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# Confluent $q$-extensions of some classical determinants

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**Abstract**

We evaluate two determinants. The first is a $q, h$-extension of the classical confluent extension of the Vandermonde determinant. The second is a similar extension of Cauchy's double alternant.


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*Keywords:* Confluent Vandermonde determinants; Alternants; $q$-analogues

## 1. Introduction

We will generalize two celebrated determinants of Cauchy, the Vandermonde determinant [4]

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1\leqslant i<j\leqslant n} (x_j - x_i),$$

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and the double alternant [5]

$$\begin{vmatrix} (x_1-y_1)^{-1} & (x_1-y_2)^{-1} & (x_1-y_3)^{-1} & \dots & (x_1-y_n)^{-1} \\ (x_2-y_1)^{-1} & (x_2-y_2)^{-1} & (x_2-y_3)^{-1} & \dots & (x_2-y_n)^{-1} \\ (x_3-y_1)^{-1} & (x_3-y_2)^{-1} & (x_3-y_3)^{-1} & \dots & (x_3-y_n)^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (x_n-y_1)^{-1} & (x_n-y_2)^{-1} & (x_n-y_3)^{-1} & \dots & (x_n-y_n)^{-1} \end{vmatrix}$$

$$= \frac{\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)(y_i-y_j)}{\prod_{1\leqslant i,j\leqslant n}(x_i-y_j)} = (-1)^{\binom{n}{2}}\frac{\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)(y_j-y_i)}{\prod_{1\leqslant i,j\leqslant n}(x_i-y_j)}.$$

One can also insert derivatives of some rows into the Vandermonde determinant. For example,

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 0 & 1 & 2x_2 & 3x_2^2 & 4x_2^3 & 5x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \end{vmatrix} = 2(x_2-x_1)^6(x_3-x_1)^3(x_3-x_2)^2.$$

Perhaps better, we can let the $i$th entry in the $j$th row corresponding to the variable $x_k$ be $\binom{i-1}{j-1}x_k^{i-j}$; then the above example becomes

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 1 & 3x_1 & 6x_1^2 & 10x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 0 & 1 & 2x_2 & 3x_2^2 & 4x_2^3 & 5x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \end{vmatrix} = (x_2-x_1)^6(x_3-x_1)^3(x_3-x_2)^2.$$

Determinants of this type are often called confluent Vandermonde determinants. The most general one was evaluated by Schendel in 1891 [9], but a result nearly as good was given by Weihrauch in 1889 [10], and there is earlier work in this direction by Meray [7], among others. Of course, [8] is our source for all this historical information.

One may similarly extend Cauchy's double alternant. When each entry occurs to the first and second powers the result is due to Brioschi [3]; for example,

$$\begin{vmatrix} (x_1-y_1)^{-1} & (x_1-y_1)^{-2} & (x_1-y_2)^{-1} & (x_1-y_2)^{-2} \\ (x_2-y_1)^{-1} & (x_2-y_1)^{-2} & (x_2-y_2)^{-1} & (x_2-y_2)^{-2} \\ (x_3-y_1)^{-1} & (x_3-y_1)^{-2} & (x_3-y_2)^{-1} & (x_3-y_2)^{-2} \\ (x_4-y_1)^{-1} & (x_4-y_1)^{-2} & (x_4-y_2)^{-1} & (x_4-y_2)^{-2} \end{vmatrix}$$
$$= \frac{(x_2-x_1)(x_3-x_1)(x_4-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3)(y_2-y_1)^4}{\prod_{i=1}^4 (x_i-y_1)^2(x_i-y_2)^2}.$$

The analogous special case of the confluent Vandermonde determinant was a problem in the *American Mathematical Monthly* several years ago [1]. It dates back to Besso in 1882 [2].

In this paper we obtain $q, h$-analogues of these confluent extensions of Cauchy's determinants. The $h$ is in the sense of the calculus of finite differences (this may become clearer further on), and we now describe the $q$ aspect. The $q$-analogue of the number $k$ is

$$[k] := \begin{cases} \frac{1-q^k}{1-q} & \text{if } q \neq 1, \\ k & \text{if } q = 1. \end{cases}$$

If $k$ is a positive integer then

$$[k] = 1 + q + q^2 + \cdots + q^{k-1},$$

but in what follows $k$ will sometimes be a negative integer. Next, build $q$-factorials and $q$-binomial coefficients out of these in the obvious way: for nonnegative integers $n$ define the $q$-factorial by

$$n!_q := [1][2]\cdots[n], \quad \text{where } 0!_q := 1,$$

and the $q$-binomial coefficient as

$$\binom{n}{k}_q := \begin{cases} \frac{n!_q}{k!_q(n-k)!_q} & \text{if } n \text{ and } k \text{ are integers with } 0 \leqslant k \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

