Ramsey Properties for Classes of Relational Systems

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A class $\mathcal{K}$ of relational systems (of the same type) has the $\mathcal{R}$-Ramsey property if for every $S \in \mathcal{K}$ there is $T \in \mathcal{K}$ such that to every 2-coloring of $T$ (relational subsystems of $T$ isomorphic to $S$) we can find a monochromatic $U$ for some $U \in T$. Extending recent results by Ježek and Nešetřil we prove it for (a) every class $\mathcal{K}$ of finite reflexive relational systems closed for products and $\mathcal{R} \in \mathcal{K}$ a singleton, (b) every abstract class $\mathcal{K}$ of finite relational systems with the strong amalgamation property and $\mathcal{R} \in \mathcal{K}$ such that the sets from $S$ are disjoint for all $S \in \mathcal{K}$. Finally we prove: Let $\mathcal{K}$ be an abstract class of finite reflexive or areflexive relational systems with the strong amalgamation property. If $\mathcal{K}$ has the $\mathcal{R}$-Ramsey property, then $\mathcal{R}$ is constant.

1. Ramsey properties of categories were introduced in [5, 9]. Here we study it for the categories of relational systems (of the same type) whose morphisms are embeddings. A weaker notion, namely the vertex partition property for relational systems has been studied in [12]. The paper is closely related to Ježek and Nešetřil's paper [8] in the sense that most of the constructions are more or less straight-forward extensions of those from [8]. Since relational systems form a much wider class than universal algebras and presently are rather underdeveloped and even neglected, we feel that the extension may be justified. However, the credit for the intricate proofs belong to Ježek and Nešetřil [8]. No attempt was made to compare the results with the many Ramsey theorems from the literature. Because relational systems seem to be not widely known, we have strived to define all terminology and present complete proofs. The paper was prepared during the first author's stay at CRMA (November 1981–January 1982). The financial assistance provided by NSERC Canada operating grant A-9128 and FCAC Quebec 'subvention d’équipe' E-539 are gratefully acknowledged.

2. Let $\mathbb{N} = \{0, 1, \ldots \}$ denote the set of non-negative integers and let $l := \mathbb{N} \setminus \{0\}$. For $n \in l$ and a set $A$ an $n$-ary relation on $A$ means a subset of $A^n$ (i.e. a set of $n$-tuples $(a_1, \ldots, a_n)$ with all $a_i \in A$). A type $\Delta$ is a pair $(D, \cdot)$ with $D$ a set and $d \rightarrow d'$ a map from $D$ into $l$. A relational $\Delta$-system (on $A$) is $R = (A, \mathcal{R})$ where $\mathcal{R} = \{\rho_d : d \in D\}$ and each $\rho_d$ is a $d'$-ary relation on $A$. The set $A$ is the base set of $R$; if $A$ is finite we say that $R$ is finite. For $B \subseteq A$ call $R \upharpoonright B := (B, R \upharpoonright B)$ (where $R \upharpoonright B := \{\rho_d \upharpoonright B : d \in D\}$) and $\rho_d \upharpoonright B = \rho \cap B^d$ for all $d \in D$) a relational $\Delta$-subsystem of $R$. The set of relational $\Delta$-subsystems of $R$ is denoted by $\text{Sub} \ R$. Let $R_i = (A_i, R_i)$ $(i = 1, 2)$ be two relational $\Delta$-systems. They are isomorphic, in symbols $R_1 \cong R_2$, if there is a bijection $\varphi$ from $A_1$ onto $A_2$ such that $\varphi(\rho_{1d}) = \rho_{2d}$ for all $d \in D$ (where, as usual, $\varphi(\sigma) := \{(\varphi(a_1), \ldots, \varphi(a_n)) : (a_1, \ldots, a_n) \in \sigma\}$ for every $n$-ary relation $\sigma$ on $A_1$); the map $\varphi$ will be referred to as an isomorphism of $R_1$ onto $R_2$. The (direct) product $R_1 \times R_2$ is $(A_1 \times A_2, R_1 \times R_2)$ where $R_1 \times R_2 := \{\rho_{1d} \times \rho_{2d} : d \in D\}$ and

