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Positive solution and its asymptotic behaviour of stochastic functional Kolmogorov-type system

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ABSTRACT

In general, population systems are often subject to environmental noise. This paper considers the stochastic functional Kolmogorov-type system

$$dx(t) = \mathrm{diag}\big(x_1(t), \ldots, x_n(t)\big)\big[f(x_t)\,dt + g(x_t)\,dw(t)\big].$$

Under the traditionally diagonally dominant condition, we study existence and uniqueness of the global positive solution of this stochastic system, and its asymptotic bound properties and moment average boundedness in time. These properties are natural requirements from the biological point of view. As the special cases, we also discuss some stochastic Lotka–Volterra systems.

1. Introduction

It is well known that many-species Lotka–Volterra system holds many desired properties (for example, existence and uniqueness of the positive solution, global stability and so on) when the intraspecific competition term dominates the interspecific interactions, namely, the community matrix is diagonally dominant. For delay Lotka–Volterra system, these nice properties still hold if the undelayed intraspecific competition dominates both the delayed intraspecific competition as well as the interspecific interactions (see [3,6] and references therein).

As the generalized Lotka–Volterra system, the n-dimensional deterministic Kolmogorov-type system for n interacting species is described by the following differential equation

$$\dot{x}(t) = \mathrm{diag}\big(x_1(t), \ldots, x_n(t)\big) f\big(x(t)\big), \tag{1.1}$$

where $x = (x_1, \ldots, x_n)^T$, $\mathrm{diag}(x_1, \ldots, x_n)$ represents the $n \times n$ matrix with all elements zero except those on the diagonal which are $x_1, \ldots, x_n$, $f = (f_1, \ldots, f_n)^T$. There is an extensive literature concerned with the dynamics of this system and we here only mention [4,5,11,14]. Tang and Kuang [13] consider the following functional form of Eq. (1.1)

$$\dot{x}(t) = \mathrm{diag}\big(x_1(t), \ldots, x_n(t)\big) f(x_t), \tag{1.2}$$

where $x_t \in C([-\tau, 0]; \mathbb{R}^n)$ is defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$.

Recall that $f_i(x_t)$ in (1.2) represents the inherent net birth rate of ith species. It is often affected by various unpredictable factors. According to the well-known central limit theorem, the sum of these factors follows a normal distribution. We can therefore replace the net birth rate $f_i(x_t)$ by

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$$f_i(x_t) + g_i(x_t)\dot{w},$$

where $\dot{w}$ is a white noise (i.e., $w(t)$ is a Brownian motion) and $g_i(x_t)$ represents the intensity of the noise, which not only depends on the current state, but also the history state of n species. Hence (1.2) becomes the following n-dimensional stochastic functional Kolmogorov-type system

$$dx(t) = \mathrm{diag}\big(x_1(t), \ldots, x_n(t)\big)\big[f(x_t)\,dt + g(x_t)\,dw(t)\big] \tag{1.3}$$

on $t \geqslant 0$, where $w(t)$ is a scalar Brownian motion, and

$$f = (f_1, \ldots, f_n)^T : C\big([-\tau, 0]; \mathbb{R}^n\big) \to \mathbb{R}^n, \qquad g = (g_1, \ldots, g_n)^T : C\big([-\tau, 0]; \mathbb{R}^n\big) \to \mathbb{R}^n.$$

Recently, stochastic population systems have received increasing attention. Refs. [1,9] reveal that the environmental noise may suppress the potential population explosion and guarantee the global positive solution to stochastic delay Lotka–Volterra system, and moreover, also shows that the stochastic Lotka–Volterra model produces many desired properties, for example, stochastically ultimate boundedness and the moment boundedness. Under another environmental noise perturbation, [2,12] reveal that the stochastic Lotka–Volterra system behaves similarly to the corresponding deterministic system. Ref. [10] reviews these two classes of the models and indicates clearly that different structure of environmental noise may have different effects on the population dynamics. By introducing the more general stochastic perturbations in the system (1.3), [15] clearly obtains the conditions for the different effects. These conditions show that if the environmental noise intensity is strongly dependent on the population size, this noise may suppress the population explosion and guarantee the global positive solution and the model produces several desired asymptotic properties. When the environmental noise intensity is weakly dependent on the population size, the stochastic system behaves similarly to the deterministic one and asymptotic properties are also independent of this noise.

This paper mainly focuses on the behaviors of the stochastic system (1.3) under the diagonally dominant condition, which is a classical condition in deterministic population systems. Under this condition, this paper establishes the existence-and-uniqueness theorem of the global positive solution to Eq. (1.3) and examine the asymptotic properties of this global positive solution, including the moment boundedness, stochastically ultimate boundedness and the moment average boundedness in time. These are the desired properties for a population system.

In next section, we give some necessary notations and lemmas. In Section 3, under the diagonally dominant condition, we examine that the system (1.3) almost surely admits a unique global positive solution. Section 4 shows that this global positive solution holds the desired ultimate boundedness under diagonally dominant condition. Section 5 gives the moment average boundedness in time. As applications of our results, Section 6 discusses some stochastic Lotka–Volterra systems.

2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^T A)}$. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_{++} = (0, +\infty)$, and let $\tau > 0$. Denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau \leqslant \theta \leqslant 0} |\varphi(\theta)|$, which forms a Banach space. Let $C_+ = C([-\tau, 0]; \mathbb{R}^n_+)$ and $C_{++} = C([-\tau, 0]; \mathbb{R}^n_{++})$. For any $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n_{++}$, let $\check{c} = \max\{c_1, c_2, \ldots, c_n\}$, $\hat{c} = \min\{c_1, c_2, \ldots, c_n\}$ and $C = \mathrm{diag}(c_1, \ldots, c_n)$. For any $a \in \mathbb{R}$, let $a^+ = a \vee 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. Let $w(t)$ be a scalar Brownian motion defined on this probability space. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t+\theta)\colon -\tau \leqslant \theta \leqslant 0,\ t \geqslant 0\}$. In addition, we impose the following assumption on the coefficients f and g.

Assumption 2.1. Both f and g are locally Lipschitz continuous.

This paper often uses the function

$$V_p(x) = \sum_{i=1}^n x_i^p, \quad \text{for } x \in \mathbb{R}^n_+,\ p > 0 \tag{2.1}$$

and its properties. For the convenience of reference, we give them as lemmas (also see [15]).

Lemma 2.1. *For any $x \in \mathbb{R}^n_+$, and $p > 0$,*

$$V_1^p(x) \leqslant n^{(p-1)\vee 0} V_p(x). \tag{2.2}$$

Lemma 2.2. *For any $x \in \mathbb{R}^n_+$ and $p > 0$,*

$$V_p(x) \leqslant n^{(1-\frac{p}{2})\vee 0}|x|^p, \qquad |x|^p \leqslant n^{(\frac{p}{2}-1)\vee 0} V_p(x). \tag{2.3}$$

The following lemma shows the boundedness of polynomial functions.

Lemma 2.3. *For any $\varphi \in C(\mathbb{R}^n_+;\mathbb{R})$, $a, p > 0$, when $|x| \to \infty$, $\varphi(x) = o(|x|^p)$, then*

$$\sup_{x\in\mathbb{R}^n_+}\big[\varphi(x) - a|x|^p\big] < \infty. \tag{2.4}$$

Proof. Define $h(x) = \varphi(x) - a|x|^p$. Choose $r > 0$ such that $|\varphi(x)| < a|x|^p$ when $x \in \mathbb{R}^n_+$, $|x| > r$, which implies $h(x) < 0$. We therefore have

$$\sup_{x\in\mathbb{R}^n_+} h(x) = \sup_{x\in\mathbb{R}^n_+,\,|x|\leqslant r} h(x) < \infty,$$

as required. □

3. Positive and global solutions

It is well known that, in order for a stochastic functional differential equation to have a unique global solution for any given initial data, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (cf. Mao [8]). However, clearly, if both f and g are not bounded, the coefficients of Eq. (1.3) do not satisfy the linear growth condition. In this section, we examine the existence and uniqueness of the global positive solution to Eq. (1.3) under the diagonally dominant condition. We need the following definition of local solutions (see [7]).