The recurrences

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q \tag{1.1}$$

$$= \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q \tag{1.2}$$

are well-known and easy to verify. We collect here some other simple facts which we will require:

$$\binom{i-2}{j-1}_q [i-2] - \binom{i-1}{j-1}_q [i-j-1] = q^{i-j-1} \binom{i-2}{j-2}_q, \tag{1.3}$$

$$\binom{i-1}{j-1}_q [i-j] = \binom{i-1}{j}_q [j], \tag{1.4}$$

$$[j-1]q^i - [i]q^{j-1} = q^i[j-i-1]. \tag{1.5}$$

## 2. A confluent $q, h$-Vandermonde determinant

For nonnegative integers $n$ define the polynomial $(x)_{n,h}$ by $(x)_{0,h} = 1$ and, for $n > 0$,

$$(x)_{n,h} = x(qx+h)(q^2x+[2]h)\cdots(q^{n-1}x+[n-1]h).$$

A more proper notation would be $(x)_{n,h,q}$, but $q$ will not vary, so we will suppress it. We will need one simple property of these polynomials:

$$q^{n-1}(x-h)(x)_{n-1,h} = (x-h)_{n,hq} \quad \text{if } n \geqslant 1. \tag{2.1}$$

Let $a_1, a_2, \ldots, a_m$ be nonnegative integers. We propose to evaluate a determinant of $a_1 + a_2 + \cdots + a_m$ rows, $a_i$ of which correspond to the variable $x_i$ for each $i$. (If $a_i = 0$ then $x_i$ does not appear; we henceforth assume that the $a_i$ are positive integers.) The $i$th entry in the $j$th $x_k$ row is

$$\binom{i-1}{j-1}_q (x_k)_{i-j,h}$$

for $1 \leqslant j \leqslant a_k$ and $1 \leqslant i \leqslant a_1 + \cdots + a_m$, where this means zero if $j > i$. We will denote this determinant by $V_{a_1,\ldots,a_m}(x_1, \ldots, x_m; h)$; again we suppress the dependence on $q$. For example, $V_{2,2}(x_1, x_2; h)$ is the determinant

$$\begin{vmatrix} 1 & x_1 & x_1(qx_1+h) & x_1(qx_1+h)(q^2x_1+[2]h) \\ 0 & 1 & [2]x_1 & [3]x_1(qx_1+h) \\ 1 & x_2 & x_2(qx_2+h) & x_2(qx_2+h)(q^2x_2+[2]h) \\ 0 & 1 & [2]x_2 & [3]x_2(qx_2+h) \end{vmatrix}.$$

Krattenthaler [6] has written an excellent survey of recent work on determinants, which in particular contains two $q$-confluent Vandermonde determinants, Theorems 23 and 24. His Theorem 23 is a generalization, in a different direction, of the case $h = 0$ of our result. As the referee points out, the parameter $h$ can be removed from our determinant by making the change of variables

$$x_i = \frac{(1+y_i)h}{1-q}$$

for each $i$; the determinant that results would be more or less the same as the case $h = 1$ of ours. But this reduction does not simplify the proof much, so we will not make it.

**Theorem 1.** *With the above notation*,

$$V_{a_1,a_2,\dots,a_m}(x_1, x_2, \dots, x_m; h) \\ = q^{e_3(a_1,\dots,a_m)} \prod_{\substack{1\leqslant k<\ell\leqslant m\\ 1\leqslant i\leqslant a_k\\ 1\leqslant j\leqslant a_\ell}} (q^{i-1}x_\ell - q^{j-1}x_k - q^{i-1}[j-i]h). \tag{2.2}$$

Here $e_3(a_1, \dots, a_m)$ is the elementary symmetric function of degree 3 in $a_1, \dots, a_m$; *i.e.*,

$$e_3(a_1, \dots, a_m) = \sum_{1\leqslant i<j<k\leqslant m} a_i a_j a_k,$$

which means zero if $m < 3$.

We will establish Theorem 1 by a lengthy series of column and row operations. Begin by subtracting $(q^{i-2}x_1 + [i-2]h)$ times the $i-1$st column from the $i$th column, from $i = a_1 + \cdots + a_m$ backwards to $i = 2$. This makes the first $x_1$ row 1 followed by all zeros. If $i > 1$, the $i$th entry in the $j$th $x_k$ row becomes

$$\binom{i-1}{j-1}_q (x_k)_{i-j,h} - \binom{i-2}{j-1}_q (x_k)_{i-j-1,h}\{q^{i-2}x_1 + [i-2]h\} \\ = (x_k)_{i-j-1,h} \left\{ \begin{array}{l} \binom{i-1}{j-1}_q (q^{i-j-1}x_k + [i-j-1]h) \\ -\binom{i-2}{j-1}_q (q^{i-2}x_1 + [i-2]h) \end{array} \right\}.$$

Using (1.3) this simplifies to

$$q^{i-j-1}(x_k)_{i-j-1,h} \left\{ \binom{i-1}{j-1}_q x_k - \binom{i-2}{j-1}_q q^{j-1}x_1 - \binom{i-2}{j-2}_q h \right\}. \tag{2.3}$$