$$\rho_{1d} \times \rho_{2d} := \{(a_1, b_1), \ldots, (a_d, b_d) : (a_1, \ldots, a_d) \in \rho_{1d}, (b_1, \ldots, b_d) \in \rho_{2d}\}.$$
Next put
\[
\left( \begin{array}{c}
R_1 \\
R_2
\end{array} \right) := \{ B \subseteq A_1 : R_1 \upharpoonright B = R_2 \}, \quad \left[ \begin{array}{c}
R_1 \\
R_2
\end{array} \right] = \left\{ R_1 \upharpoonright B : B \in \left( \begin{array}{c}
R_1 \\
R_2
\end{array} \right) \right\}
\]
(the distinction between ( ) and [ ] is rather formal and we use both only for the ease of expression). Let \( \mathcal{H} \) be a class of relational \( \Delta \)-systems and \( \mathcal{R} \) a relational \( \Delta \)-system. We say that \( \mathcal{H} \) has the \( \mathcal{R} \)-Ramsey property if for every \( S \in \mathcal{H} \) there is \( T \in \mathcal{H} \) such that to every 2-coloring of \( \left( \begin{array}{c}
T \\
\mathcal{R}
\end{array} \right) \) we can find a monochromatic \( \left( \begin{array}{c}
U \\
\mathcal{R}
\end{array} \right) \), for some \( U \in \left( \begin{array}{c}
T \\
S
\end{array} \right) \). In other words, for an arbitrary splitting of the relational \( \Delta \)-subsystems of \( T \) isomorphic to \( \mathcal{R} \) into two blocks there is a copy of \( S \) embedded in \( T \) whose restrictions isomorphic to \( \mathcal{R} \) fall all within the same block.

**Example 1.** For \( n \in \mathbb{N} \) set \( n = \{1, \ldots, n\} \). Let \( D = \{0\} \), \( 0' = 2 \) and \( \mathcal{H} = \{ K_n : n \in \mathbb{N} \} \) where \( K_n = (n; n^2 \setminus \{(1,1), \ldots, (n,n)\}) \) is the complete graph on \( n \) vertices. Now \( \mathcal{H} \) has the \( \{2\}-\text{Ramsey property} \) iff for every \( s \in \mathbb{N} \) there is \( t_s \in \mathbb{N} \) such that every graph on \( t_s \) vertices has either an \( s \)-clique (i.e. a complete subgraph on \( s \) vertices) or an \( s \)-independent set (i.e. a set of \( s \) pairwise nonadjacent vertices). The truth of the latter is the starting point of Ramsey theory [13] (cf. also [6]) whose most widely known consequence is \( t_3 = 6 \): In any collection of 6 people, there are 3 people who either know each other or are mutually strangers.

**Example 2.** Let \( n \in \mathbb{N} \). A partial \( n \)-ary operation on a set \( A \) is a map \( f \) from a subset \( P(f) \) of \( A^n \) (called the domain of \( f \)) into \( A \). To \( f \) we assign the \((n+1)\)-ary relation
\[
f^0 := \{(a_1, \ldots, a_n, f(a_1, \ldots, a_n)) : (a_1, \ldots, a_n) \in P(f)\}
\]
on \( A \). A partial algebra of type \( \Delta \) is \( A = \langle A, \{f_d : d \in D\} \rangle \) where each \( f_d \) is an \( d' \)-ary partial operation. Instead of \( A \) we may consider the relational \( \Delta^0 \)-system \( A^0 := \langle A ; f_d : d \in D \rangle \) with \( \Delta^0 = (D, \#) \) where \( d^* = d' + 1 \) for all \( d \in D \). It is immediate that \( A^0_1 \) is a relational \( \Delta^0 \)-subsystem of \( A^0_2 \) iff \( A_1 \) is a relative subalgebra of \( A_2 \) (i.e. the operations of \( A_1 \) are the operations \( f_d \) of \( A_2 \) restricted to the new domain \( \{ x \in P(f_d) \cap A^d : f_d(x) \in A_1 \} \) ). Denote by \( \mathcal{B} \) the family of base sets (universes) of relative subalgebras of \( A \). Next the isomorphism of \( A^0_1 \) and \( A^0_2 \) means exactly the standard isomorphism \( \varphi : A_1 \rightarrow A_2 \) (i.e. for all \( d \in D \) we have \( \varphi(P(f_{d_1})) = P(f_{d_2}) \) and \( \varphi(f_{d_1}(a_1, \ldots, a_d)) = f_{d_2} \left( \varphi(a_1), \ldots, \varphi(a_d) \right) \) for all \( (a_1, \ldots, a_d) \in A^d \)). Thus \( \left( \begin{array}{c}
A_2 \\
A_1
\end{array} \right) := \left( \begin{array}{c}
A^0_2 \\
A^0_1
\end{array} \right) \) is the family of base sets of relative subalgebras of \( A \) isomorphic to \( A_1 \). For example if \( A_2 \) is a lattice with the usual operations \( \vee \) and \( \wedge \) then a relative sublattice \( A_1 \) is obtained by restricting \( \vee \) and \( \wedge \) to pairs \((a, b) \in A^2_1 \) such that \( a \vee b, \text{ resp. } a \wedge b, \) are in \( A_1 \) and thus not necessarily a full lattice. Nevertheless if both \( A_1 \) and \( A_2 \) are lattices, then \( \left( \begin{array}{c}
A_2 \\
A_1
\end{array} \right) \) is the collection of sublattices of \( A_2 \) which are isomorphic to \( A_1 \).

