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Existence theorems for nth-order discontinuous ordinary differential inclusions

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Abstract

In this work an existence theorem for nth-order ordinary differential inclusions is proved without the continuity of multi-valued functions. Our results are an improvement upon the existence results of Dhage et al. [B.C. Dhage, T.L. Holambe, S.K. Ntouyas, Upper and lower solutions for second order discontinuous differential inclusions, Math. Sci. Res. J. 7 (5) (2003) 206–212] and Agarwal et al. [R.P. Agarwal, B.C. Dhage, D. O'Regan, The method of upper and lower solution for differential inclusions via a lattice fixed point theorem, Dynam. Systems Appl. 12 (2003) 1–7] under weaker conditions.

Keywords: Differential inclusion; Existence theorem

1. Introduction

Let $\mathbb{R}$ be a real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Consider the second-order differential inclusion (for short DI)

$$\left.\begin{array}{l} x^{(n)}(t) \in F(t, x(t)) \text{ a.e. } t \in J \\ x^{(i)}(0) = x_i \in \mathbb{R}, \;\; i \in \{0, 1, \ldots, n-1\} \end{array}\right\} \tag{1.1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ and $\mathcal{P}_p(\mathbb{R})$ is the class of all non-empty subsets of $\mathbb{R}$ with property p.

By a solution of the DI (1.1) we mean a function $x \in AC^{(n-1)}(J, \mathbb{R})$ that satisfies $x^{(n)}(t) = v(t)$ for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in F(t, x(t))$ a.e. $t \in J$, and $x^{(i)}(0) = x_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n-1\}$ where $AC^{(n-1)}(J, \mathbb{R})$ is the space of real-valued functions whose $(n-1)$th derivative exists and is absolutely continuous on J.

The DI (1.1) has already been studied in the literature as regards existence results under different continuity conditions of F. The existence theorem for DI (1.1) for the upper semi-continuous multi-function F is proved in Dhage et al. [7]. Again the existence results for the ordinary second-order differential inclusions

$$\left.\begin{array}{l} x''(t) \in F(t, x(t)) \text{ a.e. } t \in J \\ x^{(i)}(0) = x_i \in \mathbb{R}, \;\; i = 0, 1; \end{array}\right\} \tag{1.2}$$

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have been studied in Benchohra [3] under upper semi-continuity of the multi-valued function between the given lower and upper solutions.

The case of discontinuous multi-valued function F has been treated in Agarwal et al. [1] under monotonic conditions on F and the existence of extremal solutions proved using a multi-valued lattice fixed point theorem of Dhage and O'Regan [8]. Note that the monotonic conditions used in the above papers are of very strong nature and not every Banach space is a complete lattice. These facts motivated us to pursue the study of the present work. In this work we prove existence results for the DI (1.1) under a monotonic condition which is weaker than that presented in Agarwal et al. [1].

2. Auxiliary results

We equip the real normed linear space $X = AC(J, \mathbb{R})$ of absolutely continuous real-valued functions on J with the norm $\|\cdot\|$ and the order relation $\leq$ defined by

$$\|x\| = \sup_{t \in J} |x(t)|$$

and

$$x \leq y \iff x(t) \leq y(t)\ \forall t \in J.$$

Now we introduce different kinds of order relations in $\mathcal{P}_p$ as follows:

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean "for every $a \in A$ there exists a $b \in B$ such that $a \leq b$". Again $A \overset{d}{\leq} B$ means that for each $b \in B$ there exists an $a \in A$ such that $a \leq b$. Further we have $A \overset{id}{\leq} B \iff A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. See Dhage [6] and references therein.

Definition 2.1. A mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing (resp. left monotone increasing) if $Qx \overset{i}{\leq} Qy$ (resp. $Qx \overset{d}{\leq} Qy$) for all $x, y \in X$ for which $x \leq y$. Similarly Q is called monotone increasing if it is left as well as right monotone increasing on X.

We need the following fixed point theorem in the following.

Theorem 2.1 (*Dhage [5]*). *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a right monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a, b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$ has a cluster point whenever $\{x_n\}$ is a monotone increasing sequence in $[a, b]$, then Q has a fixed point.*

In the following section we prove our main existence results for DI (1.1) under suitable conditions.

3. Existence results

We need the following definitions later.

