Theory of Generalized Hermite Polynomials

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Abstract—We introduce multivariable generalized forms of Hermite polynomials and analyze both the Gould-Hopper type polynomials and more general forms, which are analogues of the classical orthogonal polynomials, since they represent a basis in $L^2(\mathbb{R}^N)$ Hilbert space, suitable for series expansion of square summable functions of $N$ variables. Moreover, the role played by these generalized Hermite polynomials in the solution of evolution-type differential equations is investigated. The key-note of the method leading to the multivariable polynomials is the introduction of particular generating functions, following the same criteria underlying the theory of multivariable generalized Bessel functions.

1. INTRODUCTION

The theory of Generalized Hermite Polynomials (GHP) of Gould-Hopper [1] has been recently reformulated in terms of that of Generalized Bessel Functions (GBF) [2]. Many properties of these polynomials have been straightforwardly derived within this new framework, which has allowed the possibility of introducing multivariable ($>2$) GHP.

In this paper, we study the properties of GHP from a different point of view, starting from a generalization of the generating function which defines ordinary Hermite polynomials (HP). The method is, however, reminiscent of that leading to the notion of multivariable GBF and provides polynomials of the Hermite type, which can be considered a further generalization of the case already considered by Gould and Hopper. Moreover, with this procedure, it is possible to define a new class of multivariable generalized Hermite polynomials, which cannot be reduced to the ordinary forms by means of addition formulae, and represent a complete orthonormal set in the $L^2(\mathbb{R}^N)$ space of square summable functions of $N$ variables.

In these introductory remarks, we discuss the properties of the GHP defined in [1] and fix the notation: in order to make the paper self-consistent, we recall [3] the generating function of...
ordinary HP, \( H_n(x) \),

\[
\exp \left( 2x t - t^2 \right) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!},
\]

and write explicitly its series expansion

\[
\exp \left( 2x t - t^2 \right) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+r} (2x)^r}{r! m!}.
\]

Setting \( r + 2m = n \) and rearranging the summation, we get

\[
\exp \left( 2x t - t^2 \right) = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (2x)^{n-2m}}{(n-2m)! m!},
\]

so that the series defining HP is immediately achieved:

\[
H_n(x) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (2x)^{n-2m}}{(n-2m)! m!},
\]

where \( \lfloor \frac{n}{2} \rfloor \) is the truncated part of \( \frac{n}{2} \).

A preliminary example of generating function leading to GHP of the Gould-Hopper type is the following [2]:

\[
\exp(2x t - y t^2) = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}, \quad \text{and}
\]

\[
H_n(x, y) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-y)^m (2x)^{n-2m}}{(n-2m)! m!}.
\]

It is also worth stressing that \( H_n(x, y) \) can be expanded in terms of ordinary HP,

\[
H_n(x, y) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} H_{n-2r}(x) \frac{(1-y)^r}{(n-2r)! r!},
\]

and the following properties are easily inferred:

\[
\frac{\partial}{\partial x} H_n(x, y) = 2n H_{n-1}(x, y), \quad (n \geq 1),
\]

\[
\frac{\partial}{\partial y} H_n(x, y) = -n (n - 1) H_{n-2}(x, y), \quad (n \geq 2),
\]

\[
H_n(x, y) = 2x H_{n-1}(x, y) - 2(n - 1) y H_{n-2}(x, y), \quad (n \geq 2),
\]

with \( H_0(x, y) = 1 \) and \( H_1(x, y) = 2x \).