Let \( \mathcal{H} \) be a class of partial algebras of type \( \Delta \) and \( \mathcal{R} \) a partial algebra of type \( \Delta \). We say that \( \mathcal{H} \) has the \( \mathcal{R} \)-Ramsey property if the class \( \mathcal{H}^0 \) (of \( F^0 \) with \( F \in \mathcal{H} \)) has the \( \mathcal{R}^0 \)-Ramsey property. This means that to every \( S \in \mathcal{H} \) there is \( T \in \mathcal{H} \) such that to every 2-coloring of the relative subalgebras of \( T \) isomorphic to \( \mathcal{R} \) we can find a copy of \( S \) embedded in \( T \) whose relative subalgebras isomorphic to \( \mathcal{R} \) are all monochromatic. In particular, a class \( \mathcal{H} \) of (full) algebras (\( \neq \) partial algebras such that \( P(f_d) = A^d \) for all \( d \in D \)) has the \( \mathcal{R} \)-Ramsey property if to each \( S \in \mathcal{H} \) there is \( T \in \mathcal{H} \) such that to every set \( E \) of subalgebras of \( T \) isomorphic to \( \mathcal{R} \) there is a copy \( S' \) of \( S \) in \( T \) with the property that the subalgebras
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of \( S' \) isomorphic to \( R \) are either all in \( E \) or all outside \( E \) [8] (because a relative subalgebra isomorphic to a full algebra is just a plain subalgebra).

3. A relational \( \Delta \)-system \( (A; \{\rho_d: d \in D\}) \) is reflexive if \( (a, \ldots, a) \in \rho_d \) for all \( a \in A \) and \( d \in D \). For binary relations this notion is the ordinary reflexivity, and for higher relations it is the weakest form of reflexivity. For a singleton set \( A = \{r\} \) we have a unique reflexive relational \( \Delta \)-system \( R = (\{r\}; \{\rho_d: d \in D\}) \) where \( \rho_d = \{(r, \ldots, r)\} \) for all \( d \in D \). Since up to isomorphism a reflexive \( \Delta \)-system on a singleton is unique, we denote by \( \Delta \) and use it instead of \( R \).

**Proposition 1.** Let \( \mathcal{K} \) be a class of relational \( \Delta \)-systems closed under finite powers and such that \( \left( \frac{S}{\Delta} \right) \) is finite for all \( S \in \mathcal{K} \). Then \( \mathcal{K} \) has the \( \Delta \)-Ramsey property.

**Proof.** Let \( S = (S; \{\rho_d: d \in D\}) \in \mathcal{K} \), let \( \left( \frac{S}{\Delta} \right) = \{\{s_0\}, \ldots, \{s_k\}\} \) and let \( V = \{s_0, \ldots, s_k\} \). By the Hales-Jewett theorem [7] there is \( n \in \mathbb{N} \) such that to every 2-coloring of \( V^n \) (the set of all maps from \( n = \{1, \ldots, n\} \) into \( V \)) there exist \( J \subseteq n \) and \( g \in V^J \) such that the set

\[
Z := \{f \in V^n: f \upharpoonright J = g, f \upharpoonright (n \setminus J) \text{ is constant}\}
\]

