Strong Convergence for a Finite Family of Generalized Asymptotically Nonexpansive Mappings

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1. Introduction

Let \( X \) be a real Banach space and \( K \) a nonempty closed convex subset of \( X \). Let \( N \geq 1 \) is a positive integer and \( \Lambda = \{1, 2, \ldots, N\} \). Let \( T: K \to K \) be a self-mapping. Denote by \( \text{Fix}(T) \) the set of fixed points of \( T \), i.e. \( \text{Fix}(T) = \{x \in K : Tx = x\} \). A mapping \( T \) is said to be asymptotically nonexpansive if there exists a sequence \( \{u_n\} \subset [0, +\infty) \) with \( \lim_{n \to \infty} u_n = 0 \), for all \( x, y \in X \), such that

\[
\left\| T^n x - T^n y \right\| \leq (1 + u_n) \left\| x - y \right\|, \forall n \geq 1.
\]
T is said to be asymptotically quasi-nonexpansive if $\text{Fix}(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, +\infty)$ with $\lim_{n \to \infty} u_n = 0$, for all $x, y \in X, x^* \in \text{Fix}(T)$ such that

$$\|T^n x - x^*\| \leq (1 + u_n)\|x - x^*\|, \forall n \geq 1. \quad (1.2)$$

$T$ is said to be generalized asymptotically nonexpansive if there exists sequences of real numbers of $\{u_n\}, \{c_n\} \subset [0, +\infty)$ with $\lim_{n \to \infty} u_n = 0 = \lim_{n \to \infty} c_n$, for all $x, y \in X$, such that

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\| + c_n, \forall n \geq 1. \quad (1.3)$$

$T$ is said to be generalized asymptotically quasi-nonexpansive if $\text{Fix}(T) \neq \emptyset$ and there exists sequences of real numbers of $\{u_n\}, \{c_n\} \subset [0, +\infty)$ with $\lim_{n \to \infty} u_n = 0 = \lim_{n \to \infty} c_n$, for all $x, y \in X, x^* \in \text{Fix}(T)$ such that

$$\|T^n x - x^*\| \leq (1 + u_n)\|x - x^*\| + c_n, \forall n \geq 1. \quad (1.4)$$

**Remark 1.1.** If in (1.3) and (1.4), $c_n = 0$ for all $n \geq 1$, then (1.3) and (1.4) reduces to (1.1) and (1.2) respectively. If $\text{Fix}(T) \neq \emptyset$ in (1.1) and (1.3), then from (1.1) and (1.3) implies that (1.2) and (1.4) hold respectively. Furthermore, the class of generalized asymptotically (quasi-)nonexpansive mappings include the class of asymptotically (quasi-)nonexpansive mappings and the inclusion is proper (see example [4, 8]). There have many papers in the literature dealing with the approximation of fixed points for several class of nonlinear mappings (see [1-13]).

In 2001, Xu and Ori [12] first introduced an implicit iteration: For any $x_0 \in K$, the sequence $\{x_n\}$ can be generated in the following compact form as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, n \geq 1 \quad (1.5)$$

where

$$T_n = T_n \mod N \text{ and } \{\alpha_n\} \subset [0, 1].$$

Using the implicit iterative process, they proved the weak convergence theorems of approximation to a common fixed point of the finite family of nonexpansive mappings in Hilbert spaces.

In 2003, Sun [10] defined a modified implicit iteration: For any $x_0 \in K$, the modified implicit iterative process can be given in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T^{k_i}_n x_n, n \geq 1 \quad (1.6)$$

where

$$n = (k - 1)N + i, i \in \Lambda, \{\alpha_n\} \subset [0, 1].$$

Using the modified implicit iterative process, Sun [10] proved strong convergence theorems of approximation to a common fixed point for a finite family of asymptotically quasi-nonexpansive self mappings and obtained the sufficient and necessity of strong convergence.