In particular, if $k = 1$ then this simplifies further by (1.1) to

$$q^{i-j-1}(x_1)_{i-j-1,h} \binom{i-2}{j-2}_q (x_1 - h),$$

and still further by (2.1) to

$$\binom{i-2}{j-2}_q (x_1 - h)_{i-j,hq}. \tag{2.4}$$

Since the first row is now 1 followed by all zeros, we can cut off the first row and column without changing the determinant. What was the second column is now the first, and so forth, so we should increase $i$ by 1 in (2.3): the $i$th entry in the $j$th $x_k$ row, for $k > 1$, is now

$$q^{i-j}(x_k)_{i-j,h}\left\{\binom{i}{j-1}_q x_k - \binom{i-1}{j-1}_q q^{j-1}x_1 - \binom{i-1}{j-2}_q h\right\}. \tag{2.5}$$

If $k = 1$ the first $x_1$ row is also gone, so we should increase $i$ and $j$ by 1 in (2.4): the $i$th entry in the $j$th $x_1$ row is now

$$\binom{i-1}{j-1}_q (x_1 - h)_{i-j,hq}. \tag{2.6}$$

If we take $j = 1$ in (2.5) we get $q^{i-1}(x_k)_{i-1,h}(x_k - x_1)$, so that the first $x_k$ row has the factor $x_k - x_1$, for each $k > 1$. If we pull all those factors out, what remains of the $i$th entry in the first $x_k$ row is

$$q^{i-1}(x_k)_{i-1,h} = (qx_k)_{i-1,hq}.$$

If we subtract this from the second $x_k$ row (for all $k > 1$ such that there is a second $x_k$ row), we get

$$\begin{aligned} &q^{i-2}(x_k)_{i-2,h}\left\{\binom{i}{1}_q x_k - \binom{i-1}{1}_q qx_1 - h\right\} - q^{i-1}(x_k)_{i-1,h} \\ &\quad = q^{i-2}(x_k)_{i-2,h}\left\{\binom{i}{1}_q x_k - \binom{i-1}{1}_q qx_1 - h - q(q^{i-2}x_k + [i-2]h)\right\} \\ &\quad = [i-1]q^{i-2}(x_k)_{i-2,h}(x_k - qx_1 - h). \end{aligned}$$

So this row has the factor $x_k - qx_1 - h$, and if we pull it out what remains of the $i$th entry in the second $x_k$ row is

$$\binom{i-1}{1}_q q^{i-2}(x_k)_{i-2,h} = \binom{i-1}{1}_q (qx_k)_{i-2,hq}.$$

Continue in this fashion through all of the $a_k$ $x_k$ rows, for each $k > 1$. Assume inductively that the $i$th entry in the $j$th $x_k$ row becomes

$$\binom{i-1}{j-1}_q q^{i-j}(x_k)_{i-j,h}(x_k - q^{j-1}x_1 - [j-1]h), \tag{2.7}$$

and pull out the factor $x_k - q^{j-1}x_1 - [j-1]h$, so that what remains of the $i$th entry in the $j$th $x_k$ row is

$$\binom{i-1}{j-1}_q q^{i-j}(x_k)_{i-j,h} = \binom{i-1}{j-1}_q (qx_k)_{i-j,hq}. \tag{2.8}$$

Subtract this from (2.5) for all $k > 1$ such that there is a $j + 1$st $x_k$ row. This gives

$$q^{i-j-1}(x_k)_{i-j-1,h}\left\{\binom{i}{j}_q x_k - \binom{i-1}{j}_q q^j x_1 - \binom{i-1}{j-1}_q h\right\}$$
$$-\binom{i-1}{j-1}_q q^{i-j}(x_k)_{i-j,h}$$
$$= q^{i-j-1}(x_k)_{i-j-1,h}\left\{\begin{array}{l}\binom{i}{j}_q x_k - \binom{i-1}{j}_q q^j x_1 - \binom{i-1}{j-1}_q h \\ \quad -\binom{i-1}{j-1}_q q(q^{i-j-1}x_k + [i-j-1]h)\end{array}\right\}$$
$$= q^{i-j-1}(x_k)_{i-j-1,h}\left\{\begin{array}{l}\left(\binom{i}{j}_q - q^{i-j}\binom{i-1}{j-1}_q\right) x_k - \binom{i-1}{j}_q q^j x_1 \\ \quad -\binom{i-1}{j-1}_q h(1+q[i-j-1])\end{array}\right\}.$$

Using (1.2), (1.4), and (1.5), this simplifies to

$$\binom{i-1}{j}_q q^{i-j-1}(x_k)_{i-j-1,h}(x_k - q^j x_1 - [j]h),$$

which is (2.7) with $j+1$ in place of $j$. Thus (2.7) holds by induction.