Definition 3.1. A multi-valued map $F : J \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.

Definition 3.2. A multi-function $F(t, x)$ is called right monotone increasing in x almost everywhere for $t \in J$ if $F(t, x) \overset{i}{\leq} F(t, y)$ a.e. for $t \in J$, for all $x, y \in \mathbb{R}$ with $x \leq y$.

Definition 3.3. A multi-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is called L^1-Chandrabhan if

(i) $t \mapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$,
(ii) $x \mapsto F(t, x)$ is right monotone increasing almost everywhere for $t \in J$, and

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$\|F(t, x)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t, x)\} \le h_r(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Write

$$S_F^1(x) = \{v \in L^1(J, \mathbb{R}) \mid v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}$$

for some $x \in AC(J, \mathbb{R})$. The integral of the multi-function F is defined as

$$\int_0^t F(s, x(s))\,\mathrm{d}s = \left\{\int_0^t v(s)\,\mathrm{d}s : v \in S_F^1(x)\right\}.$$

Definition 3.4. A function $a \in AC^{(n-1)}(J, \mathbb{R})$ is called a strict lower solution of the DI (1.1) if for all $v \in S_F^1(a)$,

$$a^{(n)}(t) \le v(t) \text{ a.e. } t \in J$$
$$x^{(i)}(0) \le x_i,\ \ i \in \{0, 1, \dots, n-1\}.$$

Similarly a strict upper solution b to DI (1.1) is defined.

We consider the following set of hypotheses below.

(H_1) $F(t, x)$ is closed and bounded for each $t \in J$ and $x \in \mathbb{R}$.
(H_2) $S_F^1(x) \neq \emptyset$ for all $x \in AC(J, \mathbb{R})$.
(H_3) F is L^1-Chandrabhan.
(H_4) DI (1.1) has a strict lower solution a and a strict upper solution b with $a \le b$.

Hypotheses (H_1)–(H_2) are common in the literature. Some nice sufficient conditions for guaranteeing (H_2) appear in Aubin and Cellina [2], Deimling [4], and Lasota and Opial [10]. A mild hypothesis of (H_4) has been used in Halidias and Papageorgiou [9]. Hypothesis (H_3) is relatively new to the literature, but special forms have appeared in the works of several authors. See Dhage [5,6] and references therein.

Theorem 3.1. *Assume that* (H_1)–(H_4) *hold. Then the DI* (1.1) *has a solution in* $[a, b]$.

Proof. Let $X = AC(J, \mathbb{R})$ and define an order interval $[a, b]$ in X which is well defined in view of hypothesis (H_4). Now the DI (1.1) is equivalent to the integral inclusion

$$x(t) \in \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} F(s, x(s))\,\mathrm{d}s, \quad t \in J. \tag{3.1}$$

See Dhage et al. [7] and the references therein. Define a multi-valued operator $Q : [a, b] \to \mathcal{P}_p(X)$ by

$$\begin{aligned} Qx &= \left\{u \in X : u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s)\,\mathrm{d}s,\ v \in S_F^1(x)\right\} \\ &= (\mathcal{L} \circ S_F^1)(x) \end{aligned} \tag{3.2}$$

where $\mathcal{L} : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ is a continuous operator defined by

$$\mathcal{L}x(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} x(s)\,\mathrm{d}s. \tag{3.3}$$

Clearly the operator Q is well defined in view of hypothesis (H_2). We shall show that Q satisfies all the conditions of Theorem 2.1.

Step I: First, we show that Q has compact values on $[a, b]$. Observe that the operator Q is equivalent to the composition $\mathcal{L} \circ S_F^1$ of two operators on $L^1(J, \mathbb{R})$, where $\mathcal{L} : L^1(J, \mathbb{R}) \to X$ is the continuous operator defined by (3.3). To show Q has compact values, it then suffices to prove that the composition operator $\mathcal{L} \circ S_F^1$ has compact values on $[a, b]$. Let $x \in [a, b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_F(x)$. Then, by the definition of S^1_F, $v_n(t) \in F(t, x(t))$ a.e. for $t \in J$. Since $F(t, x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in F(t, x(t))$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence, let $t, \tau \in J$; then

