p \in [1, \infty]$.

Remark 1. The way in which the different constants in the estimates (2.1)–(2.5) depend on the various parameters will be made explicit on the proof.

Remark 2. If $q = 1 + 1/N$ and $u_0 \in L^1(\mathbf{R}^N)$ estimate (2.4) holds for every $t > 0$, i.e., there exists some constant $C_\infty = C_\infty(\|u_0\|_1)$ such that

$$\|u(t)\|_\infty \leqslant C_\infty t^{-N/2}, \qquad \forall t > 0. \tag{2.4b}$$

Proof. We proceed in several steps. First we consider the case $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and prove the estimates above. Then we consider the general case $u_0 \in L^1(\mathbf{R}^N)$.

Step 1. Suppose $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and let us consider the integral equation

$$u(t) = G(t) * u_0 + \int_0^t a \cdot \nabla G(t-s) * (|u|^{q-1} u(s))\, ds, \tag{2.6}$$

where $G = G(t, x)$ is the heat kernel and $*$ denotes the convolution in the space variables.

Let us introduce the operator

$$[\phi(u)](t) = G(t) * u_0 + \int_0^t a \cdot \nabla G(t-s) * (|u|^{q-1} u(s))\, ds.$$

Applying Banach fixed point Theorem to ϕ in the following closed subset of $C([0, T]; L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N))$

$$B = \{u \in C([0, T]; L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)) \colon \sup_{0<t<T} (\|u(t)\|_1 + \|u(t)\|_\infty) \leqslant R\}$$

with $R > 0$ large enough and $T > 0$ small enough we deduce that ϕ has a unique fixed point in B. Therefore, the integral equation (2.6) has a unique local (in time) solution $u = u(t, x)$ in B.

This is by now a rather standard procedure and for brevity we shall skip the details.

The solution $u = u(t, x)$ of (2.6) solves (1.1) for $t \in (0, T)$. Classical regularity results allow us to prove that

$$u \in C((0, T); W^{2,p}(\mathbf{R}^N)) \cap C^1((0, T); L^p(\mathbf{R}^N)) \tag{2.7}$$

for every $p \in (1, \infty)$.

This solution may be extended to a maximal time interval $[0, T_{\max})$ and it will be global in time (i.e., $T_{\max} = \infty$) if the following estimate holds

$$\sup_{t \in [0, T_{\max})} (\|u(t)\|_1 + \|u(t)\|_\infty) < \infty. \tag{2.8}$$

Since $\nabla u(t) \in L^p(\mathbf{R}^N)$ for every $p \in [1, \infty]$ and $t \in (0, T_{\max})$, a simple density argument shows that

$$\int_{\mathbf{R}^N} a \cdot \nabla(|u|^{q-1} u(t, x))\, \varphi(u(t, x))\, dx = 0, \qquad \forall t \in (0, T_{\max}) \tag{2.9}$$

for every continuous function $\varphi \in C(\mathbf{R})$. By an approximation argument (2.9) may be extended to $\varphi(s) = \operatorname{sgn}(s)$, $\varphi(s) = s^+$ and $\varphi(s) = s^-$.

On the other hand, since $u(t) \in W^{2,p}(\mathbf{R}^N)$ for $t > 0$, we have

$$\int_{\mathbf{R}^N} \Delta u(t, x)\, \varphi(u(t, x))\, dx = -\int_{\mathbf{R}^N} \varphi'(u(t, x))\, |\nabla u(t, x)|^2\, dx, \qquad \forall t > 0 \tag{2.10}$$

for every $\varphi \in C^1(\mathbf{R})$. In particular, if φ is nondecreasing

$$\int_{\mathbf{R}^N} \Delta u(t, x)\, \varphi(u(t, x))\, dx \leqslant 0. \tag{2.11}$$

By density, inequality (2.11) may be extended to $\varphi(s) = \text{sign}(s)$ and $\varphi(s) = s^+$.

Multiplying Eq. (1.1) by $\text{sgn}(u(t, x))$ and integrating in all of $\mathbf{R}^N$ it follows

$$\frac{d}{dt}\int_{\mathbf{R}^N} |u(t, x)|\, dx \leqslant 0, \qquad \forall t \in (0, T_{\max}) \tag{2.12}$$

and therefore

$$\|u(t)\|_1 \leqslant \|u_0\|_1, \qquad \forall t \in (0, T_{\max}). \tag{2.13}$$

Now let $m = \|u_0\|_\infty$. Multiplying Eq. (1.1) by $\text{sgn}(u-m)^+$ and $\text{sgn}(u+m)^-$ we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^N} (u-m)^+\,(t, x)\, dx \leqslant 0 \qquad \text{and} \qquad \frac{d}{dt}\int_{\mathbf{R}^N} (u+m)^-\,(t, x)\, dx \leqslant 0$$

and therefore

$$\|u(t)\|_\infty \leqslant m = \|u_0\|_\infty, \qquad \forall t \in (0, T_{\max}). \tag{2.14}$$

Therefore, estimate (2.8) holds and the solution $u = u(t, x)$ of (1.1)–(1.2) is global in time, i.e., $T_{\max} = \infty$ and

$$u \in C([0, \infty); L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)). \tag{2.15}$$

As it was mentioned above, by classical regularity arguments we deduce that

$$u \in C((0, \infty); W^{2,p}(\mathbf{R}^N) \cap C^1((0, \infty); L^p(\mathbf{R}^N)) \tag{2.16}$$

for every $p \in (1, \infty)$.

On the other hand (2.13)–(2.14) provide the estimates (2.1b) and (2.3) claimed in the proposition.

Step 2. Let us now prove the decay rates for the $L^p(\mathbf{R}^N)$-norm.

We need the following "interpolation inequality."

LEMMA 1. *For every $p \in [2, \infty)$ there exists some constant $C = C(p, N) > 0$ such that*

$$\|v\|_p^{(N(p-1)+2)p/N(p-1)} \leqslant C\, \|v\|_1^{2p/N(p-1)}\, \|\nabla(|v|^{p/2})\|_2^2 \tag{2.17}$$

for every $v \in W^{2,p}(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$.

Proof of Lemma 1. We distinguish the cases $N=1$, $N=2$, and $N\geqslant 3$. When $N=1$ we have

$$\|v\|_\infty^2 \leqslant 2\, \|v\|_2\, \|v_x\|_2. \tag{2.18}$$

Applying (2.18) with $v=|u|^{p/2}$, it follows that

$$\frac{\|u\|_\infty^{2p}}{2\,\|u\|_p^p}=\frac{\|\,|u|^{p/2}\|_\infty^4}{2\,\|\,|u|^{p/2}\|_2^2}\leqslant \|(|u|^{p/2})_x\|_2^2. \tag{2.19}$$

On the other hand

$$\|u\|_p^{2p^2/(p-1)}\leqslant \|u\|_\infty^{2p}\, \|u\|_1^{2p/(p-1)}. \tag{2.20}$$

Combining (2.19) and (2.20), inequality (2.17) follows.

When $N=2$ the following inequality holds (cf. [5, p. 165]).

$$\|v\|_{2t}^{2t}\leqslant t\, \|v\|_{2(t-1)}^{2(t-1)}\, \|\nabla v\|_2^2. \tag{2.21}$$

Applying (2.21) with $v=|u|^{p/2}$ it follows that

$$\|u\|_{pt}^{pt}\leqslant t\, \|u\|_{p(t-1)}^{p(t-1)}\, \|\nabla(|u|^{p/2})\|_2^2 \tag{2.22}$$

and, in particular, for $t=1+1/p$ we have

$$\|\nabla(|u|^{p/2})\|_2^2\geqslant \frac{p\,\|u\|_{p+1}^{p+1}}{(p+1)\,\|u\|_1}. \tag{2.23}$$

On the other hand

$$\|u\|_p^{p^2/(p-1)}\leqslant \|u\|_{p+1}^{p+1}\, \|u\|_1^{1/(p-1)}. \tag{2.24}$$

Combining (2.23) and (2.24), inequality (2.17) follows.

When $N\geqslant 3$, by Sobolev's inequality we have

$$\|v\|_{2N/(N-2)}^2\leqslant C_N\, \|\nabla v\|_2^2.$$

Applying this inequality with $v=|u|^{p/2}$ we obtain that

$$\|u\|_{pN/(N-2)}^p\leqslant C_N\, \|\nabla(|u|^{p/2})\|_2^2. \tag{2.25}$$

On the other hand, we have the following interpolation inequality

$$\|u\|_p\leqslant \|u\|_{pN/(N-2)}^{(N(p-1))/(2+N(p-1))}\, \|u\|_1^{2/(2+N(p-1))}. \tag{2.26}$$

Combining (2.25) and (2.26), inequality (2.17) follows. The proof of Lemma 1 is now completed.

We multiply Eq. (1.1) by $|u|^{p-2}u$ with $p \geqslant 2$. Integrating in $\mathbf{R}^N$ and using (2.9), (2.10) we obtain that

$$\frac{d}{dt}\int_{\mathbf{R}^N} |u(t,x)|^p \, dx + \frac{4(p-1)}{p}\int_{\mathbf{R}^N} |\nabla(|u(t,x)|^{p/2})|^2 \, dx = 0.$$

Applying (2.13) and (2.17) it follows that

$$\frac{d}{dt}\|u(t)\|_p^p + \frac{C}{\|u_0\|_1^{2p/N(p-1)}}\|u(t)\|_p^{p[N(p-1)+2]/N(p-1)} \leqslant 0$$

and integrating this differential inequality we obtain that

$$\|u(t)\|_p \leqslant (C_p t + \|u_0\|_p^{-2p/N(p-1)})^{(-N/2)(1-1/p)}, \qquad \forall t \geqslant 0 \qquad (2.27)$$

with $C_p = C/\|u_0\|_1^{2p/N(p-1)}$.

This is the estimate (2.2) in the statement of the proposition for $p \geqslant 2$. When $p \in (1, 2)$, (2.2) follows by linear interpolation from (2.1b) and (2.27) with $p = 2$.

Let us now consider two distinct solutions $u, v \in C([0, \infty]; L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N))$ in the class (2.16) associated to initial data $u_0, v_0 \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$. We multiply the equation verified by $u - v$ by $\operatorname{sgn}(u-v)$ and integrate on all of $\mathbf{R}^N$. We obtain

$$\frac{d}{dt}\int_{\mathbf{R}^N} |u(t,x) - v(t,x)| \, dx \leqslant 0, \qquad \forall t > 0 \qquad (2.28)$$

and therefore we deduce the following $L^1(\mathbf{R}^N)$-contraction property

$$\|u(t) - v(t)\|_1 \leqslant \|u_0 - v_0\|_1, \qquad \forall t \geqslant 0. \qquad (2.29)$$

Step 3. Let us now consider a general initial state $u_0 \in L^1(\mathbf{R}^N)$. We approximate u_0 by a sequence $\{u_{0,n}\} \subset L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$ such that

$$u_{0,n} \to u_0 \qquad \text{in } L^1(\mathbf{R}^N). \qquad (2.30)$$

Let $u_n = u_n(t,x)$ be the solution of (1.1) with initial value $u_{0,n}$. Combining (2.29) and (2.30) we deduce that $\{u_n\}$ is a Cauchy sequence in $C([0,\infty); L^1(\mathbf{R}^N))$.