is monochromatic. Put \( T = S^n \) and consider \( W \in \left( \frac{T}{\Delta} \right) \). Clearly \( W = \{w\} \) where \( w = (w_1, \ldots, w_n) \) and by the definition of the product each \( w_i \in V \). Therefore \( \left( \frac{T}{\Delta} \right) = V^n \) and every 2-coloring of \( \left( \frac{T}{\Delta} \right) \) is a 2-coloring of \( V^n \). Let \( Z, J \) and \( g \) be as in the Hales-Jewett Theorem. Define \( Z' = \{f \in S^n: f \upharpoonright J = g, f \upharpoonright (n \setminus J) \text{ constant}\} \). Then \( U := T \upharpoonright Z' \) is isomorphic to \( S \) under any projection \( p_j \) onto a coordinate \( j \in n \setminus J \) (because for all \( d \in D \) and \( k \in J \), by the assumption on \( \Delta \) the \( d' \)-tuple \( (g(k), \ldots, g(k)) \) is in the \( d \)-th relation of \( S \)). Moreover \( \left( \frac{U}{\Delta} \right) = Z \), thus this set is monochromatic.

**Corollary 1.** Let \( \mathcal{K} \) be a class of finite and reflexive relational \( \Delta \)-systems. If \( \mathcal{K} \) is closed under finite powers, then \( \mathcal{K} \) has the vertex partition property: For each \( S \in \mathcal{K} \) there is \( T = (T; T) \in \mathcal{K} \) such that for each subset \( B \) of \( T \) the \( \Delta \)-system \( S \) may be embedded either in \( B \) or in \( T \setminus B \).

**Proof.** Apply Proposition 1 with \( \left( \frac{T}{\Delta} \right) = T \).

**Example 3.** The class of finite posets, or every class of finite posets closed under products has the vertex partition property.

A partial algebra on \( A \) is idempotent if \( f(x, \ldots, x) \) is defined and equal \( x \) for each operation of the algebra and all \( x \in A \).

**Corollary 2.** Let \( \mathcal{K} \) be a class of finite idempotent partial algebras closed under finite powers. Then \( \mathcal{K} \) has the vertex partition property. In particular, the class of finite algebras in an idempotent variety has the vertex partition property.

**Example 4** [8]. The class of finite lattices has the vertex partition property.
4. Comment: The above proof of Proposition 1 is based on the Hales-Jewett theorem. It is worth mentioning that Proposition 1 and the Hales-Jewett theorem are two facets of the same fact.

To see it call a reflexive relational $\Delta$-system $R = (R, R)$ endorigid if it has only the trivial endomorphisms: $id_R$ and the constants $[2, 15]$.

It is not difficult to see that for an isomorphism $\varphi$ of an endorigid $R$ into a power $R^N$ (where, for example, $N$ is a non empty subset of $N$) the image $\varphi(R)$ is of the form $\{f \in R^N : f \upharpoonright J = g$ and $f \upharpoonright (N \setminus J)$ constant $\}$ for some $J \subseteq N$ and $g : N \to R$. Indeed for every $n \in N$ the compositions $p_n \circ \varphi$ of the projection $p_n$ with $\varphi$ is an endomorphism and thus either $id_R$ or constant; therefore it suffices to set $J = \{n \in N : p_n \circ \varphi$ constant $\}$ and $g(n) = p_n \circ \varphi$. As $\varphi(R)$ has the form invoked in the Hales-Jewett theorem, the fact that there is a power $R^n$ such that for each $B \subseteq R^n$ the relation $R$ may be embedded in either $B$ or $R^n \setminus B$ is just a reformulation of this theorem. Consequently the fact that $\mathcal{H} = \{R^n; n \in \mathbb{N}\}$ has the vertex partition property yields the Hales-Jewett theorem.

Now we give a few concrete examples of endorigid relations:

As simplest-minded example we have the $m$-ary relation $R$ on an $m$-elements set, say $m = \{1, 2, \ldots, m\}$, consisting of all the $(x, x_1, \ldots, x_m)$ and of $(1, 2, \ldots, m)$. There is an example consisting of binary relations: Let $R = m$ and $R = (R, (R^i)_{i \leq m})$ where $R^i$ is the binary relation such that $(x, y) \in R^i$ iff $x = y$ or $x = i$ and $y = j$. This $R$ is endorigid because if $f$ is a non constant endomorphism and $x, y$ are two elements of $R$ such that $x < y$ and $f(x) \neq f(y)$ then, as $f$ preserves $R_{xy}$, it follows $x = f(x)$ and $y = f(y)$.