More generally, the first-kind Gould-Hopper polynomials arise from the generating function

\[
\exp \left( x t + y t^m \right) = \sum_{n=0}^{\infty} \frac{H_n^{(m)}(x, y)}{n!} t^n, \quad \text{where}
\]

\[
H_n^{(m)}(x, y) = \sum_{k=0}^{\lfloor n/m \rfloor} n! \frac{1}{k! (n-m-k)!} y^k x^{n-m-k}, \quad (n, m \geq 0),
\]
and their recurrence properties are:

\[ \frac{\partial}{\partial x} H_n^{(m)}(x, y) = n H_{n-1}^{(m)}(x, y), \quad (n \geq 1), \]

\[ \frac{\partial}{\partial y} H_n^{(m)}(x, y) = \frac{n!}{(n-m)!} H_{n-m}^{(m)}(x, y), \quad (n \geq m), \tag{9} \]

\[ H_n^{(m)}(x, y) = x H_n^{(m)}(x, y) + m y \frac{(n-1)!}{(n-m)!} H_n^{(m)}(x, y), \quad (n \geq m). \]

It is worth pointing out that the following identity derives from the first two equations in (7)

\[ \frac{\partial}{\partial y} H_n(x, y) = -\frac{1}{4} \frac{\partial^2}{\partial x^2} H_n(x, y), \tag{10} \]

and, therefore, one recovers the important relation

\[ H_n(x, y) = \exp \left( -\frac{y}{4} \frac{\partial^2}{\partial x^2} \right) H_n(x, 0). \tag{11a} \]

Now, since we have, from (5b),

\[ H_n(x, 0) = (2x)^n, \tag{11b} \]

we end up with

\[ H_n(x, y) = \left[ \sum_{m=0}^{[n/2]} \frac{(-1)^m}{m!} \left( \frac{y}{4} \right)^m \frac{\partial^{2m}}{\partial x^{2m}} \right] (2x)^n, \tag{11c} \]

which can be viewed as an alternative to Rodriguez formula [3]. More generally, from (9), it follows that

\[ \frac{\partial}{\partial y} H_n^{(m)}(x, y) = \frac{\partial^m}{\partial x^m} H_n^{(m)}(x, y), \quad (n > m > 0), \quad \text{and thus,} \tag{12a} \]

\[ H_n^{(m)}(x, y) = \left[ \sum_{r=0}^{[n/m]} \frac{1}{r!} y^r \left( \frac{\partial}{\partial x} \right)^m \right] x^n. \tag{12b} \]

A simple example of the addition theorem will give a significant insight into the formalism we have just described. Starting from the generating function

\[ \exp \left[ 2(x + y) t - 2t^2 \right] = G(x, y; t), \tag{13} \]

and using the various definitions of Hermite polynomials so far introduced, we find

\[ G(x, y; t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x + y, 2), \quad \text{and also}, \tag{14a} \]

\[ G(x, y; t) = \sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} \frac{H_{n-r}(x) H_r(y)}{(n-r)! r!}, \tag{14b} \]

thus getting the following addition theorem

\[ H_n(x + y, 2) = \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x) H_r(y), \tag{14c} \]

whose r.h.s. coincides with that of the known addition formula for the usual HP (see Section 5 for further details and connections) and, furthermore,

\[ H_n(x + y + z, 3) = \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x + y, 2) H_r(z), \tag{14d} \]

or, in general form,

\[ H_n \left( \sum_{s=1}^{M} x_s, M \right) = \sum_{r=0}^{n} \binom{n}{r} H_{n-r} \left( \sum_{s=1}^{M-1} x_s, M - 1 \right) H_r(x_M). \tag{14e} \]
2. MULTIVARIABLE HERMITE POLYNOMIALS

We define a new class of two-variable GHP specified by the following generating function

\[
\exp \left( 2x t - t^2 + 2y t^2 - t^4 \right) = \exp \left[ 2x t + (2y - 1) t^2 - t^4 \right] = \sum_{n=0}^{\infty} t^n \frac{(2)H_n(x, y)}{n!}.
\]  

(15)

These polynomials, \((2)H_n(x, y)\), can be expressed in terms of the series

\[
(2)H_n(x, y) = n! \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{H_{n-2r}(x) H_r(y)}{(n - 2r)! r!} ,
\]