Let $u = u(t,x) \in C([0,\infty); L^1(\mathbf{R}^N))$ be the limit of $\{u_n\}$ in $C([0,\infty); L^1(\mathbf{R}^N))$. Clearly, $u(0,x) = u_0(x)$.

On the other hand, from (2.27) we deduce that

$$\begin{aligned}\|u_n(t)\|_p &\leqslant (C_{p,n} t + \|u_{0,n}\|_p^{-2p/N(p-1)})^{(-N/2)(1-1/p)} \\ &\leqslant (C_{p,n} t)^{(-N/2)(1-1/p)}, \qquad \forall t > 0 \qquad (2.31)\end{aligned}$$

with $C_{p,n} = C/\|u_{0,n}\|_1^{2p/N(p-1)}$ and therefore, in particular, for any $t > 0$, $\{u_n(t)\}$ is bounded in $L^p(\mathbf{R}^N)$ for every $p \in [1, \infty)$.

Since $\{u_n(t)\}$ converges in $L^1(\mathbf{R}^N)$ to $u(t)$, we deduce that $u_n(t) \to u(t)$ in $L^p(\mathbf{R}^N)$ for every $p \in [1, \infty)$ and $t > 0$.

This allows us to pass to the limit in Eq. (1.1) satisfied by u_n and to deduce that the limit $u = u(t, x)$ also satisfies (1.1). On the other hand, passing to the limit in (2.31) we obtain (2.1a).

Standard regularity arguments allow us to prove that u is a classical solution verifying

$$u \in C((0, \infty); W^{2,p}(\mathbf{R}^N)) \cap C^1((0, \infty); L^p(\mathbf{R}^N))$$

for every $p \in (1, \infty)$. The $L^1(\mathbf{R}^N)$-contraction property (2.29) also extends to this solutions. The uniqueness of solutions is a consequence of (2.29).

Step 4. Let us now prove the estimate (2.4) when $q \geqslant 1 + 1/N$.

By construction of u we have

$$u(2t) = u(t+t) = G(t) * u(t) + \int_0^t a \cdot \nabla G(t-s) * (|u|^{q-1} u(s+t))\, ds.$$

Taking $L^\infty(\mathbf{R}^N)$-norms and using (2.1) we obtain

$$\begin{aligned}
\|u(2t)\|_\infty &\leqslant \|G(t)\|_\infty \|u(t)\|_1 + |a| \int_0^t \|\nabla G(t-s)\|_r \|u(s+t)\|_{r'q}^q\, ds \\
&\leqslant C t^{-N/2} \|u_0\|_1 \\
&\quad + |a|\, C C_{r'q}^q \int_0^t (t-s)^{(-N/2)(1-1/r)-1/2} (s+t)^{(-N/2)(q-1/r')}\, ds \\
&\leqslant C \|u_0\|_1 t^{-N/2} + C |a|\, t^{1/2-(N/2)q}, \qquad \forall t > 0
\end{aligned}$$

if $r > 1$ ($r' = r/(r-1)$) is such that $(N/2)(1-1/r) + \frac{1}{2} < 1$ since

$$\|G(t)\|_\infty \leqslant C t^{-N/2}, \qquad \forall t > 0; \|\nabla G(t)\|_r \leqslant C t^{(-N/2)(1-1/r)-1/2}, \qquad \forall t > 0.$$

Therefore,

$$\|u(2t)\|_\infty \leqslant C(t^{-N/2} + t^{(1-Nq)/2}), \qquad \forall t > 0. \tag{2.32}$$

Clearly, (2.32) implies (2.4) (resp. (2.4b)) if $q > 1 + 1/N$ (resp. $q = 1 + 1/N$).

Step 5. Finally, let us prove estimate (2.5) when $q \geqslant (N+1)/N$.

For this, define $u_\lambda = u_\lambda(t, x)$ as in (1.18). As it was said in the Introduction, u_λ satisfies (1.20) and therefore for $\tau > 0$ fixed,

$$u_\lambda(t+\tau)=G(t)*u_\lambda(\tau) \tag{2.33}$$
$$+\lambda^{N(1-q)+1}\int_0^t a\cdot\nabla G(t-s)*(|u_\lambda|^{q-1}u_\lambda(s+\tau))\,ds$$

$$\nabla u_\lambda(t+\tau)=\nabla G(t)*u_\lambda(\tau) \tag{2.34}$$
$$+\lambda^{N(1-q)+1}\int_0^t a\cdot\nabla G(t-s)*\nabla(|u_\lambda|^{q-1}u_\lambda(s+\tau))\,ds$$

for every $t\geqslant 0$ and $\lambda>0$.

Taking $L^p(\mathbf{R}^N)$-norms in (2.34) we obtain

$$\|\nabla u_\lambda(t+\tau)\|_p\leqslant C\,\|u_0\|_1\,t^{(-N/2)(1-1/p)-1/2}$$
$$+C\,|a|\,\lambda^{N(1-q)+1}q\int_0^t (t-s)^{-1/2}\,\|u_\lambda(s+\tau)\|_\infty^{q-1}$$
$$\times\|\nabla u_\lambda(s+\tau)\|_p\,ds$$
$$\leqslant C\,\|u_0\|_1\,t^{(-N/2)(1-1/p)-1/2}$$
$$+C\,|a|\,q\int_0^t (t-s)^{-1/2}\,\|\nabla u_\lambda(s+\tau)\|_p\,ds \tag{2.35}$$

for every $t>0$ and $\lambda\geqslant 1$. To obtain (2.35) we have used the fact that

$$\|u_\lambda(\tau)\|_1=\|u(\lambda^2\tau)\|_1\leqslant\|u_0\|_1$$

and

$$\|u_\lambda(s+\tau)\|_\infty\leqslant C_\tau,\qquad\forall s\geqslant 0,\ \forall\lambda\geqslant 1$$

which is an easy consequence of (2.4).

Applying Gronwall's Lemma in (2.35) with $p=1$ we deduce for $t=\tau$,

$$\|\nabla u_\lambda(2\tau)\|_1\leqslant C_\tau,\qquad\forall\lambda\geqslant 1$$

which is easily seen to be equivalent to (2.5) with $p=1$.

Now, taking $L^p(\mathbf{R}^N)$-norms in (2.34) we obtain

$$\|\nabla u_\lambda(t+\tau)\|_p\leqslant C\,\|u_0\|_1\,t^{(-N/2)(1-1/p)-1/2} \tag{2.36}$$
$$+C\,|a|\,q\int_0^t (t-s)^{-(N/2)(1-1/p)-1/2}\,\|\nabla u_\lambda(s+\tau)\|_1\,ds.$$

Combining (2.5) with $p=1$ and (2.36) we deduce that

$$\|\nabla u_\lambda(2\tau)\|_p\leqslant C_\tau,\qquad\forall\lambda\geqslant 1$$

for every $p \in [1, N(N-1))$, which is equivalent to (2.5). Iterating this argument (2.5) can be proved for every $p \in [1, \infty]$.

The proof of Proposition 1 is now completed.

3. The Convection-Diffusion Equation in the Similarity Variables

Suppose that $q = (N+1)/N$. Let $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $u = u(t, x)$ be the solution of (1.1)–(1.2) we obtained in Section 2. Let us define the function

$$v(s, y) = e^{sN/2} u(e^s - 1, e^{s/2} y), \qquad \forall s \geqslant 0, \forall y \in \mathbf{R}^N. \tag{3.1}$$

By a simple calculation one sees that v satisfies

$$\begin{cases} v_s - \Delta v - \frac{1}{2} y \cdot \nabla v = (N/2) v + a \cdot \nabla(|v|^{1/N} v) & \text{in } (0, \infty) \times \mathbf{R}^N \\ v(0) = u_0. \end{cases} \tag{3.2}$$

Since $u \in C((0, \infty); W^{2,p}(\mathbf{R}^N)) \cap C^1((0, \infty); L^p(\mathbf{R}^N))$ for every $p \in (1, \infty)$, it is easy to see that v belongs to the same class.

On the other hand, from (2.2), (2.4b), and (2.5) it follows, for every $s_0 > 0$,

$$\begin{cases} \|v(s)\|_{1,p} \leqslant C(p, s_0), & \forall s \geqslant s_0, \forall p \in [1, \infty] \\ \|v(s)\|_\infty \leqslant C_\infty, & \forall s \geqslant 0. \end{cases} \tag{3.3}$$

(In (3.3) by $\|\cdot\|_{1,p}$ we denote the norm in $W^{1,p}(\mathbf{R}^N)$.)

Let us now define the following weighted L^p and Sobolev spaces,

$$L^p(K) = \left\{ f \colon \|f\|_{L^p(K)} = \left[\int_{\mathbf{R}^N} |f(x)|^p K(x)\, dx \right]^{1/p} < \infty \right\},$$

where $K(x) = \exp(|x|^2/4)$.

(a) For $p = 2$, $L^2(K)$ is a Hilbert space and the norm $\|\cdot\|_{L^2(K)}$ is induced by the inner product

$$(f, g) = \int_{\mathbf{R}^N} f(x)\, g(x)\, K(x)\, dx.$$

(b) For $k = 1, 2, \ldots$,

$$H^k(K) = \left\{ f \in L^2(K) \colon \|f\|_{H^k(K)} = \left[\sum_{|\alpha| \leqslant k} \|D^\alpha f\|^2_{L^2(K)} \right]^{1/2} < \infty \right\}.$$

By interpolation we may define $H^s(K)$ for every $s>0$. An important result for all the following is that the embedding from $H^s(K)$ into $L^2(K)$ is compact for every $s>0$. Using this compactness one can prove the following inequality (cf. [7])

$$\forall \varepsilon>0,\ \exists C_\varepsilon>0\colon \|f\|_{L^2(K)} \leqslant \varepsilon \|f\|_{H^1(K)} + C_\varepsilon \|f\|_{1,2}, \qquad \forall f\in H^1(K). \tag{3.4}$$

It is clear that the operator

$$Lf = -\Delta f - \frac{1}{2} y\cdot\nabla f = -\frac{1}{K}\operatorname{div}(K\nabla f)$$

is self-adjoint on $L^2(K)$. Now, if we consider L as an unbounded self-adjoint operator defined on $L^2(K)$ then, $D(L)=H^2(K)$ (cf. [19]).

By the compactness of the embedding $H^1(K)\to L^2(K)$ the inverse of L, L^{-1}, is bounded and compact from $L^2(K)$ into itself. The sequence of eigenvalues of L is

$$\lambda_k = \frac{N+k-1}{2}, \qquad k=1,2,3,\ldots. \tag{3.5}$$

The first eigenvalue, $N/2$, is simple and the corresponding eigenspace is spanned by $K^{-1}(x)$. We shall call

$$\varphi_1(x) = c\exp\left(-\frac{|x|^2}{4}\right)$$

with $c>0$ such that $\int_{\mathbf{R}^N}\varphi_1(x)\,dx=1$.