There is an example consisting of a ternary relation $\{\rho\}$. For $m \geq 3$, let $R = m$ and $\rho = \{(a, b, c) : 1 \leq a < b < c\} \cup \{(1, 1, 1), \ldots, (m, m, m)\}$. The projection of $\rho$ on its first, resp. last, two coordinates is $\{(x, y) : 1 \leq x \leq y \leq m\} \cup \{(m, m)\}$, resp. $\{(x, y) : 1 \leq x \leq y \leq m\} \cup \{(1, 1)\}$. Therefore each endomorphism $f$ of $R$ is non decreasing. Let $f$ be a nonconstant endomorphism of $R$. If $f$ is no injective, then there is $1 \leq i \leq m - 1$ such that either $f(i) = f(i + 1) < f(i + 2)$ or $f(i) < f(i + 1) = f(i + 2)$; However then $f(i), f(i + 1), f(i + 2) \notin \rho$ which is contradictory. Thus $f$ is injective, and then $f = id_m$ proving that $R$ is endorigid.

For more complex examples of finite or countable binary endorigid relations the reader is referred to $[2, 15]$.

A natural question is a generalization of Proposition 1 to classes of infinite reflexive relational systems closed under arbitrary powers (amounting to an infinite version of the Hales-Jewett Theorem).

5. Instead of direct products we turn to another type of product.

We say that a class $\mathcal{H}$ of relational $\Delta$-systems is closed under ordinal (or lexicographic) squares if to each $R = (R; \{\rho_d : d \in D\}) \in \mathcal{H}$ there is $S = (S, \{\sigma_d : d \in D\}) \in \mathcal{H}$ such that for each $d \in D$ we have

(a) $((r_1, s), \ldots, (r_d, s)) \in \sigma_d \iff (r_1, \ldots, r_d) \in \rho_d$ for all $s \in R$ and

(b) if $r_1, \ldots, r_d, s_1, \ldots, s_d \in R$ such that $[s_1, \ldots, s_d] > 1$ and $s_i \Rightarrow r_i = r_j$ for all $1 \leq i < j \leq d'$, then $((r_1, s_1), \ldots, (r_d, s_d)) \in \sigma_d \iff (s_1, \ldots, s_d) \in \rho_d$.

Clearly in an ordinal square each $R \times \{s\}$ is a copy of $R$; moreover we have $((r_1, s_1), \ldots, (r_d, s_d)) \in \sigma_d$ whenever $(s_1, \ldots, s_d) \in \rho_d$ and $s_1, \ldots, s_d$ are pairwise distinct while nothing is stipulated if $s_i = s_j$ and $r_i \neq r_j$. The last ambiguity does not arise for relational systems consisting of at most binary relations. For a single binary relation the ordinal square is well known: If $\rho$ is a graph (considered as a reflexive and symmetric binary relation) its square is called the $x$-join (Sabidussi [14]) while if $\rho$ is a partial order it is termed a lexicographic or ordinal square (Birkhoff [1]). The general case is discussed in Pouzet [12]. A relation $\rho$ on $A$ is areflexive if $(a, \ldots, a) \in \rho$ for no $a \in A$. The following is very easy and even does not use Ramsey theorem.
Proposition 2. Let \( \mathcal{H} \) be a class of relational \( \Delta \)-systems such that each \( R \in \mathcal{H} \) consists of reflexive or anti-reflexive relations. If \( \mathcal{H} \) is closed under ordinal squares, then \( \mathcal{H} \) has the vertex partition property.

Proof. Let \( R \in \mathcal{H} \) and let \( S \) be as above. Assume that \( S \) is bicolored. If there is \( s \in R \) such that \( T := R \times \{s\} \) is monochromatic, we are done because \( S \upharpoonright T = R \) by (a). Thus assume that each \( R \times \{r\} \) is bichromatic. Then there is \( f: R \to R \) such that \( U := \{(f(r), r) : r \in R\} \) is monochromatic. Let \( \varphi: R \to R^2 \) be defined by \( \varphi(r) = (f(r), r) \) for all \( r \in R \). Clearly \( \varphi \) is an injection. To prove that \( \varphi(\rho_d) \subseteq \sigma_d \upharpoonright U \) for all \( d \in D \), let \( (s_1, \ldots, s_d) \in \rho_d \). If \( s_1 = \cdots = s_d = s \), then by (a): \( (\varphi(s_1), \ldots, \varphi(s_d)) = ((f(s), s), \ldots, (f(s), s)) \in \sigma_d \). If \( \{s_1, \ldots, s_d\} \) has at least two elements then, since \( s_i = s_j \) implies \( \varphi(s_i) = \varphi(s_j) \), applying (b) we get \( (\varphi(s_1), \ldots, \varphi(s_d)) \in \sigma_d \). Thus \( \varphi(\rho_d) \subseteq \sigma_d \upharpoonright U \). The converse is quite similar, therefore \( \varphi \) is an isomorphism of \( R \) onto \( S \upharpoonright U \).