(16)

and the related recurrence relations are straightforwardly derived

\[
\frac{\partial}{\partial x} (2)H_n(x, y) = 2n (2)H_{n-1}(x, y),
\]

\(n \geq 1),

(17a)

\[
\frac{\partial}{\partial y} (2)H_n(x, y) = 2n (n - 1) (2)H_{n-2}(x, y),
\]

\(n \geq 2),

(17b)

\[
\frac{1}{2} (2)H_n(x, y) = x (2)H_{n-1}(x, y) + (n - 1) (2y - 1) (2)H_{n-2}(x, y)
\]

\[- 2 \frac{(n - 1)!}{(n - 4)!} (2)H_{n-4}(x, y),
\]

\(n \geq 4),

(17c)

where \((2)H_{i}(x, y)\) for \(i \in \{0, 6\}\) are explicitly given in Section 4.

The first two of the above equations yield the relation

\[
\frac{1}{2} \frac{\partial^2}{\partial x^2} (2)H_n(x, y) = \frac{\partial}{\partial y} (2)H_n(x, y),
\]

(18)

or, equivalently,

\[
(2)H_n(x, y) = \exp \left( \frac{y \frac{\partial^2}{\partial x^2}}{2} \right) (2)H_n(x, 0),
\]

(19)

with

\[
(2)H_n(x, 0) = n! \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{H_{n-2r}(x) H_r(0)}{(n - 2r)! r!} = n! \sum_{m=0}^{\lfloor n/4 \rfloor} (-1)^m \frac{H_{n-4m}(x)}{(n - 4m)! m!},
\]

(20a)

while, we have that [4, p. 255])

\[
(2)H_n(0, y) = n! \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{H_{n-2r}(0) H_r(y)}{(n - 2r)! r!} = \begin{cases} 
0, & \text{if } n \text{ is odd}, \\
(-1)^{n/2} n! \sum_{r=0}^{n/2} (-1)^r \frac{H_r(y)}{(n/2 - r)! r!} = \frac{n!}{(n/2)!} H_{n/2} (y - \frac{1}{2}) , & \text{otherwise},
\end{cases}
\]

(20b)

where use has been made of the known relations \(H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}\) and \(H_{2n+1}(0) = 0\).

We can thus introduce the three-variable extension of \((2)H_n(x, y)\) by slightly generalizing the generating function (15), namely,

\[
\exp \left( 2x t - t^2 + 2y t^2 + 2z t^3 - t^4 + 2x t^3 - t^6 \right) = \sum_{n=0}^{\infty} t^n \frac{(3,2)H_n(x, y, z)}{n!},
\]

(21a)

\[
(3,2)H_n(x, y, z) = n! \sum_{r=0}^{\lfloor n/3 \rfloor} \frac{H_{n-3r}(x, y) H_r(z)}{(n - 3r)! r!} ,
\]

(21b)
The related recurrence relations are:

\[
\frac{\partial}{\partial z} (3,2) H_n(x, y, z) = 2n (n - 1) (n - 2) (3,2) H_{n-3}(x, y, z), \quad (n \geq 3),
\]

\[
(3,2) H_n(x, y, z) = 2x (3,2) H_{n-1}(x, y, z) + 2n (n - 1) (2y - 1) (3,2) H_{n-2}(x, y, z)
\]
\[+ 6z \frac{(n - 1)! (3,2) H_{n-3}(x, y, z) - 4 \frac{(n - 1)!}{(n - 4)!} (3,2) H_{n-4}(x, y, z)}{(n - 3)!}, \quad (n \geq 3),
\]

\[
- 6 \frac{(n - 1)!}{(n - 6)!} (3,2) H_{n-6}(x, y, z), \quad (n \geq 6),
\]

with \((3,2) H_i(x, y, z) = H_i(x)\), for \(i = 0, 1\), while for \(i = 2 - 6\), \((3,2) H_i(x, y, z)\) can be easily derived from the definition relation (21b).