Recently, Shahzad and Zegeye [8] extended the results in [10,12] to generalized asymptotically quasi-nonexpansive mappings. They showed that if $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, then they obtained a necessary and sufficient conditions of the strong convergence of the iteration (1.6), and
if \( \{ \alpha_n \} \subseteq [\delta, 1-\delta] \) for some \( \delta \in (0,1) \) and there exists one numbers \( T \) in \( \{ T_i : i \in \Lambda \} \) which is either semicompact or satisfies condition \((\overline{C})\), then the iteration (1.6) converges strongly to a common fixed point.

More recently, Thakur [11] proposed a modified composite implicit iteration:

\[
\begin{align*}
    x_n &= (1-\alpha_n)x_{n-1} + \alpha_n T_i^k y_n \\
    y_n &= (1-\beta_n)x_{n-1} + \beta_n T_i^k x_n
\end{align*}
\]

(1.7)

where

\[ n = (k-1)N + i, i \in \Lambda, \{ \alpha_n \}, \{ \beta_n \} \subseteq [0,1] \]

and \( x_0 \in K \) is a given point. Using the modified composite implicit iteration, Thakur [11] established a important Lemma as follows:

**Lemma 1.1.** [11, Lemma 2.1] Let \( X \) be a real uniformly convex Banach space, \( K \) a nonempty closed convex subset of \( X \), \( \{ T_i : i \in \Lambda \} \subseteq N \) asymptotically nonexpansive self-mappings of \( K \) with \( \{ u_n^i \} \subseteq [0, +\infty), i \in \Lambda \), and \( F = \bigcap_{n=1}^N \text{Fix}(T_i) \neq \emptyset \).

Let the sequence \( \{ x_n \} \) be defined by (1.7) satisfying the conditions:

1. \[ \sum_{n=1}^{\infty} u_n < +\infty, \text{ where } u_n = \max \{ u_n^1, u_n^2, \cdots, u_n^N \}; \]
2. there exists two constants \( \tau_1, \tau_2 \in (0,1) \) such that \( \tau_1 \leq (1-\alpha_n), (1-\beta_n) \leq \tau_2 \), for all \( n \geq 1 \). Then
3. \( \lim_{n \to \infty} \| x_n - x^* \| \) exists for all \( x^* \in F \);
4. \( \lim_{n \to \infty} d(x_n, F) \) exists;
5. \( \lim_{n \to \infty} \| x_n - T_i x_n \| = 0 \), for all \( 1 \leq i \leq N \).

**Remark 1.2.** From the proof of Lemma [11, Lemma 2.1], it is easy to see that

\[
\sum_{n=1}^{\infty} \sigma_n = \sum_{n=1}^{\infty} \frac{2(\tau_2 + u_n + d_n)}{1-\tau_2} < +\infty \quad \text{of (2.2) in [11] is a flaw. Since } \tau_2 \text{ is a constant in (0, 1),}
\]

\[
\sum_{n=1}^{\infty} u_n < +\infty \text{ and } \sum_{n=1}^{\infty} d_n < +\infty \text{,we get that } \sum_{n=1}^{\infty} \sigma_n = +\infty.
\]

However, if \( \sum_{n=1}^{\infty} \sigma_n = +\infty \), then three results of Lemma can not be obtained.

In this paper, we introduce and study a new type of iterative sequence: Let \( K \) be a nonempty closed convex subset of \( X \) with \( K + K \subseteq K \). For any \( x_0 \in K \), the sequence \( \{ x_n \} \) is generated by

\[
\begin{align*}
    x_n &= (1-\alpha_n)x_{n-1} + \alpha_n T_i^k y_n \\
    y_n &= (1-\beta_n)x_{n-1} + \beta_n T_i^k x_n
\end{align*}
\]