Example 5. For every integer \( n \), the class of ordered sets with order-dimension at most \( n \) (i.e. the class of ordered sets which can be order embedded into direct products of \( n \) chains) has the vertex partition property.

6. Let \( \mathcal{H} \) be a class of relational \( \Delta \)-systems. A family \( F \subseteq \mathcal{H} \) is consistent if the restrictions \( R_1 \upharpoonright (R_1 \cap R_2) \) and \( R_2 \upharpoonright (R_1 \cap R_2) \) coincide and belong to \( \mathcal{H} \) for all \( R_i = (R_i, R_i) \in F \) \( (i = 1, 2) \). The class \( \mathcal{H} \) has the (finite) strong amalgamation property if to every (finite) consistent family \( F \subseteq \mathcal{H} \) there is \( T \in \mathcal{H} \) such that each \( R_i \in F \) is the restriction \( T \upharpoonright R_i \) of \( T \). As usual, the class \( \mathcal{H} \) is abstract if it contains all isomorphic copies of its members.

A finite family \( M \) of sets is sparse if \( |X \cap Y| = 1 \) for all pairs \( X, Y \) of distinct members of \( M \). A typical example can be obtained as follows: For integers \( n \) and \( m \) let \( V \) be an \( n \)-element set and let \( M \) be the collection of the \( n \)-element sets \( \{f \in V^\alpha : f \upharpoonright J = g, f \upharpoonright (m \setminus J) \ \text{constant}\} \) where \( J \) runs through the proper subsets of \( m := \{1, \ldots, m\} \) and \( g \) through \( V^J \). It is easy to verify that \( M \) is sparse. Thus from the Hales-Jewett theorem it follows that for every integer \( n \) there is a sparse set \( M \) of \( n \)-element sets such that every 2-coloring of the union \( UM \) of \( M \) is monochromatic on a set from \( M \) (Erdős and Hajnal [3]). We have:

Proposition 3. Let \( \mathcal{H} \) be an abstract class of relational \( \Delta \)-systems with the finite strong amalgamation property. If \( R \in \mathcal{H} \) is such that for all \( S \in \mathcal{H} \) the family \( \left( \frac{S}{R} \right) \) is finite and consists of disjoint sets then \( \mathcal{H} \) has the \( R \)-Ramsey property.

Proof. Let
\[ S = (S; \exists) \in \mathcal{H}, \quad \left( \frac{S}{R} \right) := \{R_1, \ldots, R_n\}, \quad B := R_1 \cup \cdots \cup R_m \quad C = S \setminus B \]
and \( \alpha_i \) an isomorphism of \( R_i \) onto \( R \) \( (i = 1, \ldots, n) \). By the assumption \( \{R_1, \ldots, R_m, C\} \) is a partition of \( S \) into pairwise disjoint sets. Let \( M \) be a sparse set of \( n \)-element sets such that every 2-colouring of the union \( UM \) of \( M \) is monochromatic on a set from \( M \) (Erdős and Hajnal [3]). We have:

\[ \varphi_X(y) = \begin{cases} (x_i, \alpha_i(y)) & \text{if } y \in R_i \\ (X, y) & \text{if } y \in C \end{cases} \]

Denote by \( J_X \) the image of this map \( \varphi_X \). Clearly \( \varphi_X: S \to J_X \) is a bijection. We can equip \( J_X \) with the relational \( \Delta \)-system \( J_X = \varphi_X(S) \) so that \( \varphi_X \) is an isomorphism of \( S \) onto
$I_X = (J_X; J_X)$. Note that $J_X$ belongs to the abstract class $\mathcal{K}$. Due to its construction

$$
\left( \frac{I_X}{R} \right) = \{ \{ x_1 \} \times R, \ldots, \{ x_n \} \times R \}.
$$

We prove that the family $J := \{ I_X : X \in M \}$ is consistent. Let $X := \{ x_1, \ldots, x_n \}$ and $Y := \{ y_1, \ldots, y_m \}$ be two distinct sets from $M$ such that $Z := J_X \cap J_Y \neq \emptyset$. Then $Z \subseteq (X \times R) \cap (Y \times R) = (X \cap Y) \times R$. The family $M$ being sparse, we have $X \cap Y = \{ z \}$ where $z = x_i = y_j$ for some $1 \leq i, j \leq n$ and consequently $Z = \{ z \} \times R$. We prove that $I_X | Z = I_Y | Z$. Write $S := \{ \sigma_d : d \in D \}$. For $d \in D$ consider $((x_i, q_1), \ldots, (x_i, q_d)) \in \varphi_X(\sigma_d)$. 