The operator L is an isomorphism from $H^1(K)$ into its dual $(H^1(K))^*$ and $\|\nabla v\|_{L^2(K)}$ defines a norm in $H^1(K)$, equivalent to that given above.

Let us call S_* the analytic semigroup generated by $L-(N/2)I$ on $L^2(K)$. This semigroup is given in the following way (cf. [19])

$$\forall g\in L^2(K),\ \forall s>0,\ \forall y\in\mathbf{R}^N; \qquad (S_*(s)\,g)(y) = e^{sN/2}(G(e^s-1)*g)(e^{s/2}y).$$

Observe that, formally, the operator S_* acts in the same way in $L^2(\mathbf{R}^N)$ as in $L^2(K)$. Now, since u satisfies Eq. (2.6) then v satisfies

$$v(s) = S_*(s)\,u_0 + \int_0^s S_*(s-\sigma)\,a\cdot\nabla(|v|^{1/N}v(\sigma))\,d\sigma. \tag{3.6}$$

Let us now prove the following:

Proposition 2. *Let be $q=(N+1)/N$, $a\in\mathbf{R}^N$ and $u_0\in L^2(K)\cap L^\infty(\mathbf{R}^N)$. Then if u is the solution of (1.1)–(1.2) and v is given by (3.1) we have*

$$v\in C([0,+\infty);L^2(K))\cap C((0,\infty);H^2(K))\cap C^1((0,\infty);L^2(K)).$$

Proof. As it was said above, v satisfies the integral equation (3.6). Taking $H^1(K)$-norms in (3.6) we obtain

$$\|v(s)\|_{H^1(K)} \leqslant (1+s^{-1/2})\, \|u_0\|_{L^2(K)} + \int_0^s \|S_*(s-\sigma)a\cdot\nabla(|v|^{q-1}\, v(\sigma))\|_{H^1(K)}\, ds \tag{3.7}$$

since $S_*(\cdot)$ is a semigroup generated by a self-adjoint operator and therefore

$$\|S_*(s)v\|_{L^2(K)} \leqslant \|v\|_{L^2(K)};$$

$$\|S_*(s)v\|_{H^2(K)} \leqslant \left(1+\frac{1}{s}\right) \|v\|_{L^2(K)}, \qquad \forall s>0,\ \forall v\in L^2(K)$$

and by interpolation

$$\|S_*(s)v\|_{H^1(K)} \leqslant (1+s^{-1/2})\, \|v\|_{L^2(K)}, \qquad \forall s>0,\ \forall v\in L^2(K).$$

Using (3.3) the last term in (3.7) may be estimated as follows

$$\int_0^s \|S_*(s-\sigma)a\cdot\nabla(|v|^{1/N}\, v(\sigma))\|_{H^1(K)}\, d\sigma$$

$$\leqslant q\,|a| \int_0^s (1+(s-\sigma)^{-1/2})\, \|\,|v|^{1/N}(\sigma)\,\nabla v(\sigma)\|_{L^2(K)}\, d\sigma$$

$$\leqslant q\,|a| \int_0^s (1+(s-\sigma)^{-1/2})\, \|v(\sigma)\|_\infty^{1/N}\, \|\nabla v(\sigma)\|_{L^2(K)}\, d\sigma$$

$$\leqslant C_\infty^{1/N} q\,|a| \int_0^s (1+(s-\sigma)^{-1/2})\, \|v(\sigma)\|_{H^1(K)}\, d\sigma. \tag{3.8}$$

Combining (3.7) and (3.8) and applying Gronwall's Lemma we obtain

$$\|v(s)\|_{H^1(K)} \leqslant C_T(1+s^{-1/2})\, \|u_0\|_{L^2(K)}, \qquad \forall s\in(0,T) \tag{3.9}$$

for every $T>0$.

The fact that $v\in C((0,\infty); H^1(K))$ follows in a standard way.

We now estimate the $L^2(K)$-norm of $w=v_s$ that satisfies, for every $\tau>0$,

$$\begin{cases} w_s - \Delta w - \frac{1}{2}y\cdot\nabla w = (N/2)w + qa\cdot\nabla(|v|^{1/N}\, w) & \text{in } (\tau,\infty)\times\mathbf{R}^N \\ w(\tau) = w_\tau = \Delta v(\tau) + \frac{1}{2}y\cdot\nabla v(\tau) + (N/2)\, v(\tau) & \\ \qquad + a\cdot\nabla(|v(\tau)|^{1/N}\, v(\tau)) \in (H^1(K))^* & \end{cases}$$

and therefore

$$w(s+\tau)=S_*(s)\,w_\tau+q\int_0^s S_*(s-\sigma)a\cdot\nabla(|v|^{1/N}\,w(\sigma+\tau))\,d\sigma.$$

Taking $L^2(K)$-norms in this identity we deduce

$$\|w(s+\tau)\|_{L^2(K)}\leqslant(1+s^{-1/2})\,\|w_\tau\|_{(H^1(K))^*}$$
$$+|a|\,q\int_0^s(1+(s-\sigma)^{-1/2})\,\|\,|v|^{1/N}(\sigma+\tau)\,w(\sigma+\tau)\|_{L^2(K)}\,ds$$

since

$$\|S_*(s)v\|_{L^2(K)}\leqslant(1+s^{-1/2})\,\|v\|_{(H^1(K))^*},\qquad \forall s>0,\ \forall v\in(H^1(K))^*$$

and applying Gronwall's Lemma we obtain

$$\|w(s+\tau)\|_{L^2(K)}\leqslant C_T(1+s^{-1/2})\,\|w_\tau\|_{(H^1(K))^*},\qquad \forall s\in(0,T)\tag{3.10}$$

for every $T>0$.

Therefore, $v_s(s)\in L^2(K)$ for every $s>0$. The continuity of the map $s\in(0,\infty)\to v_s(s)\in L^2(K)$ follows in a standard way.

We finally observe that

$$Lv=-\Delta v-\frac{1}{2}y\cdot\nabla v=\frac{N}{2}v-v_s+a\cdot\nabla(|v|^{1/N}v)\in C((0,\infty);L^2(K))$$

and therefore $v\in C((0,\infty);H^2(K))$.

On the other hand, $v\in C([0,\infty);L^2(K))$. Indeed, since $u_0\in L^2(K)$, $S_*(s)\,u_0\in C([0,\infty);L^2(K))$ and from (3.3) and (3.9) it is seen that the integral term on the right hand side of (3.6) belongs to $C([0,\infty);L^2(K))$.

The proof of Proposition 2 is now completed.

As it was said in the introduction, any self-similar solution u of (1.1) with $q=1+1/N$ is of the form

$$u(t,x)=t^{-N/2}f\left(\frac{x}{\sqrt{t}}\right),\tag{3.11}$$

where the profile $f=f(y)$ solves the elliptic problem

$$-\Delta f-\frac{1}{2}y\cdot\nabla f-\frac{N}{2}f=a\cdot\nabla(|f|^{1/N}f)\qquad\text{in }\mathbf{R}^N.\tag{3.12}$$

The structure of the set of solutions of (3.12) in $H^1(K)\cap L^\infty(\mathbf{R}^N)$ was studied in [4]. The following result was proved.

THEOREM 3 [4]. *Let be $M \in \mathbf{R}$, $a \in \mathbf{R}^N$. Then there exists a unique solution $f_M \in H^2(K) \cap L^\infty(\mathbf{R}^N)$ of (3.12) verifying*

$$\int_{\mathbf{R}^N} f_M(y)\, dy = M.$$

Moreover, $f_M \in C^\infty(\mathbf{R}^N)$, $f_M(y)$ decays exponentially to zero as $|y| \to \infty$ and f_M is positive if $M > 0$.

On the other hand by definition (3.1) of v it is easy to see that

$$\|v(s) - f_M\|_r = e^{s(N/2)(1-1/r)} \|u(e^s - 1) - u_M(e^s - 1)\|_r$$
$$\|v(s) - f_M\|_\infty = e^{s(N/2)} \|u(e^s - 1) - u_M(e^s - 1)\|_\infty,$$

where u_M is the self-similar solution associated to the profile f_M, i.e.,

$$u_M(t, x) = t^{-N/2} f_M\left(\frac{x}{\sqrt{t}}\right).$$

Therefore,

$$\lim_{t \to \infty} t^{(N/2)(1-1/r)} \|u(t) - u_M(t)\|_r = 0 \tag{3.13}$$

if and only if

$$\lim_{s \to \infty} \|v(s) - f_M\|_r = 0. \tag{3.14}$$

Let us observe also that f_M solves (3.12) if and only if it is a stationary solution of (3.2).

Therefore, proving (3.13) for u is equivalent to prove that its corresponding trajectory v converges to the equilibrium f_M. In the proof of Theorem 1 we shall adopt this second approach and apply La Salle's Invariance Principle. For this, we shall first prove the precompactness of the trajectory $\{v(s)\}_{s \geq 0}$ in $L^2(K)$ and then we shall construct a suitable Lyapunov functional.

4. The Self-Similar Large Time Behavior

This section is devoted to the proof of Theorem 1.

We proceed in three steps. In the first one we prove (1.25) for initial data $u_0 \in L^2(K) \cap L^\infty(\mathbf{R}^N)$. In the second one we extend it to $u_0 \in L^1(\mathbf{R}^N)$. In the last one we prove the optimal decay rate (1.26) for $u_0 \in L^2(K) \cap L^\infty(\mathbf{R}^N)$ and $M=0$.

Step 1. Let $u_0 \in L^2(K) \cap L^\infty(\mathbf{R}^N)$. In the following proposition we establish the precompactness of the trajectory $\{v(s)\}_{s \geqslant 0}$ in $L^2(K)$.

PROPOSITION 3. *Let $q=(N+1)/N$, $u_0 \in L^2(K) \cap L^\infty(\mathbf{R}^N)$, $u=u(t, x)$ be the solution of* (1.1)–(1.2) *and $v=v(s, y)$ be the solution of* (3.2) *given by* (3.1).

Then, $v(s) \in L^\infty([1, \infty); H^1(K))$ and therefore $\{v(s)\}_{s \geqslant 0}$ is relatively compact in $L^2(K)$.

Proof. We first observe that

$$M=\int_{\mathbf{R}^N} v(s, y)\, dy=\int_{\mathbf{R}^N} u_0(y)\, dy, \qquad \forall s \geqslant 0. \tag{4.1}$$

Let $\varphi_1=cK^{-1}$ be the first eigenfunction of L introduced in Section 3. Observe that

$$(h, \varphi_1)=\int_{\mathbf{R}^N} h(x)\, \varphi_1(x)\, K(x)\, dx=c \int_{\mathbf{R}^N} h(x)\, dx.$$

Therefore, every function $h \in L^2(K)$ may be written as $h=m\varphi_1+\tilde{h}$ with $\tilde{h} \in E_1^\perp$ (the orthogonal of E_1 in $L^2(K)$) and $m=\int_{\mathbf{R}^N} h(x)\, dx$.