(1)
where \( n = (k - 1)N + i, i \in \Lambda, \{ \alpha_n \}, \{ \beta_n \} \subset [0, 1] \). The iterative sequence (I) is a natural generalization of all iterations above: (i). If \( \beta_n \equiv 0 \) for all \( n \geq 1 \) in (I), then (I) reduces to (1.6). (ii). In contrast to the iteration (1.7), \( y_n \) in (I) be generated by the convex composite of \( x_n \) and \( T_{\alpha_x} \) but \( y_n \) in (1.7) be generated by the convex composite of \( x_{n+1} \) and \( T_{\alpha_x} \). The main purpose is to show the convergence theorems of approximating to a common fixed point for generalized asymptotically nonexpansive mappings and asymptotically nonexpansive mappings in Banach spaces by using a new iteration, and to give the necessary and sufficient conditions of the strong convergence of the new iteration. As one will see, we modify some flaw in the results of Thakur [11] and relax the restriction on \( \{ \alpha_n \} \) with \( \limsup_{n \to \infty} \alpha_n < 1 \) in [8, 11]. Moreover, the method of the proof is different from theirs.

2. Preliminaries

Let \( X \) be a Banach space, \( Y \subset X \) and \( x \in X \), then we denote \( d(x, Y) : = \inf_{y \in Y} \| x - y \| \). A family \( \{ T_i \}_{i \in \Lambda} \) of \( N \) self mappings of \( K \) with \( F = \bigcap_{n=1}^{N} \text{Fix}(T_i) \neq \emptyset \), is said to satisfy:

1) Condition (B) on \( K \) [1] if there is a nondecreasing function \( f : [0, +\infty) \to [0, +\infty) \) with \( f(0) = 0 \) and \( f(r) > 0 \) for all \( r \in (0, +\infty) \) and all \( x \in K \), such that

\[
\max_{i \in \Lambda} \left\{ \| x - T_i x \| \right\} \geq f(d(x, F))
\]

2) Condition (\( \overline{C} \)) on \( K \) [1] if there is a nondecreasing function

\( f : [0, +\infty) \to [0, +\infty) \) with \( f(0) = 0 \) and \( f(r) > 0 \) for all \( r \in (0, +\infty) \) and all \( x \in K \) such that

\[
\| x - T_i x \| \geq f(d(x, F)) \]

for at least one \( T_i, i \in \Lambda \).

Noting that Condition (B) and (\( \overline{C} \)) are equivalent.

**Lemma 2.1.** ([5]) Let \( \{ a_n \}, \{ b_n \} \) and \( \{ c_n \} \)

be three nonnegative real number sequences satisfying

\[
a_{n+1} \leq (1 + b_n) a_n + c_n, \forall n \geq 1, \text{where} \sum_{n=1}^{\infty} b_n < +\infty \text{and} \sum_{n=1}^{\infty} c_n < +\infty, \text{then} \lim_{n \to \infty} a_n \text{ exists. Moreover, if}
\]

there exists a subsequences \( \{ a_{n_j} \} \) of \( \{ a_n \} \) such that \( a_{n_j} \to 0 \) as \( j \to \infty \) then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 2.2.** ([6]) Let \( X \) be a uniformly convex Banach space, \( b, c \) be two constants with \( 0 < b < c < 1 \).

Suppose that \( \{ t_n \} \) be a real sequence in \([b, c]\) and \( \{ x_n \}, \{ y_n \} \) are two sequences in \( X \) such that

\[
\limsup_{n \to \infty} \| x_n \| \leq a, \limsup_{n \to \infty} \| y_n \| \leq a \text{ and} \limsup_{n \to \infty} \| t_n x_n + (1 - t_n) y_n \| = a, \text{Then}
\]

\[
\limsup_{n \to \infty} \| x_n - y_n \| = 0, \text{where} \ a \geq 0 \text{ is some constant.}
3. Main results

Throughout this section, we assume that $K$ a nonempty closed convex subset of $X$ with $K + K \subseteq K$.

**Theorem 3.1.** Let $X$ be a real Banach space and let $\{T_i : i \in \Lambda\}$ be $N$ generalized asymptotically nonexpansive self-mappings of $K$ with $\{u_{in}\}, \{c_{in}\} \subset [0, +\infty)$ such that

$$\sum_{n=1}^{\infty} u_{in} < +\infty \text{ and } \sum_{n=1}^{\infty} c_{in} < +\infty \text{ for all } i \in \Lambda.$$  

Suppose $F = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $[0, 1]$ satisfying the conditions: $\limsup_{n \to \infty} \alpha_n < 1$

and $\sum_{n=1}^{\infty} \beta_n < +\infty$. For any $x_0 \in K$, the sequence $\{x_n\}$ is generated by (1). Then the sequences $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in \Lambda\}$ if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$.