Set $q_1^* = \alpha_i^{-1}(q_k) \ (k = 1, \ldots, d')$ and observe that $(q_1^*, \ldots, q_d^*) \in \sigma_d \upharpoonright R_i$. Applying first the isomorphism $\alpha_i : R_i \rightarrow R$ and then $\alpha_j^{-1} : R \rightarrow R_j$ we obtain $(\alpha_i^{-1}(q_1), \ldots, \alpha_j^{-1}(q_d)) \in \sigma_d \upharpoonright R_j$. It follows that $((y_j, q_1), \ldots, (y_j, q_d)) \in \varphi_Y(\sigma_d)$ whereupon $\varphi_X(\sigma_d) \upharpoonright Z \subseteq \varphi_Y(\sigma_d) \upharpoonright Z$. By symmetry, we have equality whence the required $I_X \upharpoonright Z = I_Y \upharpoonright Z = R$.

The finite strong amalgamation property of $\mathcal{K}$ guarantees $\{ I_X : X \in M \} \subseteq \text{Sub } T$ for some $T \in \mathcal{K}$. Consider an arbitrary 2-coloring of $\left( \frac{T}{S} \right)$. For $m \in X \in M$ the set $\{ m \} \times R$ belongs to

$$
\left( \frac{I_X}{R} \right) \subseteq \left( \frac{T}{S} \right)
$$

and consequently we can transfer the color of $\{ m \} \times R$ to $m$. This establishes a 2-coloring of $UM$. The choice of $M$ is such that there exists a monochromatic $X \in M$. Taking into account (1) we get that $\left( \frac{I_X}{R} \right)$ is monochromatic whereupon $I_X$ isomorphic to $S$ is the required relational $\Delta$-system.

**Corollary 3.** Let $\mathcal{K}$ be an abstract class of finite relational $\Delta$-systems consisting of reflexive or areflexive relations. If $\mathcal{K}$ has the finite strong amalgamation property then $\mathcal{K}$ has the vertex partition property.

**Remark.** To get the conclusion we just need to have, for every $S \in \mathcal{K}$, some $T \in \mathcal{K}$ and a sparse set $M \subseteq \left( \frac{T}{S} \right)$ such that every 2-coloring of the union $UM$ is monochromatic on a set from $M$. This condition is also satisfied under the conditions of Corollary 1.

**Corollary 4 ([8], prop. 3.2).** Let $\mathcal{K}$ be an abstract class of finite algebras with the strong amalgamation property. Then $\mathcal{K}$ has the $R$-Ramsey property for every $R \in \mathcal{K}$ without proper subalgebras.

**Example 6.** Let $\mathcal{K}$ be the class of all finite binary relations $\rho$ with no adjacent full edges (i.e. $(x, y), (y, z), (z, y) \in \rho \Rightarrow (y, x) \notin \rho$ for $x \neq y \neq z \neq x$). Then to every $S \in \mathcal{K}$ there exists $T \in \mathcal{K}$ such that for every 2-coloring of the full edges of $T$ the digraph $T$ contains a copy of $S$ whose full edges are all of the same color.

7. An $h$-ary relation $\rho$ on a finite $A$ is constant if $\pi(\rho) = \rho$ for all permutations $\pi$ of $A$ (this terminology is due to Fraïssé [4]). For operations this notion corresponds to the homogeneity introduced by Marczewski [10]. Let $\theta$ be an equivalence on $\{ 1, \ldots, h \}$ with at most $|A|$ blocks, and define:

$$
(\theta) = \{ (x_1, \ldots, x_h) \in A^h : x_i = x_j \Leftrightarrow i \theta j \}.
$$
It is easy to see that $\rho$ is constant iff $\rho$ is the union of relations of type $(\theta)$. Thus for $h=2$ we have $(id_2) = A^2 \setminus \iota_A$, $(2^2) = \iota_A$ where $\iota_A = \{(a, a) : a \in A\}$ and therefore there are 4 constant binary relations $\phi$, $\iota_A$, $A^2 \setminus \iota_A$, $A^2$ (A complete description of constant relations is given by R. Fraïssé [4]).