Definition 3.1. Set $\mathcal{F}_t = \mathcal{F}_0$ for $-\tau \leqslant t \leqslant 0$ and let $x(t)$, $-\tau \leqslant t \leqslant \rho_e$ be a continuous $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process. It is called a local strong solution of Eq. (1.3) with initial data $\xi \in C([-\tau,0];\mathbb{R}^n)$ if $x(t) = \xi(t)$ on $-\tau \leqslant t \leqslant 0$ and

$$x(t\wedge\rho_k) = \xi(0) + \int_0^{t\wedge\rho_k} F(x_s)\,ds + \int_0^{t\wedge\rho_k} G(x_s)\,dw(s), \quad \forall t \geqslant 0$$

for each $k \geqslant 1$, where $F(\varphi) = \operatorname{diag}(\varphi_1(0),\ldots,\varphi_n(0))f(\varphi)$, $G(\varphi) = \operatorname{diag}(\varphi_1(0),\ldots,\varphi_n(0))g(\varphi)$, $\{\rho_k\}_{k\geqslant 1}$ is a nondecreasing sequence of finite stopping times such that $\rho_k \to \rho_e$ almost surely as $k \to \infty$. If moreover, $\limsup_{t\to\rho_e}|x(t)| = \infty$ is satisfied almost everywhere when $\rho_e < \infty$, it is called a maximal local strong solution and ρ_e is called the explosion time. A maximal local strong solution $x(t)$, $-\tau \leqslant t < \rho_e$ is said to be unique if for any other maximal local strong solution $\bar{x}(t)$, $-\tau \leqslant t < \bar{\rho}_e$, we have $\rho_e = \bar{\rho}_e$ and $x(t) = \bar{x}(t)$ for $-\tau \leqslant t < \rho_e$ almost surely.

Applying the standing truncation technique (see [7, Theorem 3.2.2, p. 95]) gives

Theorem 3.1. *Under Assumption* 2.1, *Eq.* (1.3) *almost surely has a unique maximal local strong solution for any initial data* $\xi \in C([-\tau,0];\mathbb{R}^n)$.

To avoid the linear growth condition, we need further conditions for both f and g. For any $\varphi \in C_{++}$ and $1 \leqslant i, j \leqslant n$, we give the following assumption:

Assumption 3.1. There exist α, $\beta > 0$, $A = [a_{ij}]$, $B = [b_{ij}]$, $\bar{A} = [\bar{a}_{ij}]$, $\bar{B} = [\bar{b}_{ij}]$, $R = [r_{ij}]$, $D = [d_{ij}] \in \mathbb{R}^{n\times n}$ and probability measures μ_{ij}, $\bar{\mu}_{ij}$, ν_{ij} such that

$$f_i(\varphi) \leqslant \sum_{j=1}^n\left[a_{ij}\varphi_j^\alpha(0) + b_{ij}\int_{-\tau}^0 \varphi_j^\alpha(\theta)\,d\mu_{ij}(\theta)\right] + o\big(|\varphi(0)|^\alpha\big),$$

$$\big|f_i(\varphi)\big| \leqslant \sum_{j=1}^n\left[\bar{a}_{ij}\varphi_j^\alpha(0) + \bar{b}_{ij}\int_{-\tau}^0 \varphi_j^\alpha(\theta)\,d\bar{\mu}_{ij}(\theta)\right] + o\big(|\varphi(0)|^\alpha\big),$$

$$\big|g_i(\varphi)\big| \leqslant \sum_{j=1}^n\left[r_{ij}\varphi_j^\beta(0) + d_{ij}\int_{-\tau}^0 \varphi_j^\beta(\theta)\,d\nu_{ij}(\theta)\right] + o\big(|\varphi(0)|^\beta\big).$$

This assumption is more general than the corresponding assumption in [15] (see Assumption 3.2 in [15]). Under this assumption, the existence and uniqueness of the global positive solution of Eq. (1.3) follows.

Theorem 3.2. *Let Assumptions* 2.1 *and* 3.1 *hold. If* $\alpha > 2\beta$ *and*

$$-a_{ii} > A_i + nB_i \quad (1 \leqslant i \leqslant n), \tag{3.1}$$

where $A_i = \sum_{j=1, j\neq i}^{n} a_{ij}^{+}$, $B_i = \sum_{j=1}^{n} b_{ij}^{+}$, *then for any initial data* $\xi \in C_{++}$, *Eq.* (1.3) *almost surely admits a unique global positive solution, namely,* $x(t,\xi) \in \mathbb{R}_{++}^{n}$ *with probability one.*

The condition (3.1) shows that the undelayed intraspecific competition dominates both the delayed intraspecific competition as well as the interspecific interactions, namely, the community matrix is diagonally dominant, which is similar to the traditional condition for deterministic Lotka–Volterra systems (cf. [6]). To prove this theorem, we need the following elementary lemma.

Lemma 3.3. *For the nonnegative constants* a_i, b_i $(1 \leqslant i \leqslant n)$, $a_i > b_i$ *if and only if there exist positive constants* c_i $(1 \leqslant i \leqslant n)$ *such that*

$$\min_{1\leqslant i\leqslant n}\{a_i c_i\} > \max_{1\leqslant i\leqslant n}\{b_i c_i\}.$$

Proof. The sufficiency is obvious, so we omit it. We now consider the necessity. Assume $n \geqslant 2$ (when $n = 1$, the result is obvious). If $n = 2$, by $a_1 > b_1$, $a_2 > b_2$, we have $a_1 a_2 > b_1 b_2$, it is obvious that

$$\frac{a_2}{b_1} > \frac{b_2}{a_1}.$$

Then there exist $c_1, c_2 > 0$ such that

$$\frac{a_2}{b_1} > \frac{c_1}{c_2} > \frac{b_2}{a_1},$$

which implies that $a_2 c_2 > b_1 c_1$, $a_1 c_1 > b_2 c_2$. We therefore have

$$\min\{a_1 c_1, a_2 c_2\} > \max\{b_1 c_1, b_2 c_2\}.$$

Let $n > 2$. Assume that there exist positive constants λ_i $(1 \leqslant i \leqslant n-1)$ such that

$$\min_{1\leqslant i\leqslant n-1}\{a_i\lambda_i\} > \max_{1\leqslant i\leqslant n-1}\{b_i\lambda_i\}, \tag{3.2}$$

which implies that there exist i_0, $j_0 \in \{1, 2, \dots, n-1\}$ such that $a_{i_0}\lambda_{i_0} = \min_{1\leqslant i\leqslant n-1}\{a_i\lambda_i\}$ and $b_{j_0}\lambda_{j_0} = \max_{1\leqslant i\leqslant n-1}\{b_i\lambda_i\}$. For $a_{i_0}\lambda_{i_0} > b_{j_0}\lambda_{j_0}$ and $a_n > b_n$, by the above result ($n = 2$), there exist μ, $c_n > 0$ such that

$$\min\{a_{i_0}\lambda_{i_0}\mu, a_n c_n\} > \max\{b_{j_0}\lambda_{j_0}\mu, b_n c_n\}. \tag{3.3}$$

Define $c_i = \lambda_i \mu$ $(1 \leqslant i \leqslant n-1)$. Then by (3.2) and (3.3),

$$\begin{aligned}
\min_{1\leqslant i\leqslant n}\{a_i c_i\} &= \min\left\{\min_{1\leqslant i\leqslant n-1}\{a_i\lambda_i\mu\}, a_n c_n\right\} \\
&= \min\{a_{i_0}\lambda_{i_0}, a_n c_n\} \\
&> \max\{b_{j_0}\lambda_{j_0}, b_n c_n\} \\
&= \max\left\{\max_{1\leqslant i\leqslant n-1}\{b_i\lambda_i\mu\}, b_n c_n\right\} \\
&= \max_{1\leqslant i\leqslant n}\{b_i c_i\},
\end{aligned}$$

which implies desired result by induction. □

Then the proof of Theorem 3.2 follows.