For each $k > 1$, the factors that come out in this reduction are

$$\prod_{j=1}^{a_k}(x_k - q^{j-1}x_1 - [j-1]h).$$

From this, (2.6) and (2.8) we have the functional equation

$$\begin{aligned} V_{a_1,a_2,\ldots,a_m}(x_1,x_2,\ldots,x_m;h) &= \prod_{k=2}^{n}\prod_{j=1}^{a_k}(x_k - q^{j-1}x_1 - [j-1]h) \\ &\quad \times V_{a_1-1,a_2,\ldots,a_m}(x_1-h, qx_2,\ldots,qx_m; qh). \end{aligned} \tag{2.9}$$

It is easy to show by induction, with the aid of (1.5), that (2.9) implies

$$\begin{aligned} V_{a_1,a_2,\ldots,a_m}(x_1,x_2,\ldots,x_m;h) &= V_{a_1-i,a_2,\ldots,a_m}(x_1-[i]h, q^i x_2,\ldots,q^i x_m; q^i h) \\ \times \prod_{\substack{2\leqslant \ell\leqslant m\\ 1\leqslant j\leqslant a_\ell}} &\left\{\begin{array}{l}(x_\ell - q^{j-1}x_1 - [j-1]h)(qx_\ell - q^{j-1}x_1 - q[j-2]h)\times\cdots \\ \quad \times(q^{i-1}x_\ell - q^{j-1}x_1 - q^{i-1}[j-i]h)t\end{array}\right\} \end{aligned}$$

for $1 \leqslant i \leqslant a_1$, and hence

$$\begin{aligned} V_{a_1,a_2,\ldots,a_m}(x_1,x_2,\ldots,x_m;h) &= V_{a_2,\ldots,a_m}(q^{a_1}x_2,\ldots,q^{a_1}x_m; q^{a_1}h) \\ &\quad \times \prod_{\substack{2\leqslant \ell\leqslant m\\ 1\leqslant i\leqslant a_1\\ 1\leqslant j\leqslant a_\ell}} (q^{i-1}x_\ell - q^{j-1}x_1 - q^{i-1}[j-i]h). \end{aligned}$$

Using this on itself,

$$
\begin{aligned}
&V_{a_2,\dots,a_m}(q^{a_1}x_2,\dots,q^{a_1}x_m;q^{a_1}h)\\
&\quad = V_{a_3,\dots,a_m}(q^{a_1+a_2}x_3,\dots,q^{a_1+a_2}x_m;q^{a_1+a_2}h)\\
&\qquad \times \prod_{\substack{3\leqslant \ell\leqslant m\\ 1\leqslant i\leqslant a_2\\ 1\leqslant j\leqslant a_\ell}} (q^{i-1}q^{a_1}x_\ell - q^{j-1}q^{a_1}x_2 - q^{i-1}[j-i]q^{a_1}h).
\end{aligned}
$$

Since there is a factor of $q^{a_1}$ inside the product, we may rewrite this as

$$
\begin{aligned}
&V_{a_2,\dots,a_m}(q^{a_1}x_2,\dots,q^{a_1}x_m;q^{a_1}h)\\
&\quad = \prod_{\ell=3}^{m} q^{a_1a_2a_\ell}\prod_{i=1}^{a_2}\prod_{j=1}^{a_\ell}(q^{i-1}x_\ell - q^{j-1}x_2 - q^{i-1}[j-i]h)\\
&\qquad \times V_{a_3,\dots,a_m}(q^{a_1+a_2}x_3,\dots,q^{a_1+a_2}x_m;q^{a_1+a_2}h).
\end{aligned}
\tag{2.10}
$$

Using (2.10) again,

$$
\begin{aligned}
&V_{a_3,\dots,a_m}(q^{a_1+a_2}x_3,\dots,q^{a_1+a_2}x_m;q^{a_1+a_2}h)\\
&\quad = \prod_{\ell=4}^{m}\prod_{i=1}^{a_3}\prod_{j=1}^{a_\ell}(q^{i-1}q^{a_1+a_2}x_\ell - q^{j-1}q^{a_1+a_2}x_3 - q^{i-1}[j-i]q^{a_1+a_2}h)\\
&\qquad \times V_{a_4,\dots,a_m}(q^{a_1+a_2+a_3}x_4,\dots,q^{a_1+a_2+a_3}x_m;q^{a_1+a_2+a_3}h),
\end{aligned}
$$

and again we can pull out a power of $q$:

$$
\begin{aligned}
&V_{a_3,\dots,a_m}(q^{a_1+a_2}x_3,\dots,q^{a_1+a_2}x_m;q^{a_1+a_2}h)\\
&\quad = \prod_{\ell=4}^{m} q^{(a_1+a_2)a_3a_\ell}\prod_{i=1}^{a_3}\prod_{j=1}^{a_\ell}(q^{i-1}x_\ell - q^{j-1}x_3 - q^{i-1}[j-i]h)\\
&\qquad \times V_{a_4,\dots,a_m}(q^{a_1+a_2+a_3}x_4,\dots,q^{a_1+a_2+a_3}x_m;q^{a_1+a_2+a_3}h).
\end{aligned}
$$