From (4.1) we deduce that

$$v(s)=M\varphi_1+\tilde{v}(s), \qquad \forall s \geqslant 0 \tag{4.2}$$

with $\tilde{v}(s) \in E_1^\perp$, $\forall s \geqslant 0$.

On the other hand, since $L\varphi_1=(N/2)\, \varphi_1$, the function $\tilde{v}$ satisfies

$$\tilde{v}_s+\left(L-\frac{N}{2}\right) \tilde{v}=a \cdot \nabla(|v|^{q-1} v) \qquad \text{in } (0, \infty) \times \mathbf{R}^N. \tag{4.3}$$

Decomposition (4.2) shows that it suffices to establish the boundedness of $\tilde{v}(s)$ in $H^1(K)$. For that, we shall use the fact that the second eigenvalue of L is $(N+1)/2$ and therefore

$$\left(\left(L-\frac{N}{2} I\right) w, w\right) \geqslant \frac{1}{N+1} \|w\|_{H^1(K)}^2, \qquad \forall w \in H^1(K) \cap E_1^\perp, \tag{4.4}$$

where $(\cdot, \cdot)$ denotes the duality pairing between $(H^1(K))^*$ and $H^1(K)$.

We multiply Eq. (4.3) by vK and integrate in all of $\mathbf{R}^N$. Using (4.4) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|\tilde{v}(s)\|^2_{L^2(K)}&+\frac{1}{N+1}\|\tilde{v}(s)\|^2_{H^1(K)}\\&\leqslant\int_{\mathbf{R}^N}a\cdot\nabla(|v|^{1/N}v(s,y))\,v(s,y)\,K(y)\,dy\\&\leqslant|a|\,q\,\|v(s)\|^{1/N}_{\infty}\int_{\mathbf{R}^N}|\nabla v(s,y)|\,|v(s,y)|\,K(y)\,dy\\&\leqslant|a|\,q\,\|v(s)\|^{1/N}_{\infty}\|\nabla v(s)\|_{L^2(K)}\|v(s)\|_{L^2(K)}.\end{aligned}$$

Using (3.3) and applying (3.4) with $\varepsilon>0$ small enough we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|\tilde{v}(s)\|^2_{L^2(K)}&+\frac{1}{N+1}\|\tilde{v}(s)\|^2_{H^1(K)}\\&\leqslant\frac{1}{4(N+1)}\|v(s)\|^2_{H^1(K)}+C\,\|v\|^2_{1,2}\\&\leqslant\frac{1}{2(N+1)}\|\tilde{v}(s)\|^2_{H^1(K)}+C,\qquad\forall s\geqslant 0.\end{aligned}$$

Since

$$\|\nabla w\|^2_{L^2(K)}\geqslant\frac{N+1}{2}\|w\|^2_{L^2(K)},\qquad\forall w\in H^1(K)\cap E_1^{\perp}$$

we obtain that

$$\frac{d}{ds}\|\tilde{v}(s)\|^2_{L^2(K)}+\frac{1}{2}\|\tilde{v}(s)\|^2_{L^2(K)}\leqslant C,\qquad\forall s\geqslant 0.$$

Integrating this inequality we obtain $v\in L^{\infty}(0,\infty;L^2(K))$. In order to obtain the estimate in $H^1(K)$ we observe that v satisfies, for $\tau>0$ fixed, the following integral equation

$$v(s+\tau)=S_*(s)\,v(\tau)+\int_0^s S_*(s-\sigma)a\cdot\nabla(|v|^{1/N}v(\sigma+\tau))\,d\sigma.$$

Taking $H^1(K)$-norms we obtain

$$\begin{aligned}\|v(s+\tau)\|_{H^1(K)}\leqslant{}&(1+s^{-1/2})\,\|v(\tau)\|_{L^2(K)}\\&+|a|\int_0^s(1+(s-\sigma)^{-1/2})\,\|\nabla(|v|^{1/N}v(\sigma+\tau))\|_{L^2(K)}\,d\sigma\\\leqslant{}&s^{-1/2}\,\|v(\tau)\|_{L^2(K)}\\&+|a|\,qC^{1/N}_{\infty}\int_0^s(1+(s-\sigma)^{-1/2})\,\|\nabla v(\sigma+\tau)\|_{L^2(K)}\,d\sigma.\end{aligned}$$

Applying Gronwall's Lemma, using the fact that $v \in L^{\infty}(0, \infty; L^2(K))$, and then taking $s = 1$ we obtain that

$$\|v(1+\tau)\|_{H^1(K)} \leqslant C, \qquad \forall \tau \geqslant 0.$$

Therefore, $v \in L^{\infty}([1, \infty); H^1(K))$.

The compactness of the embedding $H^1(K) \subset L^2(K)$ ensures that the trajectory $\{v(s)\}_{s \geqslant 1}$ is relatively compact in $L^2(K)$. The proof of Proposition 3 is now completed.

Let us introduce the ω-limit set of u_0 in $L^2(K)$,

$$\omega(u_0) = \{ g \in L^2(K) \colon \exists s_n \to \infty \text{ such that } v(s_n) \to g \text{ in } L^2(K) \}.$$

Assume for a moment that $\omega(u_0)$ is contained in the set of stationary solutions of (3.2), i.e., in the set of solutions $f \in H^1(K) \cap L^{\infty}(\mathbf{R}^N)$ of the elliptic problem (3.12).

Then, necessarily, $\omega(u_0) = \{f_M\}$ with $M = \int_{\mathbf{R}^N} u_0(y)\, dy$ and therefore

$$v(s) \to f_M \qquad \text{in } L^2(K) \text{ as } s \to \infty. \tag{4.5}$$

Indeed, since

$$\int_{\mathbf{R}^N} v(s, y)\, dy = M, \qquad \forall s > 0$$

and the embedding $L^2(K) \subset L^1(\mathbf{R}^N)$ is continuous, for every $g \in \omega(u_0)$ we have

$$\int_{\mathbf{R}^N} g(y)\, dy = M \tag{4.6}$$

and we know from Theorem 3 that there exists a unique solution of (3.12) verifying (4.6), namely $g = f_M$.

From (4.5) and the fact that $\{v(s)\}_{s \geqslant 1}$ is uniformly bounded in $W^{1,p}(\mathbf{R}^N)$ for every $p \in [1, \infty]$ we deduce that

$$v(s) \to f_M \qquad \text{in } L^r(\mathbf{R}^N)$$

for every $r \in [1, \infty]$ and this is equivalent to (1.25) as we pointed out in the end of Section 3.

Let us now prove that $\omega(u_0)$ is indeed contained in the set of solutions of (3.12). For this we use La Salle's Invariance Principle, defining first of all a suitable Lyapunov functional.

Suppose that $M \geqslant 0$ (when $M<0$ it suffices to replace v by $-v$). For any $s>0$, let be

$$\Omega(s)=\{y \in \mathbf{R}^N: v(s, y)>f_M(y)\}.$$

By the regularity of $v(s)$ and f_M (both are continuous in $\mathbf{R}^N$ for $s>0$), the set $\Omega(s)$ is open.

One can also suppose that $\Omega(s) \neq \varnothing$ for every $s>0$. Indeed, if $\Omega(s_0)=\varnothing$ for some $s_0>0$, since

$$\int_{\mathbf{R}^N} v(s, y)\, dy=\int_{\mathbf{R}^N} f_M(y)\, dy=M, \qquad \forall s>0 \tag{4.7}$$

we deduce that $v(s_0)=f_M$ and therefore $v(s)=f_M$ for every $s \geqslant s_0$.

Define then

$$\Phi(s)=\int_{\Omega(s)}(v(s, y)-f_M(y))\, dy. \tag{4.8}$$

As was proved in [4], the mapping $M \rightarrow f_M$ is continuous from $\mathbb{R}$ to $L^2(K)$, therefore Φ depends continuously on v.

We want to prove that Φ is a strictly decreasing function on $\mathbf{R}^+$. For this, observe first that $w(s)=v(s)-f_M$ verifies

$$\begin{aligned} & w_s-\Delta w-\frac{1}{2} y \cdot \nabla w \\ & \quad=\frac{N}{2} w+a \cdot \nabla(|v|^{q-1} v-|f_M|^{q-1} f_M) \qquad \text{in } (0, \infty) \times \mathbf{R}^N. \end{aligned} \tag{4.9}$$

Since $w^+(s) \in H^1(K)$ and $a \cdot \nabla(|v|^{q-1} v(s)-|f_M|^{q-1} f_M) \in L^2(K)$ for every $s>0$ we have

$$\frac{d\Phi(s)}{ds}=\Phi'(s)=\int_{\Omega(s)} \Delta(v(s, y)-f_M(y))\, dy. \tag{4.10}$$

Note that the right hand side of (4.10) is well defined since $v(s)$, $f_M \in H^2(K)$, and therefore $\Delta(v(s)-f_M) \in L^1(\mathbf{R}^N)$.

If the boundary of $\Omega(s)$ was smooth the nonincreasing character of Φ would be proved by a simple integration by parts in (4.10), i.e.,

$$\int_{\Omega(s)} \Delta(v(s, y)-f_M(y))\, dy=\int_{\Omega(s)} \frac{\partial}{\partial v}(v(s, y)-f_M(y))\, d\sigma(y) \leqslant 0.$$

However, since in general $\Omega(s)$ is not smooth, by using Sard's Lemma we approximate $\Omega(s)$ by smooth sets of the form

$$\Omega_n(s)=\{y \in \mathbf{R}^N: v(s, y)>f_M+\varepsilon_n\}$$

with $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$. The formal argument above may be applied rigorously to $\Omega_n(s)$ and we deduce

$$\int_{\Omega_n(s)} \Delta(v(s, y) - f_M(y))\, dy \leqslant 0. \tag{4.11}$$

Passing to the limit in (4.11) we conclude that Φ is nonincreasing (see [4] for the details of this argument).

In order to see that Φ is strictly decreasing, let us suppose that there exist $s_2 > s_1 > 0$ such that $\Phi(s_2) = \Phi(s_1)$. Then integrating (4.10) in $[s_1, s_2]$ we obtain

$$\int_{s_1}^{s_2} \int_{\Omega(s)} \Delta(v(s, y) - f_M(y))\, dy\, ds = 0. \tag{4.12}$$

Now, let be $h(s, y) = (v(s, y) - f_M(y))\, \mathscr{X}_\Omega(s, y)$, where $\Omega = \{(s, y) \in (s_1, s_2) \times \mathbf{R}^N : v(s, y) > f_M(y)\}$ and where $\mathscr{X}_\Omega$ denotes the characteristic function of Ω. We want to show that $h(s) \equiv 0$ for every $s \in (s_1, s_2)$ (which would imply that Ω is empty, giving a contradiction) by means of a unique continuation principle. We shall use the same argument as in [4] and therefore we shall only sketch the proof.