It is now clear that multivariable GHP can be introduced exploiting the following generating function

\[
\exp \left( \sum_{s=1}^{M} (2s \cdot t^s - t^{2s}) \right) = \sum_{n=0}^{\infty} \frac{(s)!}{n!} H_n(\{x_s\}_M),
\]

where \(\{s\}_M \equiv (M, M - 1, \ldots, 2), \{x_s\}_M \equiv x_1, x_2, \ldots, x_M\). Obviously, these polynomials can be easily constructed iterating the procedure leading to (21b). If \(m = 4\), one gets, indeed

\[
(4,3,2) H_n(x_1, x_2, x_3, x_4) = n! \sum_{r=0}^{[n/4]} \frac{(3,2) H_{n-4r}(x_1, x_2, x_3) H_r(x_4)}{(n - 4r)! r!}.
\]

The recurrence properties of \((s)_M H_n(\{x_s\}_M)\) can be written in a very concise form, namely

\[
\frac{\partial}{\partial x_s} (s)_M H_n(\{x_s\}_M) = 2 \frac{n!}{(n - s)!} (s)_M H_{n-s}(\{x_s\}_M), \quad (n \geq s),
\]

\[
\frac{n}{2} (s)_M H_n(\{x_s\}_M) = \sum_{s=1}^{M} s x_s \frac{n!}{(n - s)!} (s)_M H_{n-s}(\{x_s\}_M)
\]
\[+ \frac{M}{2} \sum_{s=1}^{M} \frac{n!}{(n - 2s)!} (s)_M H_{n-2s}(\{x_s\}_M), \quad (n \geq 2s).
\]

Finally, it is worth mentioning the possibility of introducing Gould-Hopper multivariable GHP. We start, therefore, from the obvious extension of equation (8a), which reads as follows

\[
\exp \left( \sum_{s=1}^{M} x_s t^s \right) = \sum_{n=0}^{\infty} \frac{H_n(\{x_s\}_M)}{n!}.
\]

Polynomials \(H_n^{(s)_M}(\{x_s\}_M)\) are constructed using a criterion analogous to that leading to the definition of \((s)_M H_n(\{x_s\}_M)\) and, in fact,

\[
H_n^{(3,2)}(x_1, x_2, x_3) = n! \sum_{r=0}^{[n/3]} \frac{H_{n-3r}^{(2)}(x_1, x_2)_x^r}{(n - 3r)! r!},
\]
\[
H_n^{(4,3,2)}(x_1, x_2, x_3, x_4) = n! \sum_{r=0}^{[n/4]} \frac{H_{n-4r}^{(3,2)}(x_1, x_2, x_3)_x^r}{(n - 4r)! r!},
\]

and so on. The recurrence properties of \(H_n^{(s)_M}(\{x_s\}_M)\) are those of \((s)_M H_n(\{x_s\}_M)\), the only difference being that the factor 2 appearing in equation (25) should be dropped.
3. ORTHOGONALITY, SERIES EXPANSION AND PARTIAL DIFFERENTIAL EQUATIONS

GHP can be exploited as a convenient basis for the expansion of multivariable functions. A simple and illustrative example is provided by equation (15); in fact, setting \( t = i \), we get

\[
\begin{align*}
\cos(2x) e^{-2y} &= \sum_{r=0}^{\infty} \left(-1\right)^r \frac{(2r)!}{\left(2r\right)!}, \\
\sin(2x) e^{-2y} &= \sum_{r=0}^{\infty} \left(-1\right)^r \frac{(2r)!}{\left(2r+1\right)!}.
\end{align*}
\]

which are the corresponding analogues of the monodimensional case, and (see (21a))

\[
\begin{align*}
\cos[2(x-z)] e^{-2y+1} &= \sum_{r=0}^{\infty} \left(-1\right)^r \frac{(2r+1)!}{\left(2r\right)!}, \\
\sin[2(x-z)] e^{-2y+1} &= \sum_{r=0}^{\infty} \left(-1\right)^r \frac{(2r+1)!}{\left(2r\right)!}.
\end{align*}
\]

These results are particular cases of a general theorem that we demonstrate in the following, stating that any square summable function in the Hilbert space \( L_2(\mathbb{R}^N) \), can be expanded in a series of GHP.