We see the elementary symmetric function of degree 3 starting to show up in the exponent of $q$, and by repeated use of (2.10) we eventually get Theorem 1. Let us give two examples. Since $e_3(2,2)=0$,

$$
V_{2,2}(x_1,x_2;h) = \prod_{\substack{1\leqslant k<\ell\leqslant 2\\ 1\leqslant i\leqslant a_k\\ 1\leqslant j\leqslant a_\ell}} (q^{i-1}x_\ell - q^{j-1}x_k - q^{i-1}[j-i]h),
$$

where $a_1=2$ and $a_2=2$, and after a little reduction this becomes

$$
V_{2,2}(x_1,x_2;h) = q(x_2-x_1)^2(qx_2-x_1+h)(x_2-qx_1-h).
$$

A slightly more complex example is $V_{3,2,1}(x_1, x_2, x_3; h)$. Here $e_3(3, 2, 1) = 3 \cdot 2 \cdot 1 = 6$, and (2.2) becomes

$$\begin{aligned} V_{3,2,1}(x_1, x_2, x_3; h) = q^8(x_2 - x_1)^2(qx_2 - x_1 + h)^2(q^2x_2 - x_1 + [2]h) \\ \times(x_3 - x_1) \times (x_2 - qx_1 - h)(x_3 - x_2) \\ \times(qx_3 - x_1 + h)(qx_3 - x_2 + h)(q^2x_3 - x_1 + [2]h) \end{aligned}$$

after a little reduction.

One might hope for a common generalization of Theorem 1 and Krattenthaler's Theorem 23 [6], in which one would define

$$\binom{C}{k}_q = \frac{[C][C-1]\cdots[C-k+1]}{k!_q}$$

for a nonnegative integer $k$ and an *arbitrary* parameter $C$. The definition of $(x)_{n,h}$ extends nicely to negative integer $n$ by requiring (2.1) to hold for all integers, and one could then consider the determinant which would have

$$\binom{C+i-1}{j-1}_q (x_k)_{i-j,h}$$

as the $i$th entry in the $j$th $x_k$ row. But I have not been able to evaluate it.

## 3. A confluent $q$-double alternant

For nonnegative integers $n$ define the polynomial $c_n(x, y; h)$ by

$$c_n(x, y; h) = (x - y)(x - qy - h)(x - q^2y - [2]h)\cdots(x - q^{n-1}y - [n-1]h),$$

where $c_0(x, y; h) := 1$. We will actually work with

$$c_n(x, y) := c_n(x, y; 0) = (x - y)(x - qy)(x - q^2y)\cdots(x - q^{n-1}y).$$

As the referee points out, there is no loss of generality in doing so since $c_n(u, v; h)$ reduces to $c_n(x, y)$ after the substitutions

$$u = x + \frac{h}{1-q} \quad \text{and} \quad v = y + \frac{h}{1-q}. \tag{3.1}$$

The simple property $(x - y)c_n(x, qy) = c_{n+1}(x, y)$ will be used below.

Let $a_1, a_2, \ldots, a_m$ be nonnegative integers whose sum is $s$. We will evaluate a determinant of $s$ columns, $a_j$ of which correspond to the variable $y_j$ for each $i$. (If $a_j = 0$ then $y_j$ does not appear; we henceforth assume that the $a_j$ are positive integers.) The $i$th entry in the $n$th $y_j$ column is the *reciprocal* of $c_n(x_i, y_j)$ for $1 \leqslant n \leqslant a_j$, $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant s$. We will denote this determinant by $C_{a_1,\ldots,a_m}(x_1, \ldots, x_s; y_1, \ldots, y_m)$, suppressing the dependence on $q$. For example, $C_{2,2}(x_1, x_2, x_3, x_4; y_1, y_2)$ is the determinant

$$\begin{vmatrix} (x_1-y_1)^{-1} & \{(x_1-y_1)(x_1-qy_1)\}^{-1} & (x_1-y_2)^{-1} & \{(x_1-y_2)(x_1-qy_2)\}^{-1} \\ (x_2-y_1)^{-1} & \{(x_2-y_1)(x_2-qy_1)\}^{-1} & (x_2-y_2)^{-1} & \{(x_2-y_2)(x_2-qy_2)\}^{-1} \\ (x_3-y_1)^{-1} & \{(x_3-y_1)(x_3-qy_1)\}^{-1} & (x_3-y_2)^{-1} & \{(x_3-y_2)(x_3-qy_2)\}^{-1} \\ (x_4-y_1)^{-1} & \{(x_4-y_1)(x_4-qy_1)\}^{-1} & (x_4-y_2)^{-1} & \{(x_4-y_2)(x_4-qy_2)\}^{-1} \end{vmatrix}$$
$$= \frac{(y_2-y_1)(y_2-qy_1)(qy_2-y_1)(qy_2-qy_1)\prod_{1\leqslant i<j\leqslant 4}(x_j-x_i)}{\prod_{i=1}^{4}(x_i-y_1)(x_i-qy_1)(x_i-y_2)(x_i-qy_2)}.$$