First, by using Sard's Lemma and Green's formula, from (4.12) we deduce that $h(s) \in H^2(\mathbf{R}^N)$ for every $s \in (s_1, s_2)$ and

$$\Delta h = \Delta(v - f_M)\, \mathscr{X}_\Omega. \tag{4.13}$$

Then, from (4.9) we obtain

$$h_s + Lh = \frac{N}{2} h + \tilde{V} h + \tilde{W} a \cdot \nabla h \qquad \text{in } (s_1, s_2) \times \mathbf{R}^N \tag{4.14}$$

with $\tilde{V} = V \mathscr{X}_\Omega$, $\tilde{W} = W \mathscr{X}_\Omega$, and

$$V(s, y) = \begin{cases} q \dfrac{|v|^{q-1}(s, y) - |f_M|^{q-1}(y)}{v(s, y) - f_M(y)}\, a \cdot \nabla v(s, y) & \text{if } v(s, y) \neq f_M(s, y) \\ 0 & \text{if } v(s, y) = f_M(y) \end{cases}$$

$$W(s, y) = q\, |f_M(y)|^{q-1}.$$

Clearly $\tilde{W} \in L^\infty(\mathbf{R}^N)$ and on the other hand, since f_M is strictly positive in $\mathbf{R}^N$, $v > f_M$ in Ω and the function $\psi(s) = |s|^{q-1} \in W^{1,\infty}_{\text{loc}}(\mathbf{R} - \{0\})$, we deduce that

$$\tilde{V} \in L^\infty_{\text{loc}}(\mathbf{R}^N).$$

(When $M = 0$ we define $V = 0$ and $W = q\, |v|^{q-1}$.)

On the other hand

$$h=0 \qquad \text{in the open set } \{(s_1, s_2)\times \mathbf{R}^N\}\backslash\bar{\Omega}. \tag{4.15}$$

We claim that

$$\{(s_1, s_2)\times \mathbf{R}^N\}\backslash\bar{\Omega}\neq\emptyset. \tag{4.16}$$

Indeed, if $\{(s_1, s_2)\times \mathbf{R}^N\}\backslash\bar{\Omega}=\emptyset$, we would have $v(s, y)>f_M(y)$ for every $(s, y)\in(s_1, s_2)\times\mathbf{R}^N$ and this would contradict (4.7).

Applying the unique continuation result by L. Hörmander [14, Th. 8.9.1, p. 224] we deduce that $h\equiv 0$ and this leads to a contradiction. Therefore, Φ is strictly decreasing.

We may therefore apply La Salle's Invariance Principle and we conclude that $\omega(u_0)$ contained in the set of solutions of (3.12).

The proof of (1.25) for initial data $u_0\in L^2(K)\cap L^\infty(\mathbf{R}^N)$ is completed.

Step 2. Let us now consider a general initial data $u_0\in L^1(\mathbf{R}^N)$. Let $\{u_{0,n}\}\subset L^2(K)\cap L^\infty(\mathbf{R}^N)$ be a sequence such that

$$u_{0,n}\to u_0 \qquad \text{in } L^1(\mathbf{R}^N). \tag{4.17}$$

Therefore,

$$M_n=\int_{\mathbf{R}^N} u_{0,n}(x)\,dx\to M=\int_{\mathbf{R}^N} u_0(x)\,dx. \tag{4.18}$$

If $u_n=u_n(t,x)$ is the unique solution of (1.1)–(1.2) with initial data $u_{0,n}\in L^2(K)\cap L^\infty(\mathbf{R}^N)$, by Step 1 above we know that

$$\lim_{t\to\infty} t^{(N/2)(1-1/r)}\left\| u_n(t)-t^{-N/2}f_{M_n}\left(\frac{x}{\sqrt{t}}\right)\right\|_r=0 \tag{4.19}$$

for every $r\in[1,\infty]$ and $n\in\mathbf{N}$.

On the other hand

$$\begin{aligned}\left\| u(t)-t^{-N/2}f_M\left(\frac{x}{\sqrt{t}}\right)\right\|_1 &\\ \leqslant \|u(t)-u_n(t)\|_1+\left\| u_n(t)-t^{-N/2}f_{M_n}\left(\frac{x}{\sqrt{t}}\right)\right\|_1&\\ +\|f_{M_n}(x)-f_M(x)\|_1&\end{aligned} \tag{4.20}$$

and by the $L^1(\mathbf{R}^N)$-contraction property (2.29) we know that

$$\|u(t)-u_n(t)\|_1\leqslant\|u_0-u_{0,n}\|_1, \qquad \forall t\geqslant 0. \tag{4.21}$$

From [4] we know that

$$f_{M_n} \to f_M \qquad \text{in } L^1(\mathbf{R}^N). \tag{4.22}$$

Combining (4.19)–(4.22) we deduce that

$$\lim_{t\to\infty} \|u(t) - t^{-N/2} f_M(x/\sqrt{t})\|_1 = 0. \tag{4.23}$$

This is the desired conclusion (1.25) for $r = 1$.

Let us observe that (4.23) is equivalent to

$$u_\lambda(1) \to f_M \qquad \text{in } L^1(\mathbf{R}^N) \text{ as } \lambda \to \infty \tag{4.24}$$

with $u_\lambda = u_\lambda(t, x)$ as in (1.18).

On the other hand, estimates (2.1)–(2.5) ensure that $\{u_\lambda(1)\}_{\lambda \geq 1}$ is uniformly bounded in $W^{1,p}(\mathbf{R}^N)$ for every $p \in [1, \infty]$.

By interpolation we deduce that

$$u_\lambda(1) \to f_M \qquad \text{in } L^r(\mathbf{R}^N) \text{ as } \lambda \to \infty$$

for every $r \in [1, \infty]$, which is equivalent to (1.25).

Step 3. Let us now prove (1.26) for $u_0 \in L^2(K) \cap L^\infty(\mathbf{R}^N)$ with $M = 0$. Consider the solution $v = v(s, y)$ of (3.2) associated to $u = u(t, x)$ as in (3.1).

From Step 1 we know that

$$\|v(s)\|_\infty \to 0 \qquad \text{as } s \to \infty$$

since $f_0 = 0$. Consequently, for any $\varepsilon > 0$ there exists some constant $s_\varepsilon > 0$ such that

$$\|a \cdot \nabla(|v|^{1/N} v(s))\|_{L^2(K)} \leqslant \varepsilon \, \|\nabla v(s)\|_{L^2(K)}, \qquad \forall s \geqslant s_\varepsilon. \tag{4.25}$$

On the other hand,

$$\int_{\mathbf{R}^N} v(s, y)\, dy = 0, \qquad \forall s \geqslant 0$$

and therefore

$$v(s) \in E_1^\perp, \qquad \forall s \geqslant 0.$$

Thus, applying (4.4) we deduce

$$\left(\left(L - \frac{N}{2}\right) v(s), v(s)\right) \geqslant \frac{1}{N+1} \|v(s)\|^2_{H^1(K)}, \qquad \forall s > 0. \tag{4.26}$$

Multiplying Eq. (3.2) by vK, integrating in $\mathbf{R}^N$, and using (4.25)–(4.26) we obtain

$$\frac{1}{2}\frac{d}{ds}\|v(s)\|^2_{L^2(K)}+\frac{1}{N+1}\|v(s)\|^2_{H^1(K)}$$
$$\leqslant \varepsilon\,\|\nabla v(s)\|_{L^2(K)}\,\|v(s)\|_{L^2(K)}$$
$$\leqslant \varepsilon\left(\frac{2}{N+1}\right)^{1/2}\|v(s)\|^2_{H^1(K)},\qquad \forall s\geqslant s_\varepsilon$$

since

$$\|\nabla w\|^2_{L^2(K)}\geqslant\frac{N+1}{2}\|w\|^2_{L^2(K)},\qquad \forall w\in H^1(K)\cap E_1^{\perp}. \tag{4.27}$$

Consequently, for every $\varepsilon>0$ there exists some $s'_\varepsilon>0$ such that

$$\frac{d}{ds}\|v(s)\|^2_{L^2(K)}+(1-\varepsilon)\,\|v(s)\|^2_{L^2(K)}\leqslant 0,\qquad \forall s\geqslant s'_\varepsilon. \tag{4.28}$$

Therefore,

$$\|v(s)\|_{L^2(K)}\leqslant C_\varepsilon e^{-(1-\varepsilon)s/2},\qquad \forall s\geqslant 0. \tag{4.29}$$

Multiplying in Eq. (3.2) by vK and following the arguments above we obtain

$$\|v(s)\|_{H^1(K)}\leqslant C_\varepsilon e^{-(1-\varepsilon)s/2},\qquad \forall s\geqslant 0. \tag{4.30}$$

We multiply Eq. (3.2) again by vK obtaining

$$\frac{d}{ds}\|v(s)\|^2_{L^2(K)}+\|v(s)\|^2_{L^2(K)}\leqslant 2g(s),\qquad \forall s\geqslant 0 \tag{4.31}$$

with

$$g(s)=\int_{\mathbf{R}^N} a\cdot\nabla\left(|v|^{1/N}v(s,y)\right)v(s,y)\,K(y)\,dy. \tag{4.32}$$

Clearly

$$|g(s)|\leqslant C\,\|v(s)\|_{H^1(K)}\,\|v(s)\|^{(N+1)/N}_{L^{2(N+1)/N}(K)}. \tag{4.33}$$

From (4.30) and the continuity of the embedding $H^1(K)\to L^{2(N+1)/N}(K)$ (cf. [7]) we deduce

$$|g(s)|\leqslant C_\varepsilon e^{-((1-\varepsilon)/2)(2+1/N)s},\qquad \forall s\geqslant 0. \tag{4.34}$$

From (4.34), choosing $\varepsilon > 0$ small enough we deduce that

$$\int_0^\infty e^s g(s)\, ds < \infty$$

and then, integrating inequality (4.31) we obtain that

$$\|v(s)\|_{L^2(K)} \leqslant Ce^{-s/2}, \qquad \forall s > 0. \tag{4.35}$$

From (4.35) we deduce

$$\|v(s)\|_r \leqslant Ce^{-s/2}, \qquad \forall s > 0$$

for every $r \in [1, 2]$, which implies

$$t^{(N/2)(1-1/r)+1/2} \|u(t)\|_r \leqslant C_r, \qquad \forall t > 0 \tag{4.36}$$

for every $r \in [1, 2]$.

In order to prove (1.26) for $r > 2$ we need the following technical lemma.