We consider the case \( N = 2 \) for simplicity’s sake. Given the space \( L_2(\mathbb{R}^2) \), of functions \( f(x, y) \), \((x, y) \in \mathbb{R}^2 \), such that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx \, dy < \infty,
\]

it is well known that \( L_2(\mathbb{R}^2) \) is a Hilbert space, whose inner product is

\[
\langle f, g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \overline{g(x, y)} \, dx \, dy, \quad (f, g \in L_2(\mathbb{R}^2)).
\]

The set of functions

\[
\phi_n(x, y) \equiv \exp \left[-\frac{1}{2} (x^2 + y^2)\right] (2)H_n(x, y),
\]

form an orthogonal system in \( L_2(\mathbb{R}^2) \), namely

\[
\langle \phi_n, \phi_m \rangle = A_n \delta_{n,m},
\]

where \( A_n \) is a constant, that we will specify in the following, and \( \delta_{n,m} \) is the Kronecker symbol.

Alternatively, the exponential factor \( \exp \left[-(x^2 + y^2)\right] \), multiplying the GHP, can be viewed as a measure in the norm (29), which defines the Hilbert space \( L_2(\mathbb{R}^2) \).

The orthogonality of \((2)H_n(x, y)\) in \( L_2(\mathbb{R}^2) \) follows from that of the usual Hermite polynomials in \( L_2(\mathbb{R}) \),

\[
\int_{-\infty}^{\infty} \left(e^{-t^2/2} H_n(t)\right) \left(e^{-t^2/2} H_m(t)\right) \, dt = 2^n n! \sqrt{\pi} \delta_{n,m}.
\]

In fact, it holds

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} (2)H_n(x, y) (2)H_m(x, y) \, dx \, dy = A_n \delta_{n,m}.
\]
where \( A_n \) is a constant (see (31b)), given by

\[
A_n = [n!]^2 \pi \sum_{r=0}^{[n/2]} \frac{2^{n-r}}{(n-2r)!r!} = (-i\sqrt{2})^n n! \pi H_n \left( \frac{i}{\sqrt{2}} \right),
\]

(33b)
since it is possible to reduce the double integral (33a) to separate integrations with respect to each variable because of the convolution structure of the definition (16). Following from (33a) that functions

\[
\phi_n(x, y) = \frac{\phi_n(x, y)}{\sqrt{A_n}},
\]

(33c)
form an orthonormal system in \( L_2(\mathbb{R}^2) \), it is, therefore, a simple matter to prove the orthogonality relation (31b) for each multivariable GHP in \( N \) dimensions, by a recursive procedure analogous to that mentioned above, once the orthogonality of GHP in \( L_2(\mathbb{R}^{N-1}) \) is verified. Obviously, the related weight factor is \( \exp \left[ -\sum_{k=1}^{N} x_k^2 \right] \), which multiplies \( N \)-variables GHP.

Finally, one can prove that the \( \phi_n(\{x_s\}_N) \) functions form a complete basis in the Hilbert space \( L_2(\mathbb{R}^N) \), that is the subspace generated by

\[
\phi_n(\{x_s\}_N) = \exp \left( -\frac{1}{2} \sum_{k=1}^{N} x_k^2 \right) \left( \{x_s\}_N \right) H_n(\{x_s\}_N)
\]
is everywhere dense in it. It is evident that \( H_n(x, y) \) is given by a sum of polynomials of the kind \( x^m y^n \), with \( l + m \leq n \). Conversely, every two-variable polynomial \( x^p y^q, (p + q \leq n) \), can be expressed as a linear combination of the first \( 2n \) GHP, \( H_l(x, y), 0 \leq l \leq 2n \).