**Proposition 4.** Let $\mathcal{H}$ be an abstract class of finite relational $\Delta$-systems consisting of reflexive or areflexive relations and $\mathcal{R} \in \mathcal{H}$. If $\mathcal{H}$ has the finite strong amalgamation property (or simply if every sparse subfamily $F$ of $\mathcal{H}$ is included in $\text{Sub} \ Y$ for some $Y \in \mathcal{H}$) and if $\mathcal{H}$ has the $\mathcal{R}$-Ramsey property then $\mathcal{R}$ is constant.

**Proof.** Let $\mathcal{R} \in \mathcal{H}$, $\mathcal{R} = \{1, \ldots , n\}$, $\mathcal{H}$ have the $\mathcal{R}$-Ramsey property and $\pi$ a permutation of $\{1, \ldots , n\}$. Put $R^* = \{n+1, \ldots , 2n\}$, $\mu(i) = n + i$ for $i = 1, \ldots , n$, and $R^* = \mu(\mathcal{R})$. The family $\{R, R^*\}$ being sparse there is $Y = (V, Y) \in \mathcal{H}$ such that both $\mathcal{R}$ and $R^*$ belong to $\text{Sub} \ V$. Extend the following order $1 < 2 < \cdots < n < n + \pi(1) < \cdots < n + \pi(n)$ to a linear order $< on $V$ (a linear order or chain is a partial order in which all elements are comparable). Set $m = |V|$. According to [11] (see also [8] Lemma 3.5) there is a sparse family $M$ of $m$-element chains such that every linear order on the union $UM$ of $M$ agrees with the given chain on $X$ for at least on $X \in M$. For every $X \in M$ let $\varphi_X : V \to X$ denote the unique order-preserving bijection of the linear order $< on $V$ onto the chain on $X$, and let $X = (X, \varphi_X(V))$. Now $\{X : X \in M\}$ forms a sparse family in the abstract class $\mathcal{H}$ and therefore all $\varphi_X$ are subsystems of some $S \in \mathcal{H}$. Let $T \in \mathcal{H}$ be the relational $\Delta$-system corresponding to $S$ by the $\mathcal{R}$-Ramsey property. We produce a 2-coloring of $(T_R)$. Choose an arbitrary linear order $\subset$ on $T$, take $U \in (T_R)$, and write $U = \{u_1, \ldots , u_n\}$ with $u_1 \subset \cdots \subset u_n$. Color $U$ black if the map $\psi_U$ defined by $i \to u_i$ $(i = 1, \ldots , n)$ is an isomorphism of $\mathcal{R}$ onto $U$ and color $U$ white otherwise. By the $\mathcal{R}$-Ramsey property there exists an isomorphism $\xi$ from $S$ onto $S' \in \text{Sub} \ T$ such that all elements of $(S'_R)$ are monochromatic. We transfer the colors of $(S'_R)$ to $W := \binom{S_R}{R}$ by requiring that $A$ and $\xi(A)$ have the same color. Let $\prec$ denote the linear order on $S$ which is the preimage of $\subset$ under $\xi$. Clearly $A := \{a_1, \ldots , a_n\} \in (S_R)$ with $a_1 \prec \cdots \prec a_n$ is colored black iff $\lambda_A$ defined by $i \to a_i$ $(i = 1, \ldots , n)$ is an isomorphism of $\mathcal{R}$ onto $A$. The basic property of $M$ guarantees that $\prec$ agrees with the chain on $X$ for some $X \in M$. Clearly $\varphi_X(R) \in W$ is colored black, hence all members of $W$ are black and, in particular, $\varphi_X(R')$ is black. Writing $\varphi_X(R') = B = \{b_1, \ldots , b_n\}$ where $b_1 \prec \cdots \prec b_n$ we obtain that $\lambda_B$ defined by $i \to b_i$ $(i = 1, \ldots , n)$ is an isomorphism. Hence $\xi := \mu^{-1}\varphi_X^{-1}\lambda_B$ is also an isomorphism. However $\xi(i) = \mu^{-1}\varphi_X^{-1}(b_i) = \mu^{-1}(n + \pi(i)) = \pi(i)$ $(i = 1, \ldots , n)$ show that $\pi = \xi$ is an automorphism of $\mathcal{R}$.

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**References**


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