LEMMA 2. *For every $r \in [1, \infty]$ there exists some constant $C_r > 0$ such that*

$$\|G(t) * \varphi\|_r \leqslant C_r \|\varphi\|_{L^1(\mathbf{R}^N; |x|)} t^{-(N/2)(1-1/r)-1/2}, \qquad \forall t > 0 \tag{4.37}$$

for every $\varphi \in L^1(\mathbf{R}^N; 1+|x|)$ with

$$\int_{\mathbf{R}^N} \varphi(x)\, dx = 0. \tag{4.38}$$

Proof. Using (4.38) we obtain

$$\begin{aligned}
(G(t) * \varphi)(x) &= (4\pi t)^{-N/2} \int_{\mathbf{R}^N} e^{-|x-y|^2/4t} \varphi(y)\, dy \\
&= (4\pi t)^{-N/2} \int_{\mathbf{R}^N} (e^{-|x-y|^2/4t} - e^{-|x|^2/4t})\, \varphi(y)\, dy \\
&= \frac{(4\pi t)^{-N/2}}{\sqrt{t}} \int_0^1 \int_{\mathbf{R}^N} \frac{y \cdot (x-\theta y)}{2\sqrt{t}} e^{-|x-\theta y|^2/4t} \varphi(y)\, dy\, d\theta \\
&= \frac{(4\pi t)^{-N/2}}{\sqrt{t}} \int_0^1 \int_{\mathbf{R}^N} \frac{y \cdot (x-\theta y)}{2\sqrt{t}} e^{-|x-\theta y|^2/4t} \varphi(y)\, dy\, d\theta.
\end{aligned} \tag{4.39}$$

Therefore,

$$\|G(t) * \varphi\|_1 \leqslant C_0 t^{-1/2} \| |y|\, \varphi(y)\|_1 \tag{4.40}$$

and

$$\|G(t) * \varphi\|_\infty \leqslant C_1 t^{-N/2-1/2} \| |y| \varphi(y)\|_1 \tag{4.41}$$

with

$$C_0 = (4\pi)^{-N/2} \int_{\mathbf{R}^N} |x| e^{-|x|^2/4} dx, \qquad C_1 = (4\pi)^{-N/2} \sup_{z \in \mathbf{R}^N} \{|z| e^{-|z|^2}\}.$$

The estimates in the $L^r(\mathbf{R}^N)$-norm for $r \in (1, \infty)$ are obtained by linear interpolation from (4.40) and (4.41). The proof of Lemma 2 is now completed.

Observe that $u_0 \in L^1(\mathbf{R}^N; 1+|x|)$ since $u_0 \in L^2(K)$. On the other hand, since $u_0 \in L^\infty(\mathbf{R}^N)$, combining (2.3) and (2.4b) we deduce that for every $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that

$$\|u(t)\|_\infty \leqslant C_\varepsilon t^{-N(1/2-\varepsilon)}, \qquad \forall t > 0. \tag{4.42}$$

Taking $L^r(\mathbf{R}^N)$-norms in the integral equation (2.6) verified by u and using Lemma 2 we obtain

$$\|u(t)\|_r \leqslant C_r t^{-(N/2)(1-1/r)-1/2} + |a| \int_0^t \|\nabla G(t-s) * (|u|^{1/N} u(s))\|_r \, ds. \tag{4.43}$$

We estimate the integral term as follows

$$\begin{aligned}
&\int_{t/2}^t \|\nabla G(t-s) * (|u|^{1/N} u(s))\|_r \, ds \\
&\quad \leqslant \int_{t/2}^t \|\nabla G(t-s)\|_{2r/(2+r)} \| |u|^{1/N} u(s)\|_2 \, ds \\
&\quad \leqslant C \int_{t/2}^t (t-s)^{-(N/2)(1/2-1/r)-1/2} \|u(s)\|_\infty^{1/N} \|u(s)\|_2 \, ds.
\end{aligned}$$

Using (2.4b) and (4.36) with $r=2$ we deduce

$$\begin{aligned}
&\int_{t/2}^t \|\nabla G(t-s) * (|u|^{1/N} u(s))\|_r \, ds \\
&\quad \leqslant C \int_{t/2}^t (t-s)^{-(N/2)(1/2-1/r)-1/2} s^{-N/4-1} \, ds \\
&\quad \leqslant C t^{-N/4-1} \int_{t/2}^t (t-s)^{-(N/2)(1/2-1/r)-1/2} \, ds.
\end{aligned}$$

If $(N/2)(\frac{1}{2}-1/r)+\frac{1}{2}<1$, that is, $(N-2)r<2N$, we deduce

$$\int_{t/2}^{t} \|\nabla G(t-s) * (|u|^{1/N} u(s))\|_r \, ds \leqslant C t^{-(N/2)(1-1/r)-1/2}, \qquad \forall t>0. \tag{4.44}$$

On the other hand, using (4.36) with $r=1$ and (4.42) we obtain

$$\begin{aligned}
\int_0^{t/2} & \|\nabla G(t-s) * (|u|^{1/N} u(s))\|_r \, ds \\
&\leqslant \int_0^{t/2} \|\nabla G(t-s)\|_r \, \| |u|^{1/N} u(s)\|_1 \, ds \\
&\leqslant C_\varepsilon \int_0^{t/2} (t-s)^{-(N/2)(1-1/r)-1/2} s^{-1+\varepsilon} \, ds \\
&\leqslant C_\varepsilon t^{-(N/2)(1-1/r)-1/2+\varepsilon}, \qquad \forall t>0.
\end{aligned} \tag{4.45}$$

Combining (4.43)–(4.45) we obtain (1.26) for $r \in [1, 2N/(N-2))$. Iterating this argument we obtain (1.26) for every $r \in [1, \infty]$. This concludes the proof of Theorem 1.

5. Weakly Nonlinear Behavior

This section is devoted to the proof of Theorem 2. First, we need the following technical lemma.

Lemma 3. *For every $r \in [1, \infty]$, there exists some constant $C_r > 0$ such that*

$$\|G(t) * \varphi - MG(t)\|_r \leqslant C_r \|\varphi\|_{L^1(\mathbf{R}^N; |x|)} t^{-(N/2)(1-1/r)-1/2}, \qquad \forall t>0 \tag{5.1}$$

for every $\varphi \in L^1(\mathbf{R}^N; 1+|x|)$ with $M=\int_{\mathbf{R}^N} \varphi(x)\,dx$.

The proof of this lemma is analogous to that of Lemma 2. Therefore we omit its details.

Let us consider first $u_0 \in L^1(\mathbf{R}^N; 1+|x|) \cap L^q(\mathbf{R}^N)$. From (2.6) we have

$$u(t) - MG(t) = G(t) * u_0 - MG(t) + a \int_0^t \nabla G(t-s) * (|u|^{q-1} u(s)) \, ds.$$

Taking $L^r(\mathbf{R}^N)$-norms and using Lemma 3 we obtain

$$\|u(t)-MG(t)\|_r \leqslant C_r\, \|u_0\|_{L^1(\mathbf{R}^N;|x|)}\, t^{-(N/2)(1-1/r)-1/2} + |a| \left\| \int_0^t \nabla G(t-s) * (|u|^{q-1} u(s))\, ds \right\|_r . \quad (5.2)$$

We now estimate the last term on the right hand side of (5.2) as follows. We have

$$\left\| \int_{t/2}^t \nabla G(t-s) * (|u|^{q-1} u(s))\, ds \right\|_r \leqslant \int_{t/2}^t \|\nabla G(t-s)\|_1\, \| |u|^q(s)\|_r\, ds \leqslant C \int_{t/2}^t (t-s)^{-1/2} s^{-(N/2)(q-1/r)}\, ds \leqslant C t^{-(N/2)(q-1/r)+1/2}. \quad (5.3)$$

On the other hand, using the fact that $u_0 \in L^q(\mathbf{R}^N)$ and (2.2) we obtain

$$\left\| \int_0^{t/2} \nabla G(t-s) * (|u|^{q-1} u(s))\, ds \right\|_r \leqslant \int_0^{t/2} \|\nabla G(t-s)\|_r\, \| |u|^q (s)\|_1\, ds$$

$$\leqslant C \int_0^{t/2} (t-s)^{-(N/2)(1-1/r)-1/2} (C_q s + \|u_0\|_q^{-2q/N(q-1)})^{-(N/2)(q-1)}\, ds$$

$$\leqslant C \left(\frac{t}{2}\right)^{-(N/2)(1-1/r)-1/2} \int_0^{t/2} (C_q s + \|u_0\|_q^{-2q/N(q-1)})^{-(N/2)(q-1)}\, ds. \quad (5.4)$$

We now distinguish the cases $1+1/N < q < 1+2/N$, $q = 1+2/N$, and $q > 1+2/N$.

(a) If $q > 1+2/N$,

$$\int_0^\infty (C_q s + \|u_0\|_q^{-2q/N(q-1)})^{-(N/2)(q-1)}\, ds < \infty$$

and therefore

$$\left\| \int_0^{t/2} \nabla G(t-s) * |u|^{q-1} u(s)\, ds \right\|_r \leqslant C t^{-(N/2)(1-1/r)-1/2}. \quad (5.5)$$

On the other hand, $(N/2)(q-1/r)-\frac{1}{2}>(N/2)(1-1/r)+\frac{1}{2}$. Therefore, combining (5.2)–(5.3) and (5.5) we obtain (1.28) with

$$\alpha(r, q)=\frac{N}{2}\left(1-\frac{1}{r}\right)+\frac{1}{2}.$$

(b) If $q=1+2/N$, then

$$\int_0^{t/2} (C_q s+\|u_0\|_q^{-q})^{-1}\, ds=\frac{1}{C_q}\log\left(\frac{C_q\,\|u_0\|_q^q\, t}{2}+1\right)$$

and $(N/2)(q-1/r)-\frac{1}{2}=(N/2)(1-1/r)+\frac{1}{2}$. Therefore, combining this identity with (5.2) and (5.3) we obtain (1.28) for any

$$\alpha(r, q)<\frac{N}{2}\left(1-\frac{1}{r}\right)+\frac{1}{2}.$$

(c) If $1+1/N<q<1+2/N$,

$$\int_0^{t/2} (C_q s+\|u_0\|_q^{-2q/N(q-1)})^{-(N/2)(q-1)}\, ds\leqslant Ct^{-(N/2)(q-1)+1}$$

and since $(N/2)(q-1/r)-\frac{1}{2}<(N/2)(1-1/r)+\frac{1}{2}$ we obtain (1.28) with

$$\alpha(r, q)=\frac{N}{2}\left(q-\frac{1}{r}\right)-\frac{1}{2}. \tag{5.6}$$

Finally, let us observe that the density argument we have used in Step 2 of the proof of Theorem 1 allows us to conclude (1.27) for every $u_0\in L^1(\mathbf{R}^N)$ from (1.28).

6. Further Comments

6.1. *More General Nonlinearities*

Let us consider the more general convection-diffusion equation

$$\begin{cases} u_t-\Delta u=a\cdot\nabla(g(u)) & \text{in } (0, \infty)\times\mathbf{R}^N \\ u(0)=u_0 \end{cases} \tag{6.1}$$

with $a\in\mathbf{R}^N$ and $g\in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R})$ such that $g(0)=0$.

Assume $u_0\in L^1(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$ and let us suppose the existence of the following limit

$$\lim_{|s|\to 0}\frac{g(s)}{|s|^{1/N}s}=\alpha. \tag{6.2}$$

Under these conditions, Proposition 1 can be easily extended to system (6.1). In particular, (6.1) has a unique solution $u \in C([0, \infty); L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N))$ that verifies the decay estimates (2.2), (2.4). Since we have not done any assumption on the behavior of $g'(s)$ as $|s| \to 0$, we cannot prove (2.5) by the method of Proposition 2. However, it can be proved that $\{u_\lambda(1)\}_{\lambda>0}$ is uniformly bounded in $W^{1-\varepsilon, p}(\mathbf{R}^N)$ for any $\varepsilon > 0$ and $p \in [1, \infty)$.