Proof of Theorem 3.2. For any initial data $\xi(\theta) \in C_{++}$, by Theorem 3.1, there exists a unique maximal local solution $x(t)$ on $t \in [-\tau, \tau_e)$, where τ_e is the explosion time. To show that the solution is global, we only need to prove that $\tau_e = \infty$ a.s. Let k_0 be a sufficient large positive number such that

$$\frac{1}{k_0} < \min_{-\tau\leqslant\theta\leqslant 0}\left|\xi(\theta)\right| \leqslant \max_{-\tau\leqslant\theta\leqslant 0}\left|\xi(\theta)\right| < k_0.$$

For each integer $k \geqslant k_0$, define the stopping time

$$\tau_k = \inf\{t \in [-\tau, \tau_e):\ x_i(t) \notin (1/k, k) \text{ for some } i = 1, 2, \ldots, n\}$$

with the traditional setting $\inf\emptyset = \infty$, where $\emptyset$ denotes the empty set. Clearly, τ_k is increasing as $k \to \infty$ and $\tau_k \to \tau_\infty \leqslant \tau_e$ a.s. If we can show $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s., which implies the desired result. This is also equivalent to proving that, for any $t > 0$, $\mathbb{P}(\tau_k \leqslant t) \to 0$ as $k \to \infty$. To prove this statement, we examine the condition (3.1). This condition implies that for all $i = 1, \ldots, n$,

$$a_{ii} + A_i < 0, \qquad -a_{ii} - A_i > nB_i. \tag{3.4}$$

By Lemma 3.3, there exists a $c = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n_{++}$ such that

$$\min_{1\leqslant i\leqslant n}\{c_i(|a_{ii} + A_i|)\} > \max_{1\leqslant i\leqslant n}\{nc_iB_i\},$$

which implies that

$$\max_{1\leqslant i\leqslant n}\{c_i(a_{ii} + A_i)\} < -\max_{1\leqslant i\leqslant n}\{nc_iB_i\}. \tag{3.5}$$

Then for c defined in (3.5) and any $p > 0$, define a C^2-function

$$U(x) = \sum_{i=1}^n c_i u(x_i), \tag{3.6}$$

where $u(x_i) = x_i^p - p\log x_i$. Clearly, $u(\cdot) \geqslant 0$ and $u(0^+) = u(\infty) = \infty$. Applying the Itô formula for $U(x)$ and taking expectation yield

$$\mathbb{E}U\big(x(t\wedge\tau_k)\big) = \mathbb{E}U\big(\xi(0)\big) + \mathbb{E}\int_0^{t\wedge\tau_k} \mathcal{L}U(x_s)\,ds, \tag{3.7}$$

where $\mathcal{L}U$ is defined as

$$\begin{aligned}
\mathcal{L}U(\varphi) &= \sum_{i=1}^n c_i\Big[p\big(\varphi_i^p(0) - 1\big)f_i(\varphi) + \frac{p}{2}\big((p-1)\varphi_i^p(0) + 1\big)g_i^2(\varphi)\Big] \\
&= p\sum_{i=1}^n c_i\varphi_i^p(0)f_i(\varphi) + \frac{p(p-1)}{2}\sum_{i=1}^n c_i\varphi_i^p(0)g_i^2(\varphi) + \frac{p}{2}\sum_{i=1}^n c_i\big[-2f_i(\varphi) + g_i^2(\varphi)\big] \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.8}$$

Then we estimate I_1–I_3, respectively. Note that Lemma 2.2 gives

$$\begin{aligned}
\sum_{i=1}^n c_i x_i^p &\leqslant \check{c}V_p(x) \\
&\leqslant \check{c}n^{(1-\frac{p}{2})\vee 0}|x|^p.
\end{aligned}$$

This, together with Assumption 3.1 and the Young inequality, yields

$$\begin{aligned}
I_1 &\leqslant p\sum_{i=1}^n c_i\varphi_i^p(0)\sum_{j=1}^n\Big[a_{ij}\varphi_j^\alpha(0) + b_{ij}\int_{-\tau}^0 \varphi_j^\alpha(\theta)\,d\mu_{ij}(\theta) + o\big(|\varphi(0)|^\alpha\big)\Big] \\
&= p\sum_{i=1}^n\sum_{j=1}^n c_ia_{ij}\varphi_i^p(0)\varphi_j^\alpha(0) + p\sum_{i=1}^n\sum_{j=1}^n c_ib_{ij}\int_{-\tau}^0 \varphi_i^p(0)\varphi_j^\alpha(\theta)\,d\mu_{ij}(\theta) + o\big(|\varphi(0)|^{\alpha+p}\big) \\
&\leqslant p\sum_{i=1}^n c_i\Big(a_{ii}\varphi_i^{\alpha+p}(0) + \sum_{j=1, j\neq i}^n a_{ij}^+\varphi_i^p(0)\varphi_j^\alpha(0)\Big) + p\sum_{i=1}^n\sum_{j=1}^n c_ib_{ij}^+\int_{-\tau}^0 \varphi_i^p(0)\varphi_j^\alpha(\theta)\,d\mu_{ij}(\theta) + o\big(|\varphi(0)|^{\alpha+p}\big) \\
&\leqslant p\sum_{i=1}^n c_i\Big(a_{ii}\varphi_i^{\alpha+p}(0) + \sum_{j=1, j\neq i}^n a_{ij}^+\varphi_i^p(0)\max_{1\leqslant j\leqslant n}\{\varphi_j^\alpha(0)\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{p}{\alpha+p}\sum_{i=1}^{n}\sum_{j=1}^{n}c_ib_{ij}^{+}\left[p\varphi_i^{\alpha+p}(0)+\alpha\int_{-\tau}^{0}\varphi_j^{\alpha+p}(\theta)\,d\mu_{ij}(\theta)\right]+o\left(\left|\varphi(0)\right|^{\alpha+p}\right)\\
&\leqslant p\sum_{i=1}^{n}c_i\Big(a_{ii}\varphi_i^{\alpha+p}(0)+A_i\varphi_i^{p}(0)\max_{1\leqslant j\leqslant n}\big\{\varphi_j^{\alpha}(0)\big\}\Big)\\
&+\frac{p}{\alpha+p}\left[\sum_{i=1}^{n}pc_iB_i\varphi_i^{\alpha+p}(0)+\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}c_ib_{ij}^{+}\int_{-\tau}^{0}\varphi_j^{\alpha+p}(\theta)\,d\mu_{ij}(\theta)\right]+o\left(\left|\varphi(0)\right|^{\alpha+p}\right).
\end{aligned}
$$

For any $x\in\mathbb{R}^n_{++}$ and any given i, define the function

$$h_i(x_i)=a_{ii}x_i^{\alpha+p}+A_ix_i^{p}\max_{1\leqslant j\leqslant n}\big\{x_j^{\alpha}\big\}.$$

If $x_i=\max_{1\leqslant j\leqslant n}\{x_j\}$, then

$$h_i(x_i)=(a_{ii}+A_i)x_i^{\alpha+p},$$

which implies that

$$
\begin{aligned}
\sum_{i=1}^{n}c_i\Big(a_{ii}\varphi_i^{\alpha+p}(0)+A_i\varphi_i^{p}(0)\max_{1\leqslant j\leqslant n}\big\{\varphi_j^{\alpha}(0)\big\}\Big)&\leqslant\sum_{i=1}^{n}c_i(a_{ii}+A_i)\varphi_i^{\alpha+p}(0)\\
&\leqslant\max_{1\leqslant i\leqslant n}\big\{c_i(a_{ii}+A_i)\big\}V_{\alpha+p}\big(\varphi(0)\big).
\end{aligned}
\tag{3.9}
$$