**Theorem 2.** *With the above notation*,

$$C_{a_1,\ldots,a_m}(x_1,\ldots,x_s;y_1,\ldots,y_m)$$
$$= (-1)^{\binom{s}{2}} \frac{\left(\prod_{1\leqslant i<j\leqslant s}(x_j-x_i)\right)\left(\prod_{\substack{1\leqslant k<\ell\leqslant m\\ 1\leqslant i\leqslant a_k\\ 1\leqslant j\leqslant a_\ell}}(q^{j-1}y_\ell-q^{i-1}y_k)\right)}{\prod_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant m}}(x_i-y_j)(x_i-qy_j)\cdots(x_i-q^{a_j-1}y_j)}. \tag{3.2}$$

Again our evaluation proceeds by a long series of row and column operations. We first subtract the $s$th (last) row from all the others. The $i$th entry in the $n$th $y_j$ column becomes

$$\frac{(x_s-y_j)\cdots(x_s-q^{n-1}y_j)-(x_i-y_j)\cdots(x_i-q^{n-1}y_j)}{(x_i-y_j)\cdots(x_i-q^{n-1}y_j)(x_s-y_j)\cdots(x_s-q^{n-1}y_j)}$$

for $1\leqslant n\leqslant a_j$, $1\leqslant j\leqslant m$ and $1\leqslant i\leqslant s-1$. Each column now has the common factor

$$\frac{1}{c_n(x_s,y_j)} = \frac{1}{(x_s-y_j)(x_s-qy_j)\cdots(x_s-q^{n-1}y_j)},$$

and we take all these factors out of the determinant, so the $i^{\text{th}}$ entry in the $n$th $y_j$ column is now

$$\frac{(x_s-y_j)\cdots(x_s-q^{n-1}y_j)-(x_i-y_j)\cdots(x_i-q^{n-1}y_j)}{(x_i-y_j)(x_i-qy_j)\cdots(x_i-q^{n-1}y_j)}$$

for $1\leqslant n\leqslant a_j$, $1\leqslant j\leqslant m$ and $1\leqslant i\leqslant s-1$, and the last row now has all 1's. Next, we subtract the next-to-last $y_j$ column from the last one for each $j$, then the third-to-last $y_j$ column from the next-to-last, and so forth; finally subtracting the first $y_j$ column from the second. The $i$th entry in the $n$th $y_j$ column becomes $(c_n(x_i,y_j))^{-1}$ times

$$(x_s-y_j)(x_s-qy_j)\cdots(x_s-q^{n-1}y_j)-(x_i-y_j)(x_i-qy_j)\cdots(x_i-q^{n-1}y_j)$$
$$-(x_i-q^{n-1}y_j)\{(x_s-y_j)\cdots(x_s-q^{n-2}y_j)-(x_i-y_j)\cdots(x_i-q^{n-2}y_j)\}$$

for $2 \leqslant n \leqslant a_j$, $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant s-1$; the last row now has a 1 in the first $y_j$ column for each $j$, and zeros otherwise. Two of the four terms here cancel. Combining the other two we get

$$(x_s - y_j)(x_s - qy_j)\cdots(x_s - q^{n-2}y_j)\{x_s - q^{n-1}y_j - (x_i - q^{n-1}y_j)\},$$

and therefore the $i$th entry in the $n$th $y_j$ column is now

$$\frac{(x_s - x_i)(x_s - y_j)(x_s - qy_j)\cdots(x_s - q^{n-2}y_j)}{(x_i - y_j)(x_i - qy_j)\cdots(x_i - q^{n-1}y_j)}$$

for $2 \leqslant n \leqslant a_j$, $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant s-1$. In fact the first $y_j$ column also has this form, for $n = 1$. Therefore the $i$th row has the factor $x_s - x_i$ for $1 \leqslant i \leqslant s-1$, and we take all these out. For each $j$ and for each $n \geqslant 2$ we can also take out

$$(x_s - y_j)(x_s - qy_j)\cdots(x_s - q^{n-2}y_j).$$

These factors cancel most of the ones we took out earlier, leaving only one copy of each factor in the denominator. Thus we have pulled out

$$\frac{(x_s - x_1)(x_s - x_2)\cdots(x_s - x_{s-1})}{\prod_{j=1}^{m}(x_s - y_j)\cdots(x_s - q^{a_j-1}y_j)} = \frac{\prod_{i=1}^{s-1}(x_s - x_i)}{\prod_{j=1}^{m} c_{a_j}(x_s, y_j)} \tag{3.3}$$