**Proof.** The necessity is obvious, so we will only prove the sufficiency. Taking $x^* \in F$, (1.4) implies

$$\langle T_i^k x - x^*, J(x - x^*) \rangle \leq \|T_i^k x - x^*\| \|x - x^*\| \leq (1 + u_n + c_n) \|x_n - x^*\|^2 + c_n, \quad (3.1)$$

where $n = (k - 1)N + i, i \in \Lambda$. From (1),(1.4)and (3.1) it follows that

$$\|x_n - x^*\|^2 = (1 - \alpha_n) \|x_{n-1} - x^*, J(x_n - x^*)\| + \alpha_n \langle T_i^k y_n - x^*, J(x_n - x^*)\rangle + \alpha_n (u_n + 2c_n) \|x_n - x^*\|^2 + 2\alpha_n c_n$$

$$\leq \frac{1 - \alpha_n}{2} \|x_{n-1} - x^*\|^2 + \frac{1 + \alpha_n}{2} \|x_n - x^*\|^2 + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\|$$

$$\leq \frac{1 - \alpha_n}{2} \|x_{n-1} - x^*\|^2 + \frac{1 + \alpha_n}{2} \|x_n - x^*\|^2 + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\| \|x_n - x^*\| + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\|$$

$$\leq \frac{1 - \alpha_n}{2} \|x_{n-1} - x^*\|^2 + \frac{1 + \alpha_n}{2} \|x_n - x^*\|^2 + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\|$$

$$\leq \frac{1 - \alpha_n}{2} \|x_{n-1} - x^*\|^2 + \frac{1 + \alpha_n}{2} \|x_n - x^*\|^2 + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\| \|x_n - x^*\| + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\|$$

$$\leq \frac{1 - \alpha_n}{2} \|x_{n-1} - x^*\|^2 + \frac{1 + \alpha_n}{2} \|x_n - x^*\|^2 + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\|$$

$$\leq \frac{1 - \alpha_n}{2} \|x_{n-1} - x^*\|^2 + \frac{1 + \alpha_n}{2} \|x_n - x^*\|^2 + \alpha_n (1 + u_n) \|y_n - x_n\| \|x_n - x^*\|$$

and

$$\|y_n - x_n\| = \|\beta_n \left(T_i^k x_n - x_n\right)\| \leq \beta_n (2 + u_n) \|x_n - x^*\| + \beta_n c_n \quad (3.3)$$

Substituting (3.3) into (3.2), we get

$$\|x_n - x^*\|^2 \leq \left|1 - \alpha_n\right| \|x_{n-1} - x^*\|^2 + \alpha_n \left[1 + 2\beta_n (1 + u_n) (2 + u_n) + 2\beta_n c_n (1 + u_n) + 2u_n + 4c_n\right]$$

$$+ \alpha_n \left[1 + 2\beta_n (1 + u_n) (2 + u_n) + 2\beta_n c_n (1 + u_n) + 2u_n + 4c_n\right] \|x_n - x^*\|^2 + 4\alpha_n c_n + 2\alpha_n \beta_n c_n (1 + u_n).$$

Transposing and simplifying the above inequality yield

$$\|x_n - x^*\|^2 \leq (1 + \delta_n) \|x_{n-1} - x^*\|^2 + \xi_n, \quad (3.4)$$

where
\[ \delta_n = \frac{2\alpha_n\beta_n(1+u_n)(2+u_n+c_n)+2\alpha_n(u_n+2c_n)}{1-\alpha_n-2\alpha_n\beta_n(1+u_n)(2+u_n+c_n)-2\alpha_n(u_n+2c_n)}, \]

\[ \xi_n = \frac{4\alpha_n c_n+2\alpha_n\beta_n c_n(1+u_n)}{1-\alpha_n-2\alpha_n\beta_n(1+u_n)(2+u_n+c_n)-2\alpha_n(u_n+2c_n)}. \]