When $x_i\neq\max_{1\leqslant j\leqslant n}\{x_j\}$, for $x_i\in[0,\ \max_{1\leqslant j\leqslant n}\{x_j\})$,

$$h_i'(x_i)=a_{ii}(\alpha+p)x_i^{\alpha+p-1}+pA_ix_i^{p-1}\max_{1\leqslant j\leqslant n}\big\{x_j^{\alpha}\big\},$$

which implies that there exists a unique

$$x_i^*=\left[\frac{pA_i}{(\alpha+p)|a_{ii}|}\right]^{\frac{1}{\alpha}}\max_{1\leqslant j\leqslant n}\{x_j\}$$

such that the function $h_i'(x_i^*)=0$. Noting $h_i(0)=0$ and $h_i(\max_{1\leqslant j\leqslant n}\{x_j\})=(a_{ii}+A_i)\max_{1\leqslant j\leqslant n}\{x_j^{\alpha+p}\}<0$, we have $h_i(x_i)\leqslant h_i(x_i^*)$ for all $x_i\in[0,\max_{1\leqslant j\leqslant n}\{x_j\}]$. Since there must exist some $i^*\in\{1,2,\dots,n\}$ such that $x_{i^*}=\max_{1\leqslant j\leqslant n}\{x_j\}$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}c_i\Big(a_{ii}\varphi_i^{\alpha+p}(0)+A_i\varphi_i^{p}(0)\max_{1\leqslant j\leqslant n}\big\{\varphi_j^{\alpha}(0)\big\}\Big)\\
&=c_{i^*}h_{i^*}\Big(\max_{1\leqslant j\leqslant n}\big\{\varphi_j(0)\big\}\Big)+\sum_{i=1,\,i\neq i^*}^{n}c_ih_i\big(\varphi_i(0)\big)\\
&\leqslant\left[c_{i^*}(a_{i^*i^*}+A_{i^*})+\sum_{i=1,i\neq i^*}^{n}\frac{c_i\alpha A_i}{\alpha+p}\left(\frac{pA_i}{(\alpha+p)|a_{ii}|}\right)^{\frac{p}{\alpha}}\right]\max_{1\leqslant j\leqslant n}\big\{\varphi_j^{\alpha+p}(0)\big\}\\
&\leqslant\left[\max_{1\leqslant i\leqslant n}\big\{c_i(a_{ii}+A_i)\big\}+\sum_{i=1}^{n}\frac{c_i\alpha A_i}{\alpha+p}\left(\frac{pA_i}{(\alpha+p)|a_{ii}|}\right)^{\frac{p}{\alpha}}\right]\max_{1\leqslant j\leqslant n}\big\{\varphi_j^{\alpha+p}(0)\big\}\\
&=:-K_p\max_{1\leqslant i\leqslant n}\big\{\varphi_i^{\alpha+p}(0)\big\},
\end{aligned}
\tag{3.10}
$$

where

$$-K_p=\max_{1\leqslant i\leqslant n}\big\{c_i(a_{ii}+A_i)\big\}+\sum_{i=1}^{n}\frac{c_i\alpha A_i}{\alpha+p}\left(\frac{pA_i}{(\alpha+p)|a_{ii}|}\right)^{\frac{p}{\alpha}}.$$

Noting $A_i<|a_{ii}|$, we have

$$K_\infty=\lim_{p\to\infty}K_p=-\max_{1\leqslant i\leqslant n}\big\{c_i(a_{ii}+A_i)\big\}>0.$$

In the following, we choose sufficiently large p such that $K_p > 0$. Clearly, $V_{\alpha+p}(x) \leqslant n \max_{1\leqslant i\leqslant n}\{x_i^{\alpha+p}\}$. We therefore have

$$\sum_{i=1}^{n} c_i\Big(a_{ii}\varphi_i^{\alpha+p}(0) + A_i\varphi_i^p(0)\max_{1\leqslant j\leqslant n}\big\{\varphi_j^\alpha(0)\big\}\Big) \leqslant -n^{-1}K_pV_{\alpha+p}\big(\varphi(0)\big).$$

This, together with (3.9), implies that

$$I_1 \leqslant \left(-\frac{pK_p}{n} + \frac{p^2\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}\right)V_{\alpha+p}\big(\varphi(0)\big) + \frac{\alpha p}{\alpha+p}\sum_{i=1}^{n}\sum_{j=1}^{n} c_ib_{ij}^+\int_{-\tau}^{0}\varphi_j^{\alpha+p}(\theta)\,d\mu_{ij}(\theta) + o\big(\big|\varphi(0)\big|^{\alpha+p}\big). \tag{3.11}$$

By Lemma 2.1, for $x_1, x_2, \ldots, x_n \geqslant 0$,

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leqslant n\sum_{i=1}^{n} x_i^2. \tag{3.12}$$

Then by Assumption 3.1, the Young inequality and the Hölder inequality,

$$\begin{aligned}
I_2 &\leqslant \frac{p(p-1)^+}{2}\sum_{i=1}^{n} c_i\varphi_i^p(0)\left[\sum_{j=1}^{n}\left(r_{ij}\varphi_j^\beta(0) + d_{ij}\int_{-\tau}^{0}\varphi_j^\beta(\theta)\,d\nu_{ij}(\theta)\right) + o\big(\big|\varphi(0)\big|^\beta\big)\right]^2 \\
&\leqslant p(p-1)^+\sum_{i=1}^{n} c_i\varphi_i^p(0)\left[\sum_{j=1}^{n}\left(r_{ij}\varphi_j^\beta(0) + d_{ij}\int_{-\tau}^{0}\varphi_j^\beta(\theta)\,d\nu_{ij}(\theta)\right)\right]^2 + o\big(\big|\varphi(0)\big|^{2\beta+p}\big) \\
&\leqslant 2np(p-1)^+\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\varphi_i^p(0)\left[r_{ij}^2\varphi_j^{2\beta}(0) + d_{ij}^2\left|\int_{-\tau}^{0}\varphi_j^\beta(\theta)\,d\nu_{ij}(\theta)\right|^2\right] + o\big(\big|\varphi(0)\big|^{2\beta+p}\big) \\
&\leqslant 2np(p-1)^+\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\left[r_{ij}^2\varphi_i^p(0)\varphi_j^{2\beta}(0) + d_{ij}^2\int_{-\tau}^{0}\varphi_i^p(0)\varphi_j^{2\beta}(\theta)\,d\nu_{ij}(\theta)\right] + o\big(\big|\varphi(0)\big|^{2\beta+p}\big) \\
&\leqslant \frac{2np(p-1)^+}{2\beta+p}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[pc_i\big(r_{ij}^2+d_{ij}^2\big)\varphi_i^{2\beta+p}(0) + 2\beta c_ir_{ij}^2\varphi_j^{2\beta+p}(0) + 2\beta c_id_{ij}^2\int_{-\tau}^{0}\varphi_j^{2\beta+p}(\theta)\,d\nu_{ij}(\theta)\right] \\
&\quad + o\big(\big|\varphi(0)\big|^{2\beta+p}\big).
\end{aligned}$$

Noting $2\beta < \alpha$, by Lemma 2.2,

$$\begin{aligned}
p\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\big(r_{ij}^2+d_{ij}^2\big)\varphi_i^{2\beta+p}(0) &\leqslant np\max_{1\leqslant i,j\leqslant n}\big\{c_i\big(r_{ij}^2+d_{ij}^2\big)\big\}\sum_{i=1}^{n}\varphi_i^{2\beta+p}(0) \\
&\leqslant np\max_{1\leqslant i,j\leqslant n}\big\{c_i\big(r_{ij}^2+d_{ij}^2\big)\big\}n^{(1-\frac{2\beta+p}{2})\vee 0}\big|\varphi(0)\big|^{2\beta+p} \\
&= o\big(\big|\varphi(0)\big|^{\alpha+p}\big).
\end{aligned}$$