so far. The determinant that remains is the same as the one we started with, except for the last row, where we have 1's in the first $y_j$ column for each $j$ and 0's otherwise. To make the last row 1 followed by zeros we subtract the first $y_1$ column from the first $y_j$ column for each $j$, $2 \leqslant j \leqslant m$. The other entries in these columns become

$$\frac{1}{x_i - y_j} - \frac{1}{x_i - y_1} = \frac{y_j - y_1}{(x_i - y_1)(x_i - y_j)}$$

for $1 \leqslant i \leqslant s-1$ and $2 \leqslant j \leqslant m$. We can factor $y_j - y_1$ out of the determinant for each of these $j$. We would like also to factor out $(x_i - y_1)^{-1}$, but this would only be possible if every $a_j = 1$. For any $j$ for which $a_j \geqslant 2$, we need to subtract the new first $y_j$ column from the second $y_j$ column. The $i$th entry in the second $y_j$ column becomes

$$\frac{1}{(x_i - y_j)(x_i - qy_j)} - \frac{1}{(x_i - y_1)(x_i - y_j)} = \frac{qy_j - y_1}{(x_i - y_1)(x_i - y_j)(x_i - qy_j)}$$

for $1 \leqslant i \leqslant s-1$ and all $j \geqslant 2$ such that $a_j \geqslant 2$. We can pull out the factor $qy_j - y_1$, and then subtract the new second $y_j$ column from the third $y_j$ column whenever there is one. The third $y_j$ column then becomes

$$\frac{1}{(x_i - y_j)(x_i - qy_j)(x_i - q^2y_j)} - \frac{1}{(x_i - y_1)(x_i - y_j)(x_i - qy_j)}$$
$$= \frac{q^2y_j - y_1}{(x_i - y_1)(x_i - y_j)(x_i - qy_j)(x_i - q^2y_j)}$$

for $1 \leqslant i \leqslant s-1$ and all $j \geqslant 2$ such that $a_j \geqslant 3$, and again we can pull out a numerator factor. Proceed in this way until the $a_j$th column is reached for every $j$. By this time the $i$th row has $(x_i - y_1)^{-1}$ as a factor for $1 \leqslant i \leqslant s-1$, so pull all those factors out. The last row is 1 followed by all zeros, so we expand on this row, getting a factor of $(-1)^{s+1}$ from the only nonzero term. The new determinant is exactly like the previous one, except that the first column and last row have been cut off. If $a_1 = 1$ then the new determinant would be $C_{a_2,\dots,a_m}(x_1, \dots, x_{s-1}; y_2, \dots, y_m)$, but in general it is $C_{a_1-1,a_2,\dots,a_m}(x_1, \dots, x_{s-1}; qy_1, y_2, \dots, y_m)$. The factors that we have pulled out so far are the ones in (3.3), and more recently

$$(-1)^{s+1} \frac{\prod_{j=2}^{m}(y_j - y_1)(qy_j - y_1)\cdots(q^{a_j-1}y_j - y_1)}{(x_1 - y_1)(x_2 - y_1)\cdots(x_{s-1} - y_1)}. \tag{3.4}$$

Noting that

$$(y_j - y_1)(qy_j - y_1)\cdots(q^{a_j-1}y_j - y_1) = (-1)^{a_j} c_{a_j}(y_1, y_j),$$

we see that the factors we have so far are

$$(-1)^{s-1} \frac{\left(\prod_{i=1}^{s-1}(x_s - x_i)\right)\left(\prod_{\substack{2\leqslant \ell\leqslant m\\ 1\leqslant j\leqslant a_\ell}}(q^{j-1}y_\ell - y_1 + [j-1]h)\right)}{(x_1 - y_1)(x_2 - y_1)\dots(x_{s-1} - y_1)\prod_{j=1}^{m} c_{a_j}(x_s, y_j)}. \tag{3.5}$$

In other words, if we temporarily set $X$ equal to the quantity in (3.5), then we have the functional equation

$$\begin{aligned} &C_{a_1,a_2,\dots,a_m}(x_1, \dots, x_{s-1}, x_s; y_1, y_2, \dots, y_m) \\ &\quad = X \cdot C_{a_1-1,a_2,\dots,a_m}(x_1, \dots, x_{s-1}; qy_1, y_2, \dots, y_m). \end{aligned} \tag{3.6}$$