Polynomials \( x^p y^q \), even if multiplied by the scaling factor \( \exp \left[ -\frac{1}{2} (x^2 + y^2) \right] \), are dense in the Hilbert space of square summable functions \( \hat{f}(x, y) \), which are restrictions of the \( L_2(\mathbb{R}^2) \)-functions to subintervals of \( \mathbb{R}^2 \), being defined as follows:

\[
\hat{f}(x, y) = \begin{cases} f(x, y), & \text{if } x \in (-P, P), \ y \in (-Q, Q), \ (P, Q \in \mathbb{R}) \\ 0, & \text{otherwise} \end{cases}
\]

(34)
The closure of the set \( \{\hat{f}(x, y)\} \) coincides with \( L_2(\mathbb{R}^2) \), since

\[
\lim_{P, Q \to \infty} \int_{-P}^{P} \int_{-Q}^{Q} |\hat{f}(x, y)|^2 \, dx \, dy = (f, f).
\]

(35)
In other words, it is always possible to find a real number \( \varepsilon \), such that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y) - \hat{f}(x, y)|^2 \, dx \, dy < \varepsilon^2.
\]

(36)
This demonstration can be easily extended to the \( N \)-variables case.

It follows that the orthogonal system of GHP is complete and every function \( g(\{x_s\}_N) \in L_2(\mathbb{R}^N) \), admits a Fourier series expansion with respect to GHP,

\[
g(\{x_s\}_N) = \sum_{k=0}^{\infty} c_k \exp \left( \frac{1}{2} \sum_{l=1}^{N} x_l^2 \right) \left( \{x_s\}_N \right) H_k(\{x_s\}_N),
\]

(37)
of which equations (28) represent particular examples, showing the effectiveness of the result.

Moreover, a further point of remarkable importance is the role that the GHP may have in the solution of partial differential equations. From the operational identity (11a) and equation (11b), it follows that

\[
\exp \left( -\frac{y}{4} \frac{\partial^2}{\partial x^2} \right) \eta(x) = \sum_{n=0}^{\infty} \frac{\eta^{(n)}(0)}{n!} H_n(x, y),
\]

(38)
where we have assumed that the function \( \eta(x) \) may be expanded in Maclaurin series.
The role of the above identity in the solution, for instance, of a Schrödinger-type equation will be clarified below. We consider the case

\[ i \frac{\partial}{\partial y} \psi(x, y) = a(y) \frac{\partial^2}{\partial x^2} \psi(x, y), \]  
(39)

where \(a(y)\) is a continuous function of \(y\) and \(g(x)\) can be expanded in Maclaurin series, according to (38). We get

\[ \psi(x, y) = \sum_{n=0}^{\infty} \frac{g^{(n)}(y)}{n!} H_n(x, R(y)), \]

(40)

\[ R(y) = 4i \int_0^y a(y') dy'. \]

As a consequence, any function solution of (39), can be expanded in series of GHP of the type discussed above. A typical example may be provided by the operational identity

\[ \exp \left( -it \frac{\partial^2}{\partial x^2} \right) e^{-x^2} = \frac{1}{\sqrt{1 - it}} \exp \left( -\frac{x^2}{1 - it} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} H_{2n}(x, 4it). \]

(41)

Further examples of application of GHP to the solution of parabolic equations will be presented elsewhere.

4. THE TWO-DIMENSIONAL CASE AND NUMERICAL RESULTS

In previous sections, starting from suitable forms of generating functions, we have described some multivariable generalizations of the usual Hermite polynomials, \(H_n(x)\). We now consider, in more detail, the two-variable case and the connections relating the involved generalized Hermite polynomials together with the relevant explicit expressions.