Similarly,

$$2\beta\sum_{i=1}^{n}\sum_{j=1}^{n} c_ir_{ij}^2\varphi_j^{2\beta}(0) = o\big(\big|\varphi(0)\big|^{\alpha+p}\big).$$

These imply that

$$I_2 \leqslant \frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^{n}\sum_{j=1}^{n} c_id_{ij}^2\int_{-\tau}^{0}\varphi_j^{2\beta+p}(\theta)\,d\nu_{ij}(\theta) + o\big(\big|\varphi(0)\big|^{\alpha+p}\big). \tag{3.13}$$

By Assumption 3.1 and the condition $2\beta < \alpha$,

$$\begin{aligned} I_3 &\leqslant p\sum_{i=1}^{n} c_i \big|f_i(\varphi)\big| + \frac{p}{2}\sum_{i=1}^{n} c_i g_i^2(\varphi) \\ &\leqslant p\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\left[\bar{a}_{ij}\varphi_j^{\alpha}(0) + \bar{b}_{ij}\int_{-\tau}^{0}\varphi_j^{\alpha}(\theta)\,d\bar{\mu}_{ij}(\theta)\right] + o\big(\big|\varphi(0)\big|^{\alpha}\big) \\ &\quad + 2np\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\left[r_{ij}^2\varphi_j^{2\beta}(0) + d_{ij}^2\int_{-\tau}^{0}\varphi_j^{2\beta}(\theta)\,d\nu_{ij}(\theta)\right] + o\big(\big|\varphi(0)\big|^{2\beta}\big). \end{aligned}$$

Note that

$$p\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{a}_{ij}\varphi_j^{\alpha}(0) = o\big(\big|\varphi(0)\big|^{\alpha+p}\big),$$

$$2np\sum_{i=1}^{n}\sum_{j=1}^{n} c_j r_{ij}^2\varphi_j^{2\beta}(0) = o\big(\big|\varphi(0)\big|^{\alpha+p}\big).$$

We therefore have

$$I_3 \leqslant p\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{b}_{ij}\int_{-\tau}^{0}\varphi_j^{\alpha}(\theta)\,d\bar{\mu}_{ij}(\theta) + 2np\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_{ij}^2\int_{-\tau}^{0}\varphi_j^{2\beta}(\theta)\,d\nu_{ij}(\theta) + o\big(\big|\varphi(0)\big|^{\alpha+p}\big). \tag{3.14}$$

Substituting I_1–I_3 into (3.8) yields

$$\begin{aligned} \mathcal{L}U(\varphi) &= H\big(\varphi(0)\big) + \frac{\alpha p}{\alpha+p}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i b_{ij}^{+}\left[\int_{-\tau}^{0}\varphi_j^{\alpha+p}(\theta)\,d\mu_{ij}(\theta) - \varphi_j^{\alpha+p}(0)\right] \\ &\quad + \frac{4\beta np(p-1)^{+}}{2\beta+p}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_{ij}^2\left[\int_{-\tau}^{0}\varphi_j^{2\beta+p}(\theta)\,d\nu_{ij}(\theta) - \varphi_j^{2\beta+p}(0)\right] \\ &\quad + p\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{b}_{ij}\left[\int_{-\tau}^{0}\varphi_j^{\alpha}(\theta)\,d\bar{\mu}_{ij}(\theta) - \varphi_j^{\alpha}(0)\right] \\ &\quad + 2np\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_{ij}^2\left[\int_{-\tau}^{0}\varphi_j^{2\beta}(\theta)\,d\nu_{ij}(\theta) - \varphi_j^{2\beta}(0)\right], \end{aligned}$$

where

$$\begin{aligned} H(x) &= \left(-\frac{pK_p}{n} + \frac{p^2\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}\right)V_{\alpha+p}(x) + \frac{\alpha p}{\alpha+p}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i b_{ij}^{+}x_j^{\alpha+p} + \frac{4\beta np(p-1)^{+}}{2\beta+p}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_{ij}^2 x_j^{2\beta+p} \\ &\quad + p\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{b}_{ij}x_j^{\alpha} + 2np\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_{ij}^2 x_j^{2\beta} + o\big(|x|^{\alpha+p}\big). \end{aligned}$$

Noting

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_{ij}^2 x_j^{2\beta+p} = o\big(|x|^{\alpha+p}\big),$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\bar{b}_{ij}x_j^{\alpha} = o\big(|x|^{\alpha+p}\big),$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_{ij}^2 x_j^{2\beta} = o\big(|x|^{\alpha+p}\big)$$

we therefore have

$$\begin{aligned} H(x) &= \left(-\frac{pK_p}{n}+\frac{p^2\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}\right)V_{\alpha+p}(x)+\frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_ib_{ij}^+x_j^{\alpha+p}+o\big(|x|^{\alpha+p}\big) \\ &\leqslant \left(-\frac{pK_p}{n}+\frac{p^2\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}+\frac{\alpha p\max_{1\leqslant j\leqslant n}\{\sum_{i=1}^n c_ib_{ij}^+\}}{\alpha+p}\right)V_{\alpha+p}(x)+o\big(|x|^{\alpha+p}\big) \\ &=: -n^{-1}pb_pV_{\alpha+p}(x)+o\big(|x|^{\alpha+p}\big), \end{aligned} \tag{3.15}$$

where

$$b_p = K_p-\frac{n(p\max_{1\leqslant i\leqslant n}\{c_iB_i\}+\alpha\max_{1\leqslant j\leqslant n}\{\sum_{i=1}^n c_ib_{ij}^+\})}{\alpha+p}.$$

By (3.5), we have

$$\begin{aligned} \lim_{p\to\infty} b_p &= K_\infty - n\max_{1\leqslant i\leqslant n}\{c_iB_i\} \\ &= -\max_{1\leqslant i\leqslant n} c_i(a_{ii}+A)-\max_{1\leqslant i\leqslant n}\{nc_iBi\}>0, \end{aligned}$$

which implies that for sufficiently large p, $b_p>0$. Then Lemma 2.3 gives that there exists a constant $\bar{H}$ such that $H(x)\leqslant\bar{H}$. So we have

$$\begin{aligned} \mathbb{E}U\big(x(t\wedge\tau_k)\big) \leqslant{}& \mathbb{E}U\big(\xi(0)\big)+\bar{H}t+\frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_ib_{ij}^+\mathbb{E}\left[\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{\alpha+p}(s+\theta)\,d\mu_{ij}(\theta)-x_j^{\alpha+p}(s)\right)ds\right] \\ &+\frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_id_{ij}^2\mathbb{E}\left[\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{2\beta+p}(s+\theta)\,d\nu_{ij}(\theta)-x_j^{2\beta+p}(s)\right)ds\right] \\ &+p\sum_{i=1}^n\sum_{j=1}^n c_i\bar{b}_{ij}\mathbb{E}\left[\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{\alpha}(s+\theta)\,d\bar{\mu}_{ij}(\theta)-x_j^{\alpha}(s)\right)ds\right] \\ &+2np\sum_{i=1}^n\sum_{j=1}^n c_id_{ij}^2\mathbb{E}\left[\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{2\beta}(s+\theta)\,d\nu_{ij}(\theta)-x_j^{2\beta}(s)\right)ds\right]. \end{aligned}$$