$$\begin{aligned}
|\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| &\le \left| \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} - \sum_{i=0}^{n-1} \frac{x_i \tau^i}{i!} \right| \\
&\quad + \left| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s - \int_0^\tau \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s \right| \\
&\le |q(t) - q(\tau)| + \left| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s - \int_0^t \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s \right| \\
&\quad + \left| \int_0^t \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s - \int_0^\tau \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s \right| \\
&\le |q(t) - q(\tau)| + \int_0^t \left| \frac{(t-s)^{n-1}}{(n-1)!} - \frac{(\tau-s)^{n-1}}{(n-1)!} \right| |v_n(s)|\,\mathrm{d}s \\
&\quad + \left| \int_t^\tau \left| \frac{(\tau-s)^{n-1}}{(n-1)!} \right| |v_n(s)|\,\mathrm{d}s \right| \qquad (3.4) \\
&\le |q(t) - q(\tau)| + \int_0^T \left| \frac{(t-s)^{n-1}}{(n-1)!} - \frac{(\tau-s)^{n-1}}{(n-1)!} \right| |v_n(s)|\,\mathrm{d}s \\
&\quad + \left| \int_t^\tau \frac{T^{n-1}}{(n-1)!} |v_n(s)|\,\mathrm{d}s \right| \qquad (3.5)
\end{aligned}$$

where $q(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!}$.

Since $v_n \in L^1(J, \mathbb{R})$, the right hand side of (3.4) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Arzelá–Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S^1_F)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S^1_F)(x)$ is compact-valued. Therefore, Q is a compact-valued multi-valued operator on $[a, b]$.

Step II: Secondly we show that Q is right monotone increasing and maps $[a, b]$ into itself. Let $x, y \in [a, b]$ be such that $x \le y$. Since $x \mapsto F(t, x)$ is right monotone increasing, one has $F(t, x) \overset{i}{\le} F(t, y)$. As a result we have that $S^1_F(x) \overset{i}{\le} S^1_F(y)$. Hence $Q(x) \overset{i}{\le} Q(y)$. From ($H_3$) it follows that $a \le Qa$ and $Qb \le b$. Now Q is right monotone increasing, so we have

$$a \le Qa \overset{i}{\le} Qx \overset{i}{\le} Qb \le b$$

for all $x \in [a, b]$. Hence Q defines a multi-valued operator $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$.

Step III: Finally let $\{x_n\}$ be a monotone increasing sequence in $[a, b]$ and let $\{y_n\}$ be a sequence in $Q([a, b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$. We shall show that $\{y_n\}$ has a cluster point. This is achieved by showing that $\{y_n\}$ is a uniformly bounded and equi-continuous sequence.

Case I: First we show that $\{y_n\}$ is a uniformly bounded sequence. By definition of $\{y_n\}$ there is a $v_n \in S^1_F(x_n)$ such that

$$y_n(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s, \quad t \in J.$$

Therefore

$$|y_n(t)| \le \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s)\,\mathrm{d}s$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \|F(s, x_n(s))\|_{\mathcal{P}} \, ds \\
&\leq \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} h_r(s) \, ds \\
&\leq \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{T^{n-1}}{(n-1)!} h_r(s) \, ds \\
&\leq \sum_{i=0}^{n-1} \frac{x_i T^i}{i!} + \frac{T^{n-1}}{(n-1)!} \|h_r\|_{L^1}
\end{aligned}
$$

for all $t \in J$, where $r = \|a\| + \|b\|$.

Taking the supremum over t,

$$
\|y_n\| \leq \sum_{i=0}^{n-1} \frac{x_i T^i}{i!} + \frac{T^{n-1}}{(n-1)!} \|h_r\|_{L^1}
$$

which shows that $\{y_n\}$ is a uniformly bounded sequence in $Q([a, b])$.