As for the first GHP (5b), it can be remarked that this is a fictitious generalization since they can be immediately reduced to the ordinary case according to the relation

\[ H_n(x, y) = y^{n/2} H_n(x y^{-1/2}). \]

(42)

As far as the GHP of Gould-Hopper type, \(H_n^{(m)}(x, y)\), are concerned, we add the following result, which holds in the particular case \(m = 1\),

\[ H_n^{(1)}(x, y) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^{n-k} y^k = (x + y)^n, \]

(43)

while, for \(m = 2\), it can be easily seen that

\[ H_n^{(2)}(x, y) = H_n \left( \frac{x}{2i\sqrt{y}} \right) \equiv (-y)^{n/2} H_n \left( \frac{x}{2i\sqrt{y}} \right). \]

(44)

Then, we consider the more interesting case of GHP represented by \((2)_n(x, y)\), introduced in Section 2, which, due to the related orthogonality property, can give a valid contribution to the development of the approximation theory of two-variable functions. In particular, it may be interesting to know the explicit expressions of \((2)_n(x, y)\), that we report hereafter up to the first \(n \leq 6\) orders:

\( (2)_0(x, y) = 1, \)
\( (2)_1(x, y) = 2x, \)
\( (2)_2(x, y) = 4x^2 - 2 + 4y, \)
\( (2)_3(x, y) = 8x^3 - 12x + 24xy, \)
\( (2)_4(x, y) = 16x^4 - 48x^2 - 12 + 96x^2y - 48y + 48y^2, \)
\( (2)_5(x, y) = 32x^5 - 160x^3 - 120x + 320x^3y - 480xy + 480xy^2, \)
\( (2)_6(x, y) = 64x^6 - 460x^4 - 720x^2 + 600 + 960x^4y - 2880x^2y + 2880x^2y^2 - 720y - 1440y^2 + 960y^3. \)
From these expressions, it comes out that each \( (2)H_n(x, y) \) has the coefficients of the terms containing the \( x \) variable only, which are, in modulus, the same as those of the corresponding \( H_n(x) \).

Moreover, these GHP, considered as functions of \( x \), have the same parity of the related index \( n \), just as in the corresponding ordinary case of \( H_n(x) \). This property holds, more generally, for every \( (2)H_n(x, y) \), whenever \( n \in \mathbb{N} \), as can be easily inferred from the related structure in \( x \) of equation (16). In other words, \( (2)H_n(x, y) \) for \( n \) even (odd) involves only even (odd) powers of \( x \).

It follows that if a function \( f(x, y) \) admits expansion in terms of \( (2)H_n(x, y) \), as those encountered in the previous section, we have that, if \( f(x, y) \) is even or odd in \( x \), its expansion involves only \( (2)H_n(x, y) \) of order even or odd, respectively, as it is also shown in (28). This property, giving the dependence of the parity in \( x \) of \( (2)H_n(x, y) \) on the index \( n \), outlines the analogy with the ordinary case.

In addition, the following symmetry holds:

\[
(2)H_n(-x, y) = (-1)^n (2)H_n(x, y),
\]

which is the corresponding analogue of the well-known relation

\[
H_n(-x) = (-1)^n H_n(x),
\]

and allows us to reduce the amount of computation in the evaluation of \( (2)H_n(x, y) \).

In this respect, considering the highly increasing behaviour of \( (2)H_n(x, y) \) for increasing values of \( x \) and \( y \), instead of \( (2)H_n(x, y) \), as a numerical example, we consider the exponentially scaled Hermite polynomials represented by functions \( \phi_n(x, y) \), equation (31a), which can be written as follows:

\[
\phi_n(x, y) = \frac{[n/2]}{\phi_{n-2r}(x) \phi_r(y)} \frac{(n-2r)!r!}{(n-2r)!r!}, \quad \text{where}
\]

\[
\phi_m(\xi) = e^{-\xi^2/2} H_m(\xi).
\]

Figures (1)-(6) show the behavior of \( \phi_n(x, y) \), for \( n = 1, 2, 3, 5, 8, 10 \), in the range \(-5 \leq x, y \leq 5\), and their symmetry properties; the location of the relevant zeros, which are the same as those of the corresponding \( (2)H_n(x, y) \), is also illustrated in the figures.