By the Fubini Theorem and a substitution technique, we may estimate that

$$\begin{aligned} \int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{\alpha+p}(s+\theta)\,d\mu_{ij}(\theta)-x_j^{\alpha+p}(s)\right)ds &= \int_{-\tau}^0 d\mu_{ij}(\theta)\int_\theta^{t\wedge\tau_k+\theta} x_j^{\alpha+p}(s)\,ds-\int_0^{t\wedge\tau_k} x_j^{\alpha+p}(s)\,ds \\ &\leqslant \int_{-\tau}^0 d\mu_{ij}(\theta)\int_{-\tau}^{t\wedge\tau_k} x_j^{\alpha+p}(s)\,ds-\int_0^{t\wedge\tau_k} x_j^{\alpha+p}(s)\,ds \\ &\leqslant \int_{-\tau}^0 \xi_j^{\alpha+p}(\theta)\,d\theta. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{2\beta+p}(s+\theta)\,d\nu_{ij}(\theta)-x_j^{2\beta+p}(s)\right)ds \leqslant \int_{-\tau}^0 \xi_j^{2\beta+p}(\theta)\,d\theta, \\ &\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{\alpha}(s+\theta)\,d\bar{\mu}_{ij}(\theta)-x_j^{\alpha}(s)\right)ds \leqslant \int_{-\tau}^0 \xi_j^{\alpha}(\theta)\,d\theta, \end{aligned}$$

$$\int_0^{t\wedge\tau_k}\left(\int_{-\tau}^0 x_j^{2\beta}(s+\theta)\,d\nu_{ij}(\theta)-x_j^{2\beta}(s)\right)ds\leqslant\int_{-\tau}^0\xi_j^{2\beta}(\theta)\,d\theta.$$

We therefore have

$$\begin{aligned}\mathbb{E}U\big(x(t\wedge\tau_k)\big)\leqslant{}&\mathbb{E}U\big(\xi(0)\big)+\bar{H}t+\frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_i b_{ij}^+\mathbb{E}\int_{-\tau}^0\xi_j^{\alpha+p}(\theta)\,d\theta\\&+\frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_i d_{ij}^2\mathbb{E}\int_{-\tau}^0\xi_j^{2\beta+p}(\theta)\,d\theta\\&+p\sum_{i=1}^n\sum_{j=1}^n c_i\bar{b}_{ij}\mathbb{E}\int_{-\tau}^0\xi_j^{\alpha}(\theta)\,d\theta+2np\sum_{i=1}^n\sum_{j=1}^n c_i d_{ij}^2\mathbb{E}\int_{-\tau}^0\xi_j^{2\beta}(\theta)\,d\theta\\=:{}&K_t,\end{aligned}$$

where K_t is independent of k. By the definition of τ_k, $x_i(\tau_k)=k$ or $1/k$ for some $i=1,2,\dots,n$, so

$$\begin{aligned}\mathbb{P}(\tau_k\leqslant t)\big[u(1/k)\wedge u(k)\big]&\leqslant\mathbb{P}(\tau_k\leqslant t)U\big(x(t\wedge\tau_k)\big)\\&\leqslant\mathbb{E}U\big(x(t\wedge\tau_k)\big)\\&\leqslant K_t,\end{aligned}$$

which implies that

$$\limsup_{k\to\infty}\mathbb{P}(\tau_k\leqslant t)\leqslant\lim_{k\to\infty}\frac{K_t}{u(k)\wedge u(1/k)}=0,$$

as required. □

In this theorem, the diagonally dominant condition (3.1) is independent of the environmental noise. The key condition is $\beta<\alpha/2$. By Assumption 3.1, this condition implies that the intensity of environmental noise weakly depends on the population size, so the environmental noise does not play a crucial role for the global positive solution (also see [15]).

4. Asymptotic bound properties

Theorem 3.2 shows that the solutions of Eq. (1.3) will remain in the positive cone $\mathbb{R}^n_{++}$. This nice positive property admits us to further examine how the solutions vary in $\mathbb{R}^n_{++}$ in more detail under the diagonally dominant condition. Comparison with nonexplosion of the solution, stochastically ultimate boundedness is more interesting from the biological point of view. To discuss stochastically ultimate boundedness, we first examine the pth moment boundedness.

Theorem 4.1. *Let Assumptions* 2.1 *and* 3.1 *hold. If* $\alpha>2\beta$ *and the condition* (3.1) *hold, then for any* $q>0$, *there exists a constant* $\bar{K}_q$ *independent of the initial data such that the global positive solution* $x(t)$ *of Eq.* (1.3) *has the property*

$$\limsup_{t\to\infty}\mathbb{E}\big|x(t)\big|^q\leqslant\bar{K}_q. \tag{4.1}$$

Proof. For any $p>0$, define a C^2-function $\bar{V}_p:\mathbb{R}^n_{++}\to\mathbb{R}_+$ by

$$\bar{V}_p(x)=\sum_{i=1}^n c_i x_i^p, \tag{4.2}$$

where c is defined in (3.5). Applying the Itô formula to $e^t\bar{V}_p(x(t))$ and taking expectation yield

$$\mathbb{E}\bar{V}_p\big(x(t)\big)=e^{-t}\mathbb{E}\bar{V}_p\big(\xi(0)\big)+e^{-t}\mathbb{E}\int_0^t e^s\big[\mathcal{L}\bar{V}_p(x_s)+\bar{V}_p\big(x(s)\big)\big]\,ds, \tag{4.3}$$

where $\mathcal{L}\bar{V}_p$ is defined by

$$\mathcal{L}\bar{V}_p(\varphi)=\sum_{i=1}^n\left[pc_i\varphi_i^p(0)f_i(\varphi)+\frac{p(p-1)}{2}c_i\varphi_i^p(0)g_i^2(\varphi)\right].$$

By Assumption 3.1 and the computation of I_1 and I_2,

$$\begin{aligned}\mathcal{L}\bar{V}_p(\varphi)+\bar{V}_p\big(\varphi(0)\big)&\leqslant I_1+I_2+\sum_{i=1}^n c_i\varphi_i^p(0)\\&\leqslant \Phi\big(\varphi(0)\big)+\frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_ib_{ij}^+\left[\int_{-\tau}^0\varphi_j^{\alpha+p}(\theta)\,d\mu_{ij}(\theta)-e^\tau\varphi_j^{\alpha+p}(0)\right]\\&\quad+\frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_id_{ij}^2\left[\int_{-\tau}^0\varphi_j^{2\beta+p}(\theta)\,d\nu_{ij}(\theta)-e^\tau\varphi_j^{2\beta+p}(0)\right],\end{aligned}$$

where

$$\begin{aligned}\Phi(x)=\left(-\frac{pK_p}{n}+\frac{p^2\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}\right)V_{\alpha+p}(x)+\frac{\alpha pe^\tau}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_ib_{ij}^+x_j^{\alpha+p}\\+\frac{4\beta np(p-1)^+e^\tau}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_id_{ij}^2x_j^{2\beta+p}+\sum_{i=1}^n c_ix_i^p+o\big(|x|^{\alpha+p}\big).\end{aligned}$$

Noting that $2\beta<\alpha$, we therefore have

$$\begin{aligned}\Phi(x)&=\left(-\frac{pK_p}{n}+\frac{p^2\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}+\frac{\alpha pe^\tau\max_{1\leqslant j\leqslant n}\{\sum_{i=1}^n c_ib_{ij}^+\}}{\alpha+p}\right)V_{\alpha+p}(x)+o\big(|x|^{\alpha+p}\big)\\&=:-n^{-1}p\tilde{b}_pV_{\alpha+p}(x)+o\big(|x|^{\alpha+p}\big),\end{aligned}$$

where

$$\tilde{b}_p=K_p-\frac{np\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}-\frac{\alpha ne^\tau\max_{1\leqslant j\leqslant n}\{\sum_{i=1}^n c_ib_{ij}^+\}}{\alpha+p}.$$

By the definition of K_p,

$$\lim_{p\to\infty}\tilde{b}_p=K_\infty-\max_{1\leqslant i\leqslant n}\{nc_iB_i\}>0.$$