As with (2.8), we can solve (3.6) by iteration. If we use (3.6) on itself $a_1$ times to eliminate $y_1$ we find that

$$\begin{aligned} &C_{a_1,a_2,\dots,a_m}(x_1, \dots, x_s; y_1, y_2, \dots, y_m) \\ &\quad = (-1)^{sa_1 - \binom{a_1+1}{2}} \\ &\qquad \times \frac{\left(\prod_{\substack{s-a_1+1\leqslant j\leqslant s\\ 1\leqslant i<j}}(x_j - x_i)\right)\left(\prod_{\substack{2\leqslant \ell\leqslant m\\ 1\leqslant j\leqslant a_\ell\\ 1\leqslant i\leqslant a_1}}(q^{j-1}y_\ell - q^{i-1}y_1)\right)}{\left(\prod_{i=1}^{s} c_{a_1}(x_i, y_1)\right)\left(\prod_{\substack{s-a_1+1\leqslant i\leqslant s\\ 2\leqslant j\leqslant m}} c_{a_j}(x_i, y_j)\right)} \\ &\qquad \times C_{a_2,\dots,a_m}(x_1, \dots, x_{s-a_1}; y_2, \dots, y_m). \end{aligned} \tag{3.7}$$

Rather than inflicting any further iteration on the reader, we complete the proof of Theorem 2 by showing that the right side of (3.2) satisfies (3.7), for then we could use induction from $C_0 = 1$. More precisely, we will show that if we divide the right side of (3.2) by what it would be with $a_1$ and $y_1$ deleted and $s$ replaced by $s - a_1$, then we get the factors in (3.7).

We look at each of the four constituents of (3.7) in turn. The powers of $-1$ are

$$(-1)^{\binom{s}{2}-\binom{s-a_1}{2}},$$

and these work out since

$$\binom{s}{2}-\binom{s-a_1}{2}=sa_1-\binom{a_1+1}{2}.$$

The Vandermonde factor of the $x_i$'s gives

$$\frac{\prod_{1\leqslant i<j\leqslant s}(x_j-x_i)}{\prod_{1\leqslant i<j\leqslant s-a_1}(x_j-x_i)}=\prod_{\substack{s-a_1+1\leqslant j\leqslant s\\ 1\leqslant i<j}}(x_j-x_i).$$

The denominator factors are

$$\frac{\left(\prod_{\substack{1\leqslant i\leqslant s-a_1\\ 2\leqslant j\leqslant m}} c_{a_j}(x_i,y_j)\right)\left(\prod_{\substack{s-a_1+1\leqslant i\leqslant s\\ 2\leqslant j\leqslant m}} c_{a_j}(x_i,y_j)\right)\left(\prod_{i=1}^{s}c_{a_1}(x_i,y_1)\right)}{\prod_{\substack{1\leqslant i\leqslant s-a_1\\ 2\leqslant j\leqslant m}} c_{a_j}(x_i,y_j)}$$

$$=\left(\prod_{\substack{s-a_1+1\leqslant i\leqslant s\\ 2\leqslant j\leqslant m}} c_{a_j}(x_i,y_j)\right)\left(\prod_{i=1}^{s}c_{a_1}(x_i,y_1)\right).$$

Finally, the $y_j$ numerator factors become

$$\frac{\prod_{\substack{1\leqslant k<\ell\leqslant m\\ 1\leqslant i\leqslant a_k\\ 1\leqslant j\leqslant a_\ell}}(q^{j-1}y_\ell-q^{i-1}y_k)}{\prod_{\substack{2\leqslant k<\ell\leqslant m\\ 1\leqslant i\leqslant a_k\\ 1\leqslant j\leqslant a_\ell}}(q^{j-1}y_\ell-q^{i-1}y_k)}=\prod_{\substack{2\leqslant \ell\leqslant m\\ 1\leqslant i\leqslant a_1\\ 1\leqslant j\leqslant a_\ell}}\left(q^{j-1}y_\ell-q^{i-1}y_1\right).$$

Thus we get precisely the factors we see in (3.7), and this proves Theorem 2.

If we denote by $C_{a_1,a_2,\ldots,a_m}(x_1,\ldots,x_s;y_1,y_2,\ldots,y_m;h)$ the corresponding determinant with the $i$th entry in the $n^{\text{th}}$ $y_j$ column being the reciprocal of $c_n(x_i,y_j;h)$ for $1\leqslant n\leqslant a_j$, $1\leqslant j\leqslant m$, then by the referee's remark (3.1) we have

**Theorem 3.** *With the above notation*,

$$\begin{aligned}&C_{a_1,\ldots,a_m}(x_1,\ldots,x_s;y_1,\ldots,y_m;h)\\&\quad=(-1)^{\binom{s}{2}}\\&\qquad\times\frac{\left(\prod_{1\leqslant i<j\leqslant s}(x_j-x_i)\right)\left(\prod_{\substack{1\leqslant k<\ell\leqslant m\\ 1\leqslant i\leqslant a_k\\ 1\leqslant j\leqslant a_\ell}}(q^{j-1}y_\ell-q^{i-1}y_k+q^{i-1}[j-i]h)\right)}{\prod_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant m}}(x_i-y_j)(x_i-qy_j-h)\cdots(x_i-q^{a_j-1}y_j-[a_j-1]h)}.\end{aligned}$$

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