From the conditions of Theorem, it implies that \( \sum_{n=1}^{\infty} \delta_n < +\infty \)
and \( \sum_{n=1}^{\infty} \xi_n < +\infty. \) Consequently, by (3.4) and lemma 2.1, we obtain that \( \lim_{n \to \infty} \|x_n - x^*\| \) exists and \( \|x_n - x^*\| \) is bounded. From (3.4), we find
\[ d(x_n, F)^2 \leq (1 + \delta_n) d(x_{n-1}, F)^2 + \xi_n, \forall n \geq n_0. \] (3.5)
It follows from (3.5) and lemma 2.1 that \( \lim_{n \to \infty} d(x_n, F) \) exists. Furthermore, since \( \lim_{n \to \infty} d(x_n, F) = 0 \), we have
\[ \lim_{n \to \infty} d(x_n, F) = 0. \]

Next we will show that the sequence \( \{x_n\} \) is a Cauchy sequence in \( K \). Indeed, since
\[ \sum_{n=1}^{\infty} \delta_n < +\infty, 1 + t \leq \exp\{t\}, \text{ for all } t > 0 \text{ and } (3.4), \]
we discover
\[ \|x_n - x^*\|^2 \leq \exp\{\delta_n\}\|x_{n-1} - x^*\|^2 + \xi_n, \forall n \geq n_0. \] (3.6)
From (3.6), it implies that for any positive integers \( m, n \geq n_0, \)
\[ \|x_{n+m} - x^*\|^2 \leq \exp\left\{\sum_{i=n+1}^{n+m} \delta_i\right\}\|x_{n} - x^*\|^2 + \exp\left\{\sum_{i=n+1}^{n+m} \delta_i\right\}, \]
\[ \cdot \sum_{i=n+1}^{\infty} \xi_i \leq M \left(\|x_n - x^*\|^2 + \sum_{i=n+1}^{n+m} \xi_i\right), \]
where \( M = \exp\left\{\sum_{n=1}^{\infty} \delta_n\right\} < +\infty. \)

Since \( \lim_{n \to \infty} d(x_n, F) = 0 \) and \( \sum_{n=1}^{\infty} \xi_n < +\infty, \) there exists a positive integer \( n_1 \geq n_0 \) such that for any given \( \varepsilon > 0, \)
\[ d(x_n, F)^2 < \frac{\varepsilon^2}{8(M+1)}, \sum_{i=n+1}^{\infty} \xi_i < \frac{\varepsilon^2}{4M}, \forall n \geq n_1. \]
Hence there exists \( y^* \in F \) such that
Consequently, for all \( n \geq n_1 \) and for all \( m \geq 1 \), we have
\[
\left\| x_{n+m} - x_n \right\|^2 \leq 2\left( \left\| x_{n+m} - y^* \right\|^2 + \left\| x_n - y^* \right\|^2 \right) \leq \varepsilon^2.
\]
This implies that \( \{x_n\} \) is a cauchy sequence in \( K \). By the completeness of \( K \), we can assume that \( x_n \to z^* \in K \). It remains to show that \( z^* \in F \). Noting that
\[
\left| d(z^*, F) - d(x_n, F) \right| \leq \left\| z^* - x_n \right\|,
\]
for all \( n \). Since \( \lim_{n \to \infty} x_n = z^* \) and \( \lim_{n \to \infty} d(x_n, F) = 0 \), we conclude that \( z^* \in F \).