So, for sufficiently large p, we have $\tilde{b}_p>0$, which implies that there exists a positive constant $\widetilde{H}>0$ such that $\Phi(x)\leqslant\widetilde{H}$. We therefore obtain

$$\begin{aligned}\mathbb{E}\bar{V}_p\big(x(t)\big)\leqslant e^{-t}\mathbb{E}\bar{V}_p\big(\xi(0)\big)+e^{-t}\mathbb{E}\int_0^t e^s\Bigg[\widetilde{H}+\frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_ib_{ij}^+\left(\int_{-\tau}^0 x_j^{\alpha+p}(s+\theta)\,d\mu_{ij}(\theta)-e^\tau x_j^{\alpha+p}(s)\right)\\+\frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_id_{ij}^2\left(\int_{-\tau}^0 x_j^{2\beta+p}(s+\theta)\,d\nu_{ij}(\theta)-e^\tau x_j^{2\beta+p}(s)\right)\Bigg]ds.\end{aligned}$$

Applying the Fubini Theorem and a substitution technique yields

$$\begin{aligned}\int_0^t e^s\int_{-\tau}^0 x_j^{\alpha+p}(s+\theta)\,d\mu_{ij}(\theta)\,ds-\int_0^t e^{s+\tau}x_j^{\alpha+p}(s)\,ds&=\int_{-\tau}^0 d\mu_{ij}(\theta)\int_\theta^{t+\theta}e^{s-\theta}x_j^{\alpha+p}(s+\theta)\,ds-\int_0^t e^{s+\tau}x_j^{\alpha+p}(s)\,ds\\&\leqslant\int_{-\tau}^0 d\mu_{ij}(\theta)\int_{-\tau}^t e^{s+\tau}x_j^{\alpha+p}(s+\theta)\,ds-\int_0^t e^{s+\tau}x_j^{\alpha+p}(s)\,ds\\&=e^\tau\int_{-\tau}^0 e^\theta\xi_j^{\alpha+p}(\theta)\,d\theta.\end{aligned}$$

Similarly,

$$\int_0^t \int_{-\tau}^0 e^s x_j^{2\beta+p}(s+\theta)\,d\nu_{ij}(\theta)\,ds - \int_0^t e^{s+\tau} x_j^{2\beta+p}(s)\,ds \leqslant e^\tau \int_{-\tau}^0 e^\theta \xi_j^{2\beta+p}(\theta)\,d\theta.$$

We therefore have

$$\begin{aligned}\mathbb{E}\bar V_p\big(x(t)\big) \leqslant{}& e^{-t}\mathbb{E}\bar V_p\big(\xi(0)\big) + \big(1-e^{-t}\big)\widetilde H + e^{-t}\frac{\alpha p e^\tau}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_j b_{ij}^+ \mathbb{E}\int_{-\tau}^0 e^\theta \xi_j^{\alpha+p}(\theta)\,d\theta \\ &+ e^{-t}\frac{4\beta np(p-1)^+e^\tau}{2\beta+p}\sum_{i=1}^n\sum_{i=1}^n c_i d_{ij}^2 \mathbb{E}\int_{-\tau}^0 e^\theta \xi_j^{2\beta+p}(\theta)\,d\theta.\end{aligned}$$

Clearly,

$$\limsup_{t\to\infty} \mathbb{E}\bar V_p\big(x(t)\big) \leqslant \widetilde H.$$

Lemma 2.2 gives

$$\begin{aligned}\limsup_{t\to\infty}\mathbb{E}\big|x(t)\big|^p &\leqslant \limsup_{t\to\infty} n^{(\frac{p}{2}-1)\vee 0}\mathbb{E}V_p\big(x(t)\big) \\ &\leqslant \limsup_{t\to\infty}\frac{n^{(\frac{p}{2}-1)\vee 0}}{\hat c}\mathbb{E}\bar V_p\big(x(t)\big) \\ &\leqslant \frac{n^{(\frac{p}{2}-1)\vee 0}}{\hat c}\widetilde H.\end{aligned}$$

For any $q>0$, choose sufficiently large p such that $p>q$. By the Lyapunov inequality,

$$\mathbb{E}\big|x(t)\big|^q \leqslant \big(\mathbb{E}\big|x(t)\big|^p\big)^{\frac{q}{p}},$$

which implies that

$$\limsup_{t\to\infty}\mathbb{E}\big|x(t)\big|^q \leqslant \left[\frac{n^{(\frac{p}{2}-1)\vee 0}}{\hat c}\widetilde H\right]^{\frac{q}{p}}.$$

Then the desired assertion (4.1) follows by setting $\bar K_q = [n^{(\frac{p}{2}-1)\vee 0}\hat c^{-1}\widetilde H]^{q/p}$. □

From the pth moment boundedness, the stochastically ultimate boundedness will follow directly. We describe it as the following theorem.

Theorem 4.2. *Let Assumptions* 2.1 *and* 3.1 *hold. If* $\alpha > 2\beta$ *and the condition* (3.1) *hold, for any* $\epsilon \in (0,1)$, *there is a positive constant* $M = M(\epsilon)$ *such that for any initial data* $\xi \in C_{++}$, *the solution* $x(t)$ *of Eq.* (1.3) *has the property that*

$$\limsup_{t\to\infty}\mathbb{P}\big\{\big|x(t)\big| \leqslant M\big\} \geqslant 1-\epsilon, \tag{4.4}$$

namely, $x(t)$ *is ultimately bounded.*

Proof. Theorem 4.1 shows that for any $p>0$, $\limsup_{t\to\infty}\mathbb{E}|x(t)|^p \leqslant K_p$. Now for any $\epsilon\in(0,1)$, let $M = K_p^{1/p}/\epsilon^{1/p}$. Then by the Chebyshev inequality

$$\mathbb{P}\big\{\big|x(t)\big| > M\big\} \leqslant \frac{\mathbb{E}|x(t)|^p}{M^p}.$$

Hence

$$\limsup_{t\to\infty}\mathbb{P}\big\{\big|x(t)\big| \leqslant M\big\} \geqslant 1-\epsilon,$$

as required. □

5. Moment average in time

The result in the previous section shows that the solution of Eq. (1.3) will be stochastically ultimately bounded. That is, the solution will be ultimately bounded with large probability. The following result shows that the moment average in time of the solution of Eq. (1.3) will be bounded under the diagonally dominant condition.

Theorem 5.1. *Let Assumptions* 2.1 *and* 3.1 *hold. Under the conditions* (H1) *and* (H2), *if* $\alpha > 2\beta$ *and the condition* (3.1) *hold, then there exists a constant* $\underline{p}$ *such that for any* $p > \underline{p}$, *there exists a constant* $\bar{K}_p^*$ *independent of the initial data such that the global positive solution* $x(t)$ *of Eq.* (1.3) *has the property*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}\big|x(s)\big|^{\alpha+p}\,ds \leqslant \bar{K}_p^*. \tag{5.1}$$

Proof. Directly applying the Itô formula to $\bar{V}_p(x)$ as defined in (4.2) and taking expectation give

$$\mathbb{E}\bar{V}_p\big(x(t)\big) = \mathbb{E}\bar{V}_p\big(\xi(0)\big) + \int_0^t \mathcal{L}\bar{V}_p(x_s)\,ds. \tag{5.2}$$

Repeating the proof of Theorem 4.1 yields

$$\begin{aligned}\mathcal{L}\bar{V}_p(\varphi) &\leqslant I_1 + I_2\\ &\leqslant \Psi\big(\varphi(0)\big) + \frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_i b_{ij}^+\bigg[\int_{-\tau}^0 \varphi_j^{\alpha+p}(\theta)\,d\mu_{ij}(\theta) - \varphi_j^{\alpha+p}(0)\bigg]\\ &\quad + \frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_i d_{ij}^2\bigg[\int_{-\tau}^0 \varphi_j^{2\beta+p}(\theta)\,d\nu_{ij}(\theta) - \varphi_j^{2\beta+p}(0)\bigg],\end{aligned}$$

where

$$\begin{aligned}\Psi(x) &= \bigg(-\frac{pK_p}{n} + \frac{p^2\max_{1\leqslant i\leqslant n}\{c_iB_i\}}{\alpha+p}\bigg)V_{\alpha+p}(x) + \frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_i b_{ij}^+ x_j^{\alpha+p}\\ &\quad + \frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_i d_{ij}^2 x_j^{2\beta+p} + o\big(|x|^{\alpha+p}\big).\end{aligned}$$