Next we show that $\{y_n\}$ is an equi-continuous sequence in $Q([a, b])$. Let $t, \tau \in J$. Then we have

$$
\begin{aligned}
|y_n(t) - y_n(\tau)| &\leq \left| \sum_{i=0}^{n-1} \frac{x_i t_1^i}{i!} - \sum_{i=0}^{n-1} \frac{x_i \tau^i}{i!} \right| \\
&\quad + \left| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s) \, ds - \int_0^\tau \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s) \, ds \right| \\
&\leq |q(t) - q(\tau)| + \left| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_n(s) \, ds - \int_0^t \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s) \, ds \right| \\
&\quad + \left| \int_0^t \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s) \, ds - \int_0^\tau \frac{(\tau-s)^{n-1}}{(n-1)!} v_n(s) \, ds \right| \\
&\leq |q(t) - q(\tau)| + \int_0^t \left| \frac{(t-s)^{n-1}}{(n-1)!} - \frac{(\tau-s)^{n-1}}{(n-1)!} \right| |v_n(s)| \, ds \\
&\quad + \left| \int_t^\tau \left| \frac{(\tau-s)^{n-1}}{(n-1)!} \right| |v_n(s)| \, ds \right| \\
&\leq |q(t) - q(\tau)| + \int_0^T \left| \frac{(t-s)^{n-1}}{(n-1)!} - \frac{(\tau-s)^{n-1}}{(n-1)!} \right| h_r(s) \, ds \\
&\quad + |p(t) - p(\tau)|
\end{aligned}
$$

where $p(t) = \frac{T^{n-1}}{(n-1)!} \int_0^t h_r(s) \, ds$.

From the above inequality it follows that

$$
|y_n(t) - y_n(\tau)| \to 0 \quad \text{as } n \to \infty.
$$

This shows that $\{y_n\}$ is an equi-continuous sequence in $Q([a, b])$. Now $\{y_n\}$ is uniformly bounded and equi-continuous, so it has a cluster point in view of the Arzelá–Ascoli theorem. Now the desired conclusion follows by an application of Theorem 2.1. □

To prove the next result, we need the following definitions.

Definition 3.5. A multi-function $\beta : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called L^1_X-Chandrabhan if

(i) $t \mapsto \beta(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $\beta(t, x)$ is right monotone increasing in x almost everywhere for $t \in J$, and

(iii) there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$\|\beta(t, x)\| = \sup\{|u| : u \in \beta(t, x)\} \leq h(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$.

Remark 3.1. Note that if the multi-function $\beta(t, x)$ is L^1_X-Chandrabhan, then it is measurable in t and integrally bounded on $J \times \mathbb{R} \times \mathbb{R}$, and so, by a selection theorem, S^1_β has non-empty values, that is,

$$S^1_\beta(x) = \{u \in L^1(J, \mathbb{R}) \mid u(t) \in \beta(t, x) \text{ a.e. } t \in J\} \neq \emptyset$$

for all $x \in \mathbb{R}$. See Deimling [4] and the references therein.

Theorem 3.2. *Assume that* (H_1) *and*

(H_5) *F is L^1_X-Chandrabhan*

hold. Then the DI (1.1) *has a solution on J.*

Proof. Obviously the hypotheses (H_2) and (H_3) of Theorem 3.1 hold in view of Remark 3.1. Define two functions $a, b : J \to \mathbb{R}$ by

$$a(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} - \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} h(s)\,\mathrm{d}s,$$

and

$$b(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} h(s)\,\mathrm{d}s.$$

It is easy to verify that a and b are respectively the strictly lower and upper solutions of the DI (1.1) on J with $a \leq b$. Thus (H_4) holds and now the desired conclusion follows by an application of Theorem 3.1. □

Example 3.3. Let $J = [0, 1]$ and define a multi-function $F : J \times \mathbb{R} \to \mathbb{R}$ by

$$F(t, x) = \begin{cases} \{0\} \text{ if } x < 0, \\ [0, p(t)[x]] \text{ if } x \in [0, 2] \\ \{3\} \text{ if } x > 2, \end{cases} \tag{3.6}$$

for all $t \in J$, where $p \in L^1(J, \mathbb{R}^+)$ and $[x]$ is a greatest integer not greater than x. Now consider the DI

$$\left.\begin{array}{l} x^{(n)}(t) \in F(t, x(t)) \text{ a.e. } t \in J \\ x^{(i)}(0) = \dfrac{1}{1+i} \in \mathbb{R},\ i \in \{0, 1, \ldots, n-1\} \end{array}\right\}. \tag{3.7}$$

Clearly the multi-function F satisfies all the hypotheses of Theorem 3.1 with

$$a(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!(1+i)} \quad \text{and} \quad b(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!(1+i)} + 3\frac{T^n}{(n-1)!}$$

for $t \in J$.

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