**Lemma 3.1.** Let \( X \) be a real uniformly convex Banach space and \( \{T_i : i \in \Lambda\} \) \( \Lambda \) uniformly L-lipschitzian, generalized asymptotically nonexpansive self-mappings of \( K \) with \( \{u_{in}\}, \{c_{in}\} \subseteq [0, +\infty) \) such that \( \sum_{n=1}^{\infty} u_{in} < +\infty \) and \( \sum_{n=1}^{\infty} c_{in} < +\infty \) for all \( i \in \Lambda \). Suppose
\[
F = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset.
\]
For any \( x_0 \in K \), the sequence \( \{x_n\} \) be generated by (I). Then

\[
\lim_{n \to \infty} \left\| x_n - T_i x_n \right\| = 0, \text{ for all } i \in \Lambda.
\]

**Proof.** By Theorem 3.1, we have \( \lim_{n \to \infty} \left\| x_n - x^* \right\| \) exists. Without loss of generality, we may assume that
\[
\lim_{n \to \infty} \left\| x_n - x^* \right\| = d, \text{ where } d \geq 0 \text{ is a real number. Obviously, } \{x_n\} \text{ is a bounded sequence.}
\]

Rewriting the limit
\[
\lim_{n \to \infty} \left\| x_n - x^* \right\| = \lim_{n \to \infty} \left\| (1-\alpha_n)(x_{n-1} - x^*) + \alpha_n (T^i y_n - x^*) \right\| = d. \text{ Again}
\]
\[
\left\| T^i y_n - x^* \right\| \leq (1-\beta_n)(1+u_n)\left\| x_n - x^* \right\| + \beta_n (1+u_n)\left\| y_n - x^* \right\| + (1+u_n)c_n.
\]
Therefore
\[
\limsup_{n \to \infty} \left\| T^i y_n - x^* \right\| \leq d.
\]

Lemma 2.2 implies
\[
\lim_{n \to \infty} \left\| x_{n-1} - T^i y_n \right\| = 0.
\]

By (I), we get
\[
\left\| y_n - x_n \right\| = \beta_n \left\| x_n - T^i y_n \right\| \leq \beta_n \left( (2+u_n)\left\| x_n - x^* \right\| + c_n \right) \to 0
\]
(as \( n \to \infty \)). Hence we obtain
\[
\left\| x_{n-1} - T^i x_n \right\| \leq \left\| x_{n-1} - T^i y_n \right\| + (1+u_n)\left\| y_n - x_n \right\| + c_n \to 0 \text{ (as } n \to \infty \)).
Further, we discover $\|x_n - x_{n-1}\| \leq \alpha_n \|x_{n+1} - T^k_{x_n}\| \to 0$ (as $n \to \infty$) and $\|x_n - x_{n+i}\| \to 0$ (as $n \to \infty$ and $i \in \Lambda$). Consequently, we have

$$\|x_{n-1} - T^k_{x_n}\| \leq \|x_{n-1} - T^k_{x_n}\| + L\|x_n - x_{n-N}\| + L\|x_n - x_{n-N}\| \to 0$$

($n \to \infty$),

where

$$i = n - (k - 1)N \quad \text{and} \quad n - N = (k - 2)N + i.$$

Therefore, we have

$$\|x_n - T^k_{x_n}\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T^k_{x_n}\| \to 0$$

(as $n \to \infty$). For all $j \in \Lambda$ and $i + j \leq N$, it implies that

$$\|x_n - T^k_{x_n}\| \leq \|x_n - x_{n-j}\| + \|x_{n-j} - T^k_{x_n}\| \to 0(n \to \infty).$$

Then for all $i \in \Lambda$, we conclude $\lim_{n \to \infty} \|x_n - T^k_{x_n}\| = 0$.

**Theorem 3.2.** Let $X$ be a real uniformly convex Banach space and $\{T_i : i \in \Lambda\}$ $N$ uniformly $L$-lipschitzian, generalized asymptotically nonexpansive self-mappings of $K$ with $\{u_{in}\}$, $\{c_{in}\} \subset [0, +\infty)$ such that $\sum_{n=1}^{\infty} u_{in} < +\infty$ and $\sum_{n=1}^{\infty} c_{in} < +\infty$ for all $i \in \Lambda$.

Suppose $F = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$ and $T$ satisfies Condition ($\overline{C}$). Let $\{\alpha_n\}$ and $\{\beta_n\}$ is as in Lemma 3.1.