By the same way as before, we have

$$\Psi(x) = -n^{-1} p b_p V_{\alpha+p}(x) + o\big(|x|^{\alpha+p}\big).$$

The definition of b_p implies that there exists a $\underline{p}$ such that $b_p > 0$ for all $p > \underline{p}$. Clearly, Lemma 2.3 gives that there exists a constant $\widetilde{H}^*$ such that

$$\begin{aligned}\Psi(x) + \frac{pb_p}{2n}V_{\alpha+p}(x) &\leqslant -\frac{pb_p}{2n}V_{\alpha+p}(x) + o\big(|x|^{\alpha+p}\big)\\ &\leqslant \widetilde{H}^*,\end{aligned}$$

which implies that

$$\begin{aligned}\frac{pb_p}{2n}\int_0^t \mathbb{E}\sum_{i=1}^n x_i^{\alpha+p}(s)\,ds &\leqslant \mathbb{E}\bar{V}_p\big(x(t)\big) + \frac{pb_p}{2n}\int_0^t \mathbb{E}\sum_{i=1}^n x_i^{\alpha+p}(s)\,ds\\ &\leqslant \mathbb{E}\bar{V}_p\big(\xi(0)\big) + \widetilde{H}^* t + \frac{\alpha p}{\alpha+p}\sum_{i=1}^n\sum_{j=1}^n c_i b_{ij}^+\mathbb{E}\int_{-\tau}^0 \xi_j^{\alpha+p}(\theta)\,d\theta\\ &\quad + \frac{4\beta np(p-1)^+}{2\beta+p}\sum_{i=1}^n\sum_{j=1}^n c_i d_{ij}^2\mathbb{E}\int_{-\tau}^0 \xi_j^{2\beta+p}(\theta)\,d\theta.\end{aligned}$$

By Theorem 4.1, we therefore have

$$\frac{pb_p}{2n}\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\sum_{i=1}^n x_i^{\alpha+p}(s)\,ds \leqslant \widetilde{H}^*.$$

Lemma 2.2 gives

$$\frac{pb_p}{2n}n^{(1-\frac{\alpha+p}{2})\wedge 0}\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\big|x(s)\big|^{\alpha+p}\,ds \leqslant \widetilde{H}^*. \tag{5.3}$$

Then we may obtain the desired assertion by setting $\bar{K}_p^* = 2\widetilde{H}^* n^{\frac{\alpha+p}{2}\vee 1}/(pb_p)$. □

6. Stochastic functional Lotka–Volterra systems

In this section, we apply the results in the previous sections to study the following generalized n-dimensional stochastic functional Lotka–Volterra system

$$dx_i(t) = x_i(t)\left[\left(a_i + \sum_{j=1}^n a_{ij}x_j^\alpha(t) + \sum_{j=1}^n \int_{-\tau}^0 b_{ij}(\theta)x_j^\alpha(t+\theta)\,d\mu_{ij}(\theta)\right)dt + \left(q_i + \sum_{j=1}^n q_{ij}x_j^\beta(t)\right)dw(t)\right]$$
$$(1 \leqslant i \leqslant n), \tag{6.1}$$

which is a special case of Eq. (1.3). This equation may be regarded as the stochastically perturbed equation of the generalized functional Lotka–Volterra system

$$\dot{x}_i(t) = x_i(t)\left(a_i + \sum_{j=1}^n a_{ij}x_j^\alpha(t) + \sum_{j=1}^n \int_{-\tau}^0 b_{ij}(\theta)x_j^\alpha(t+\theta)\,d\mu_{ij}(\theta)\right) \quad (1 \leqslant i \leqslant n). \tag{6.2}$$

Define $a = (a_1, \dots, a_n)^T$, $q = (q_1, \dots, q_n)^T \in \mathbb{R}^n$, $A = [a_{ij}]$, $Q = [q_{ij}] \in \mathbb{R}^{n\times n}$, $B(\theta) = [b_{ij}(\theta)] \in C([-\tau, 0]; \mathbb{R}^{n\times n})$, $f = (f_1, \dots, f_n)^T$ and $g = (g_1, \dots, g_n)^T$ and

$$\begin{cases} f_i(\varphi) = a_i + \sum_{j=1}^n a_{ij}\varphi_j^\alpha(0) + \sum_{j=1}^n \int_{-\tau}^0 b_{ij}(\theta)\varphi_j^\alpha(t)\,d\mu_{ij}(\theta), & \text{for } \varphi \in C_{++}, \\ g_i(x) = q_i + \sum_{j=1}^n q_{ij}x_j^\beta(t), & \text{for } x \in \mathbb{R}^n_{++}. \end{cases} \tag{6.3}$$

Clearly, f and g satisfy Assumption 3.1 if we choose $\bar{b}_{ij} = \sup_{-\tau\leqslant\theta\leqslant 0}\{b_{ij}(\theta)\}$ and $d_{ij} = 0$ for $1 \leqslant i, j \leqslant n$. By Theorems 3.2, 4.2 and 5.1, we have following theorem.

Theorem 6.1. *If $\alpha > 2\beta > 0$ and*

$$\sum_{j=1}^n a_{ij}^+ + n\sum_{j=1}^n \bar{b}_{ij}^+ < -a_{ii} \quad (1 \leqslant i \leqslant n), \tag{6.4}$$

then for any initial data $\xi \in C_{++}$, Eq. (6.1) *admits a unique global positive solution $x(t)$ and this solution has the properties* (4.4) *and* (5.1).

If we choose $\alpha = 1$, μ_{ij} is the Dirac measure in the point $-\tau$ and define $b_{ij} = b_{ij}(-\tau)$ and $B = [b_{ij}]$, Eq. (6.1) may be rewritten as the following stochastic delay Lotka–Volterra system

$$dx(t) = \mathrm{diag}\big(x_1(t), \dots, x_n(t)\big)\big[\big(a + Ax(t) + Bx(t-\tau)\big)\,dt + g\big(x(t)\big)\,dw(t)\big], \tag{6.5}$$

where g is defined as (6.3). Applying Theorem 6.1 gives

Theorem 6.2. *Let $\beta < 1/2$ or $Q \equiv 0$. If*

$$\sum_{j=1}^{n} a_{ij}^{+} + n \sum_{j=1}^{n} b_{ij}^{+} < -a_{ii} \quad (1 \leqslant i \leqslant n), \tag{6.6}$$

then for any initial data $\xi \in C_{++}$, Eq. (6.5) *admits a unique global positive solution $x(t)$ and this solution has the properties* (4.4) *and* (5.1)*.*

To close this paper, let us recall the existing results on stochastic Lotka–Volterra system and compare them with our results. In [2], Bahar and Mao study the stochastic delay Lotka–Volterra system

$$dx(t) = \operatorname{diag}\big(x_1(t), \dots, x_n(t)\big)\big[\big(a + Ax(t) + Bx(t-\tau)\big)\,dt + q\,dw(t)\big] \tag{6.7}$$

and obtain

Theorem 6.3. *(See [2, Theorems 2.1 and 3.1].) Assume that there exist $c = (c_1, \dots, c_n)^T \in \mathbb{R}_{++}^n$ and $\theta > 0$ such that*

$$\lambda_{\max}^{+}\left(\frac{1}{2}\big(CA + A^T C\big) + \frac{1}{4\theta} CBB^T C + \theta I\right) \leqslant 0, \tag{6.8}$$

where I is the $n \times n$ identity matrix. Then for any given initial data $\xi \in C_{++}$, there is a unique global positive solution $x(t)$ to Eq. (6.7) *and this solution is ultimately bounded.*

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