It is worth remarking that \( \phi_0(x, y) \) reduces to the known Gaussian distribution, being \( (2)H_0(x, y) = 1 \).

5. CONCLUDING REMARKS

Considering the structure of \( (2)H_n(x, y) \), expressed by equation (16), we point out that these generalized polynomials are defined as a sort of discrete convolution of ordinary HP, namely of Hermite polynomials having a lower number of variables. This fact allows generalizations to the multivariable case and also to index values \( m > 2 \).

In fact, starting from a generalized generating function, such as (15)

\[
\exp \left( 2x t - t^2 + 2y t^m - t^{2m} \right) = \sum_{n=0}^{\infty} \frac{(m)H_n(x, y) t^n}{n!},
\]

we obtain

\[
(m)H_n(x, y) = n! \sum_{r=0}^{[n/m]} \frac{H_{n-2mr}(x) H_r(y)}{(n-2mr)!r!},
\]
which, in the particular case $m = 1$, reduces to the known addition theorem for ordinary HP (see [5, p. 1035]), since one recovers equation (13),

$$
\exp \left( 2x t - t^2 + 2y t - t^2 \right) = \sum_{n=0}^{\infty} \frac{(1) H_n(x, y) t^n}{n!},
$$

(52)
and, considering also (14c), one gets

$$(1) H_n(x, y) = \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x) H_r(y) = H_n(x + y, 2) = 2^n \sqrt{2} H_n \left( \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \right).$$ (53)

As for the convolution structure of $(m) H_n(x, y)$ and, in particular, of $(2) H_n(x, y)$, it is worth mentioning that we have already encountered this kind of structure in the generalization of...
another important special function, namely the Bessel function, whose importance for physical applications has been already stated [6,7], and which, in the two-dimensional case, takes the form

\[ (2) J_n(x, y) = \sum_{l=-\infty}^{\infty} J_{n-2l}(x) J_l(y). \] (54)
Recalling the related generating function [6,7]
\[
\exp \left[ \frac{x}{2} \left( t - \frac{1}{t} \right) + \frac{y}{2} \left( t^2 - \frac{1}{t^2} \right) \right] = \sum_{n=-\infty}^{\infty} t^n \mathcal{H}_n(x, y),
\]
we can easily obtain the following expression:
\[
(2)_n J_n(2x, 2y) = \sum_{l=0}^{\infty} \frac{H_{n+l}^{(2)}(x, y) H_{l}^{(2)}(-x, -y)}{l!(n+l)!}, \quad (n \geq 0),
\]
relating generalized Bessel functions (GBF), equation (54), to GHP of the form \( \mathcal{H}_n^{(2)}(x, y) \). The condition, \( n \geq 0 \), in (56) is not restrictive since, due to the reflection property of \( \mathcal{J}_n(x, y) \), we can obtain GBF in terms of \( \mathcal{H}_n^{(2)}(x, y) \) for every integer \( n \).

The above results, though far from completing the argument, can give a notion of the usefulness of the present method for the identification of suitable generalizations of known functions. Moreover, they represent a starting point for the development of a unified theory of multivariable orthogonal polynomials, which will be the subject of forthcoming works, actually in preparation. These works are according to the lines indicated for \( \mathcal{H}_n(x) \) by Hermite in two classic seminal papers published more than a century ago (1864) in the proceedings of the French Academy of Sciences in Paris and just mentioned by Appell and Kampé de Fériet [8], then unfortunately disattended by further studies.

REFERENCES