When $\alpha \neq 0$ we should expect a self-similar behavior of solutions as it was the case (Theorem 1) when $g(s) = |s|^{1/N} s$. When $\alpha = 0$ we should expect a weakly nonlinear behavior as in the case where $g(s) = |s|^{q-1} s$ with $q > 1 + 1/N$.

We have the following results.

THEOREM 4. *Assume* $\alpha \neq 0$ *and let us denote by* $\{u_M\}$ *the family of self-similar solutions of*

$$u_t - \Delta u = \alpha a \cdot \nabla(|u|^{1/N} u) \qquad \text{in } (0, \infty) \times \mathbf{R}^N.$$

Let be $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ *such that* $M = \int_{\mathbf{R}^N} u_0(x)\, dx$. *Then, the solution* $u = u(t, x)$ *of* (6.1) *verifies* (1.25) *for every* $r \in [1, \infty]$.

THEOREM 5. *Assume* $\alpha = 0$ *and let be* $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ *such that* $M = \int_{\mathbf{R}^N} u_0(x)\, dx$.

Then, the solution $u = u(t, x)$ *of* (6.1) *verifies* (1.27) *for every* $r \in [1, \infty]$.

Remark 3. We can also prove a faster dacay rate than (1.25) or (1.27), in the spirit of (1.26) or (1.28), depending on the rate of convergence to zero of the function $(g(s) - \alpha |s|^{1/N} s)/|s|^{1/N} s$ as $|s| \to 0$.

Proof of Theorem 4. We just give an outline of the proof.

Step 1. Let us consider first $u_0 \in L^2(K) \cap L^\infty(\mathbf{R}^N)$ and define $v = v(s, y)$ as in (3.1). In this case v satisfies

$$\begin{cases} v_s - \Delta v - \dfrac{1}{2} y \cdot \nabla v = \dfrac{N}{2} v + e^{(N+1)s/2} a \cdot \nabla(g(e^{-Ns/2} v)) & \text{in } (0, \infty) \times \mathbf{R}^N \\ v(0) = u_0. \end{cases} \tag{6.3}$$

On the other hand, $u = u(t, x)$ verifies (2.2) and (2.4) and, as we pointed out above, $\{u_\lambda(1)\}_{\lambda>0}$ is uniformly bounded in $W^{1-\varepsilon, p}(\mathbf{R}^N)$. Therefore, for every $s_0 > 0$ we have

$$\|v(s)\|_{1-\varepsilon, p} \leqslant C_{\varepsilon, p}(s_0), \qquad \forall s \geqslant s_0; \qquad \|v(s)\|_\infty \leqslant C_\infty, \qquad \forall s \geqslant 0.$$

On the other hand, the method of proof of Proposition 2, allows us to prove that $v \in C([0, +\infty); L^2(K))$.

Let us see that in fact $v \in L^\infty(0, \infty; L^2(K))$. We decompose v as in (4.2) and we observe that $\tilde{v} = \tilde{v}(s, y)$ satisfies

$$\tilde{v}_s - \Delta\tilde{v} - \frac{1}{2} y \cdot \nabla\tilde{v} = \frac{N}{2}\tilde{v} + e^{(N+1)s/2} a \cdot \nabla(g(e^{-Ns/2} v)) \qquad \text{in } (0, \infty) \times \mathbf{R}^N. \tag{6.4}$$

Multiplying in (6.4) by vK and using (4.4) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\|\tilde{v}(s)\|^2_{L^2(K)} + \frac{1}{N+1}\|\tilde{v}(s)\|^2_{H^1(K)} \\
&\leqslant |a|\, e^{(N+1)s/2} \left| \int_{\mathbf{R}^N} \nabla(g(e^{-sN/2} v(s, y)))\, v(s, y)\, K(y)\, dy \right| \\
&\leqslant |a|\, e^{(N+1)s/2} \int_{\mathbf{R}^N} |g(e^{sN/2} v(s, y))| \\
&\quad \times \left(|\nabla v(s, y)|\, K(y) + \frac{|y|}{2} |v(s, y)|\, K(y) \right) dy \\
&\leqslant C \int_{\mathbf{R}^N} (|v(s, y)|\, |\nabla v(s, y)|\, K(y) + |v(s, y)|^2\, |y|\, K(y))\, dy
\end{aligned} \tag{6.5}$$

since $|g(s)| \leqslant C|s|^{1+1/N}$ for $|s| \leqslant 1$ and $v \in L^\infty((0, \infty) \times \mathbf{R}^N)$.

Combining (6.5) with (3.4) and the inequality (cf. [7, Lemma 1.5])

$$\frac{1}{16}\int_{\mathbf{R}^N} |v(y)|^2\, |y|^2\, K(y)\, dy \leqslant \int_{\mathbf{R}^N} |\nabla v(y)|^2\, K(y)\, dy, \qquad \forall v \in H^1(K)$$

we obtain, as in the proof of Theorem 1,

$$\frac{d}{ds}\|\tilde{v}(s)\|^2_{L^2(K)} + \frac{1}{2}\|\tilde{v}(s)\|^2_{L^2(K)} \leqslant C, \qquad \forall s \geqslant 1.$$

Integrating this inequality we obtain $v \in L^\infty(1, \infty; L^2(K))$ and since $v \in C([0, \infty); L^2(K))$ we deduce $v \in L^\infty(0, \infty; L^2(K))$.

Let us now see that $v \in L^\infty(1, \infty; H^{1-\varepsilon}(K))$ for every $\varepsilon > 0$. Given $\tau > 0$, we introduce the integral equation verified by v,

$$\begin{aligned}
v(s+\tau) = S_*(s)\, v(\tau) + \int_0^s S_*(s-\sigma)\, e^{(N+1)(\sigma+\tau)/2} a \\
\cdot \nabla(g(e^{-(\sigma+\tau)N/2} v(\sigma+\tau)))\, d\sigma.
\end{aligned} \tag{6.6}$$

Taking $H^{1-\varepsilon}(K)$-norms in (6.6) and using the fact that

$$\|S_*(s)v\|_{H^{1-\varepsilon}(K)} \leqslant (1+s^{-(1-\varepsilon)/2})\,\|v\|_{L^2(K)}, \qquad \forall v \in L^2(K),\ \forall s>0$$

we obtain

$$\begin{aligned}
&\|v(s+\tau)\|_{H^{1-\varepsilon}(K)} \\
&\quad\leqslant (1+s^{-(1-\varepsilon)/2})\,\|v(\tau)\|_{L^2(K)} \\
&\qquad + |a| \int_0^s (1+(s-\sigma)^{-(1-\varepsilon/2)})\, e^{(N+1)(\sigma+\tau)/2}\, \|g(e^{-(\sigma+\tau)N/2} v(\sigma+\tau))\|_{L^2(K)}\, d\sigma \\
&\quad\leqslant s^{-(1-\varepsilon)/2}\,\|v(\tau)\|_{L^2(K)} \\
&\qquad + C \int_0^s (1+(s-\sigma)^{-(1-\varepsilon/2)})\, \|\,|v|^{1+1/N}(\sigma+\tau)\|_{L^2(K)}\, d\sigma \\
&\quad\leqslant (1+s^{-(1-\varepsilon)/2})\,\|v(\tau)\|_{L^2(K)} \\
&\qquad + C \int_0^s (1+(s-\sigma)^{-(1-\varepsilon/2)})\, \|v(\sigma+\tau)\|_{L^2(K)}\, d\sigma.
\end{aligned} \tag{6.7}$$

Taking $s=1$ in (6.7) we obtain

$$\|v(1+\tau)\|_{H^{1-\varepsilon}(K)} \leqslant C\,\|v\|_{L^\infty(\tau,\infty;L^2(K))}, \qquad \forall \tau \geqslant 0$$

and therefore $v \in L^\infty([1,\infty); H^{1-\varepsilon}(K))$.

The compactness of the imbedding $H^{1-\varepsilon}(K) \subset L^2(K)$ for $\varepsilon \in (0,1)$ ensures that the trajectory $\{v(s)\}_{s \geqslant 0}$ is relatively compact in $L^2(K)$.

We now decompose $v = v(s, y)$ as follows

$$v(s,y) = f_M(y) + w(s,y), \tag{6.8}$$

where $w \in C([0,\infty); L^2(K))$ solves

$$\begin{cases} w_s - \Delta w - \dfrac{1}{2} y \cdot \nabla w = \dfrac{N}{2} w + e^{(N+1)s/2} a \cdot \nabla(g(e^{-Ns/2}(w+f_M))) \\ \qquad\qquad - \alpha a \cdot \nabla(|f_M|^{1/N} f_M) \qquad \text{in } (0,\infty) \times \mathbf{R}^N \\ w(0) = u_0 - f_M = w_0 \in E_1^\perp. \end{cases} \tag{6.9}$$

We introduce the ω-limit set of the orbit $\{w(s)\}_{s \geqslant 0}$,

$$\omega(w_0) = \{f \in L^2(K) \colon \exists s_n \to \infty \text{ such that } w(s_n) \to f \text{ in } L^2(K)\}.$$

The problem reduces to proving that $\omega(w_0) = \{0\}$. Indeed, this would imply

$$w(s) \to 0 \qquad \text{in } L^2(K) \text{ as } s \to \infty$$

or equivalently

$$v(s) \to f_M \qquad \text{in } L^2(K) \text{ as } s \to \infty. \tag{6.10}$$

In order to prove that $\omega(w_0) = \{0\}$ we use the following result due to V. A. Galaktionov and J. L. Vazquez [10] on the ω-limit sets of perturbed dynamical systems.

THEOREM 6 [10]. *Consider a dynamical system in a Banach space X given by the evolution equation*

$$w_t = A(w) \tag{6.11}$$

and a perturbation

$$w_t = B(t, w). \tag{6.12}$$

Assume that the following three conditions hold:

(a) *The orbits $\{w(t)\}_{t \geq 0}$ of* (6.12) *are relatively compact in X. Moreover, if we let $w^\tau(t) = w(t+\tau)$, $t, \tau > 0$, the sets $\{w^\tau\}_{\tau > 0}$ are relatively compact in $L^\infty_{\text{loc}}(0, \infty; X)$.*

(b) *Given a solution $w \in C([0, \infty); X)$ of* (6.12) *and if $t_j \to \infty$ is such that $w^{t_j}(t)$ converges to a function $v(t)$ in $L^\infty_{\text{loc}}(0, \infty; X)$, then v is a solution of* (6.11).