Then the sequence $\{x_n\}$ by (1) converges strongly to a common fixed point of $\{T_i : i \in \Lambda\}$.

Proof. By Theorem 3.1, we obtain that $\lim_{n \to \infty} \|x_n - x^\star\|$ and $\lim_{n \to \infty} d(x_n, F)$ exists. Let one of $\{T_i : i \in \Lambda\}$, say $T_{r}, r \in \Lambda$ satisfies $\|x - T_{r}x\| \geq f(d(x, F))$. Also by Lemma 3.1, we have $\lim_{n \to \infty} \|x_n - T_{r}x_n\| = 0$, so we have $\lim_{n \to \infty} d(x_n, F) = 0$. Therefore Theorem 3.1 guarantees that $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in \Lambda\}$.

If we take $\beta_n \equiv 0$ for all $n \geq 1$ in Theorem 3.1 and Theorem 3.2, then we obtain:

**Corollary 3.1.** ([8, Theorem 3.3]) Let $X$, $\{T_i : i \in \Lambda\}$ and $\{\alpha_n\}$ be as the assumptions of Theorem 3.1. For any $x_0 \in K$, the sequence $\{x_n\}$ be given by (1.6). Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in \Lambda\}$ if and only if

$$\lim inf_{n \to \infty} d(x_n, F) = 0.$$

**Corollary 3.2.** ([8, Theorem 3.8]) Let $X$, $\{T_i : i \in \Lambda\}$ and $\{\alpha_n\}$ be as assumptions of Theorem 3.2. For any $x_0 \in K$, the sequence $\{x_n\}$ be defined by (1.6). Then $\{x_n\}$ converges strongly to a common fixed
point of \( \{ T_i : i \in \Lambda \} \).

If we take \( c_n = 0 \), for all \( n \geq 1 \), in (1.3), then we have:

**Theorem 3.3.** Let \( X \) be a real Banach space and \( \{ T_i : i \in \Lambda \} \) \( N \) asymptotically nonexpansive self-mappings of \( K \) with \( \{ u_{in} \} \subseteq [0, +\infty) \) such that \( \sum_{n=1}^{\infty} u_{in} < +\infty \) for all \( i \in \Lambda \). Suppose \( F = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset \).

Let \( \{ \alpha_n \} \), \( \{ \beta_n \} \) be two real sequences in \([0,1]\) satisfying the conditions:

\[
\limsup_{n \to \infty} \alpha_n < 1, \quad \text{and} \quad \sum_{n=1}^{\infty} \beta_n < +\infty.
\]

For any \( x_0 \in K \), the sequence \( \{ x_n \} \) is generated by (1). Then \( \{ x_n \} \) converges strongly to a common fixed point of \( \{ T_i : i \in \Lambda \} \) if and only if \( \liminf_{n \to \infty} d(x_n, F) = 0 \).

**Lemma 3.2.** Let \( X \) be a real uniformly convex Banach space and \( \{ T_i : i \in \Lambda \} \) be \( N \) asymptotically nonexpansive self-mappings of \( K \) with \( \{ u_{in} \} \subseteq [0, +\infty) \) such that \( \sum_{n=1}^{\infty} u_{in} < +\infty \) for all \( i \in \Lambda \).

Suppose \( F = \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset \). Let \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) is as in Lemma 3.1. For any \( x_0 \in K \), and the sequence \( \{ x_n \} \) is given by (1). Then \( \lim_{n \to \infty} \| x_n - T_i x_n \| = 0 \), for all \( i \in \Lambda \).

**Remark 3.1.** In contrast to the result in [11, Lemma 2.1], Lemma 3.2 give a right proof which the flaw of [11]. Furthermore, the conditions of sequences \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) be different from that of [11]. If we take \( \beta_n \equiv 0 \) for all \( n \geq 1 \), we obtain the corresponding results in [10].

**Acknowledgements**

The authors are grateful to an anonymous referee for careful reading of the manuscript and helpful suggestions.

**References**