(c) *The ω-limit set of Eq.* (6.11) *in X,*

$$\Omega = \{ f \in X : \exists w \in C([0, \infty); X) \text{ solution of } (6.11) \text{ and a sequence } t_j \to \infty \text{ such that } w(t_j) \to f \text{ in } X\}$$

is compact in X and uniformly stable in the following sense: *for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if w is a solution of* (6.11) *with $d(w(0), \Omega) \leq \delta$ (by d we denote the distance in X) then*

$$d(w(t), \Omega) \leq \varepsilon, \qquad \forall t > 0.$$

Under these circumstances, the ω-limit sets of the solutions $w \in C([0, \infty); X)$ of (6.12) *are contained in Ω.*

In our context, the Banach space X is $X = L^2(K) \cap E_1^{\perp}$ endowed with the norm of $L^2(K)$. Equation (6.9) plays the role of (6.12) and (6.11) is the following one

$$w_s - \Delta w - \frac{1}{2} y \cdot \nabla w = \frac{N}{2} w + \alpha a \cdot \nabla(|w + f_M|^{1/N} (w + f_M))$$
$$- \alpha a \cdot \nabla(|f_M|^{1/N} f_M) \qquad \text{in } (0, \infty) \times \mathbf{R}^N. \quad (6.13)$$

We have proved above the precompactness of the orbits of (6.3) in $L^2(K)$ which is equivalent to the precompactness of the orbits of (6.9) in $L^2(K)$. On the other hand, since $\{w(s)\}_{s \geq 0}$ is relatively compact in $L^2(K)$ it is easy to see that $\{w^{\tau}(s)\}_{\tau > 0}$ is relatively compact in $L^{\infty}_{\mathrm{loc}}(0, \infty; L^2(K))$. Therefore, the hypothesis (a) of Theorem 6 is satisfied. Using (6.2) is easy to see that (b) holds. Finally, from Step 1 of the proof of Theorem 1 we know that the ω-limit set of the system (6.13) reduces to the zero solution, i.e., $\Omega = \{0\}$. In order to prove its stability we first observe that by multiplying in (6.13) by $\operatorname{sgn}(w)$ and integrating by parts we obtain

$$\|w(t)\|_1 \leq \|w(0)\|_1, \qquad \forall t \geq 0. \quad (6.14)$$

Then, multiplying in (6.13) by wK, using (6.4), and the interpolation inequality

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0\colon \|f\|_{L^2(K)} \leq \varepsilon \|f\|_{H^1(K)} + C_\varepsilon \|f\|_1, \qquad \forall f \in H^1(K) \quad (6.15)$$

(what is a consequence of the compactness (resp. the continuity) of the embedding $H^1(K) \subset L^2(K)$ (resp. $L^2(K) \subset L^1(\mathbf{R}^N)$)) with ε small enough we obtain

$$\frac{d}{ds} \|w(s)\|^2_{L^2(K)} + \frac{1}{2} \|w(s)\|^2_{L^2(K)} \leq C \|w(s)\|_1^2 \leq C \|w(0)\|^2_{L^2(K)} \quad (6.16)$$

from where the stability property easily follows.

The hypotheses of Theorem 6 being satisfied we may ensure that $\omega(w_0) = \{0\}$ and therefore (6.10) holds. In particular

$$v(s) \to f_M \text{ in } L^1(\mathbf{R}^N) \text{ as } s \to \infty. \quad (6.17)$$

Since $\{v(s)\}_{\geq 1}$ is uniformly bounded in $W^{1-\varepsilon, p}(\mathbf{R}^N)$, by interpolation we deduce that

$$v(s) \to f_M \text{ in } L^r(\mathbf{R}^N) \text{ as } s \to \infty \quad (6.18)$$

for every $r \in [1, \infty]$. This is equivalent to (1.25) and therefore, the proof is concluded for $u_0 \in L^2(K) \cap L^\infty(\mathbf{R}^N)$.

Step 2. The density argument of Step 2 of the proof of Theorem 1 allows us to extend (1.25) to every initial data $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. The proof of Theorem 4 is now completed.

Proof of Theorem 5. The proof is similar to that of Theorem 2 and therefore we shall only give an outline.

Let us consider first initial data $u_0 \in L^1(\mathbf{R}^N; 1+|x|) \cap L^\infty(\mathbf{R}^N)$. We have the following integral equation

$$u(t) - MG(t) = G(t) * u_0 - MG(t) + \int_0^t a \cdot \nabla G(t-s) * g(u(s))\, ds. \tag{6.19}$$

Taking $L^r(\mathbf{R}^N)$-norms in (6.19) and using Lemma 3 we obtain

$$\begin{aligned} \|u(t) - MG(t)\|_r \leqslant\, & C_r \|u_0\|_{L^1(\mathbf{R}^N; |x|)}\, t^{-(N/2)(1-1/r)-1/2} \\ & + |a| \left\| \int_0^t \nabla G(t-s) * g(u(s))\, ds \right\|_r. \end{aligned} \tag{6.20}$$

We now estimate the integral term of (6.20) as follows. Since $\alpha = 0$ we have

$$g(s) = |s|^{1/N} s\delta(s) \tag{6.21}$$

with $\delta(s) \to 0$ as $|s| \to 0$.

Combining (2.3) and (2.4) we deduce that u satisfies (2.4b). Using (2.1a) and (2.4b) we obtain

$$\begin{aligned} & \int_{t/2}^t \|\nabla G(t-s) * g(u(s))\, ds\|_r \\ & \leqslant \int_{t/2}^t \|\nabla G(t-s)\|_1 \, \| |u|^{1+1/N}(s)\, \delta(s)\|_r \, ds \\ & \leqslant C \|\delta(u)\|_{L^\infty((t/2,t) \times \mathbf{R}^N)} \int_{t/2}^t (t-s)^{-1/2} s^{-(N/2)(1-1/r)-1/2}\, ds \\ & = \varphi(t)\, t^{-(N/2)(1-1/r)}, \qquad \forall t > 0 \end{aligned} \tag{6.22}$$

with

$$\varphi(t) = C \|\delta(u)\|_{L^\infty((t/2,t) \times \mathbf{R}^N)} \to 0 \qquad \text{as} \quad t \to \infty.$$

On the other hand

$$\left\| \int_0^{t/2} \nabla G(t-s) * g(u(s))\, ds \right\|_r$$

$$\leqslant \int_0^{t/2} \|\nabla G(t-s)\|_r \, \|g(u(s))\|_1 \, ds$$

$$\leqslant C \int_0^{t/2} (t-s)^{-(N/2)(1-1/r)-1/2} \, \|g(u(s))\|_1 \, ds$$

$$\leqslant C \left(\frac{t}{2}\right)^{-(N/2)(1-1/r)-1/2} \int_0^{t/2} \|g(u(s))\|_1 \, ds. \tag{6.23}$$

Combining (6.20), (6.22), and (6.23) the problem reduces to proving that

$$t^{-1/2} \int_0^t \|g(u(s))\|_1 \, ds \to 0 \qquad \text{as} \quad t \to \infty. \tag{6.24}$$

Given $\varepsilon > 0$ we fix $T > 0$ large enough such that

$$\|\delta(u(t))\|_\infty \leqslant \varepsilon, \qquad \forall t \geqslant T$$

and therefore, from (2.4) we deduce that for $t \geqslant T$

$$\int_T^t \|g(u(s))\|_1 \, ds \leqslant \varepsilon \int_T^t \| \, |u|^{1+1/N}(s)\|_1 \, ds$$

$$\leqslant \varepsilon C \int_T^t s^{-1/2} \, ds \leqslant \varepsilon C t^{1/2}, \qquad \forall t \geqslant T. \tag{6.25}$$

On the other hand

$$t^{-1/2} \int_0^T \|g(u(s))\|_1 \, ds \to 0 \qquad \text{as} \quad t \to \infty. \tag{6.26}$$

Combining (6.25) and (6.26) we deduce (6.24) and this concludes the proof for $u_0 \in L^1(\mathbf{R}^N; 1+|x|) \cap L^\infty(\mathbf{R}^N)$.

Finally, (1.27) can be extended to initial data $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ by the usual density argument.

6.2. *Initial Data that Tend to a Constant State as* $|x| \to \infty$

In this section we show how the earlier results may be used to describe the large time behavior of solutions of (1.1) whose initial value tend to a constant state as $|x| \to \infty$.

Let $u_0 \in L^\infty(\mathbf{R}^N)$ such that the following limit exists

$$\lim_{|x| \to \infty} u_0(x) = l \tag{6.27}$$

and with $v_0(x) = u_0(x) - l \in L^1(\mathbf{R}^N)$, $M = \int_{\mathbf{R}^N} v_0(x)\, dx$.

Then, $u = u(t, x)$ solves (1.1)–(1.2) if and only if

$$v(t, x) = u(t, x) - l \tag{6.28}$$

solves

$$\begin{cases} v_t - \Delta v = a \cdot \nabla(g(v)) & \text{in } (0, \infty) \times \mathbf{R}^N \\ v(0) = v_0 \end{cases} \tag{6.29}$$

with

$$g(s) = |s + l|^{q-1}(s + l) - |l|^{q-1}\, l. \tag{6.30}$$

The method of proof of Proposition 1 allows us to prove that (6.29) has a unique classical solution $v \in C([0, \infty); L^1(\mathbf{R}^N))$. Therefore, (1.1)–(1.2) has a unique solution

$$u(t, x) = v(t, x) + l. \tag{6.31}$$

On the other hand, (6.31) shows that the large time behavior of u may be understood in terms of v.

Let us observe that, since $l \neq 0$, then

$$g(s) = q\, |l|^{q-1}\, s + \psi(s) \tag{6.32}$$

with ψ smooth at the origin and such that

$$|\psi(s)| \leqslant C\, |s|^2, \qquad \forall s \in \mathbf{R}: |s| \leqslant 1. \tag{6.33}$$

When $N = 1$ the large time behavior of (6.29) is well known (cf. for instance [21]). Let us consider the case $N > 1$.

Since v solves (6.29), using (6.32) we deduce that the function

$$w(t, x) = v(t, x - q\, |l|^{q-1}\, at) \tag{6.34}$$

verifies

$$\begin{cases} w_t - \Delta w = a \cdot \nabla(\psi(w)) & \text{in } (0, \infty) \times \mathbf{R}^N \\ w(0) = v_0. \end{cases} \tag{6.35}$$

From (6.33) we deduce (since $N>1$) that

$$\lim_{|s|\to 0} \frac{\psi(s)}{|s|^{1/N} s} = 0 \tag{6.36}$$

and therefore, from Theorem 5,

$$t^{N(1-1/r)/2} \|w(t, x) - MG(t, x)\|_r \to 0 \quad \text{as} \quad t \to \infty \tag{6.37}$$

for every $r \in [1, \infty]$. But (6.37) is equivalent to

$$t^{N(1-1/r)/2} \|u(t, x) - l - MG(t, x + q\,|l|^{q-1}\,at)\|_r \to 0 \quad \text{as} \quad t \to \infty. \tag{6.38}$$

Let us note that, in the obtention of (6.38), we have not used the fact that $q>1$.

We have proved the following result.

THEOREM 7. *Let be $N>1$, $q>0$, and $u_0 \in L^\infty(\mathbf{R}^N)$ such that the limit (6.27) exists with $l \neq 0$ and $v_0(x) = u_0(x) - l \in L^1(\mathbf{R}^N)$ with $M = \int_{\mathbf{R}^N} v_0(x)\,dx$.*

Then the solution $u = u(t, x)$ of (1.1)–(1.2) *satisfies* (6.38) *for every $r \in [1, \infty]$.*

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