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Topology and its Applications 155 (2008) 1265–1296

Topology and its Applications

www.elsevier.com/locate/topol

Asymptotic dimension

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Received 25 March 2007; received in revised form 31 October 2007

Abstract

The asymptotic dimension theory was founded by Gromov [M. Gromov, Asymptotic invariants of infinite groups, in: Geometric Group Theory, vol. 2, Sussex, 1991, in: London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295] in the early 90s. In this paper we give a survey of its recent history where we emphasize two of its features: an analogy with the dimension theory of compact metric spaces and applications to the theory of discrete groups.

MSC: primary 55M10; secondary: 20F69

Keywords: Dimension; Asymptotic dimension; Discrete group

Counting dimensions we are definitely not counting “things”.

Yu. Manin [71]

1. Introduction

Dimension is a basic concept in mathematics. It came from Greek geometry and made its way to all branches of modern mathematics and science. Now dimension can be related to almost every mathematical object.

In this survey we consider the asymptotic dimension of metric spaces, in particular, discrete finitely generated groups taken with word metrics. Asymptotic dimension theory bears a great deal of resemblance to dimension theory in topology. In topology there are three definitions of dimension: the covering dimension (dim), the large inductive dimension (Ind) and the small inductive dimension (ind). All three notions agree for separable metric spaces. In view of the fact that $\mathrm{Ind}\, X = \dim X$ for all metrizable spaces whereas $\mathrm{ind}\, X$ can be different, the dimension ind is less important outside the class of separable metric spaces. For this reason, it and its asymptotic analog will be avoided in this survey. We refer to [50] for a definition of the asymptotic version of ind. The role of the asymptotic analog of Ind

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[1] The author was partially supported by NSF grant DMS-0604494.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.02.011

is limited to one application in this stage of the theory. Thus, we will use the word "dimension" to mean the covering dimension and the term "asymptotic dimension" for the asymptotic analog of the covering dimension.

Just as topological dimension is invariant under homeomorphisms, its asymptotic analog is invariant under coarse isometries. Here a *coarse isometry* is an equivalence in the coarse category. *The coarse category* can be described as follows. It is the category whose objects are metric spaces (X, d_X) and whose morphisms are (not necessarily continuous) maps $f:(X, d_X) \to (Y, d_Y)$ that are *metrically proper* (i.e., the preimage of every bounded set is bounded), and are *uniformly expansive* (i.e., there is a function $\rho:\mathbb{R} \to \mathbb{R}$ such that $d_Y(f(x), f(x')) \leqslant \rho(d_X(x, x'))$). Two morphisms are said to be equivalent if they are in finite distance from each other, i.e., there is a constant $D > 0$ such that $d_Y(f(x), g(x)) < D$ for all $x \in X$. The typical example of a coarse isometry is the inclusion of the integers into the reals $\mathbb{Z} \to \mathbb{R}$ supplied with the standard metric $d(x, y) = |x - y|$. Also, we note that every bounded metric space is coarsely equivalent to a point. For more details, see Sections 3.2 and 12.

Asymptotic dimension was introduced by Gromov as an invariant of a finitely generated group [61]. A (finite) symmetric generating set S on a group Γ defines the word metric d_S by taking the distance $d_S(x, y)$ to be the length of the shortest presentation of $x^{-1}y$ in the alphabet S. Such a metric is left-invariant. For a finitely generated group Γ, any choice of generating sets gives rise to coarsely equivalent metric spaces and hence, the asymptotic dimension $\operatorname{asdim} \Gamma$ is a group invariant. Significant attention to asymptotic dimension was brought by a theorem of Guoliang Yu [94] that proves the Novikov higher signature conjecture for manifolds whose fundamental group has finite asymptotic dimension. Later the integral Novikov conjecture and some of its relatives were proved for groups with finite asymptotic dimension [3,29,32,34,42]. Unfortunately not all finitely presented groups have finite asymptotic dimension. For example, Thompson's group F has infinite asymptotic dimension since it contains $\mathbb{Z}^n$ for all n. Many relatives of the Novikov conjecture (such as the Borel conjecture) are of great interest for groups with finite cohomological dimension. This makes the problem similar to the old Alexandroff Problem from dimension theory: *Does the cohomological dimension of a space coincide with its covering dimension?* In particular, this makes the asymptotic dimension theory a more attractive subject.

This paper consists of two parts. In the first part we survey the development of the asymptotic dimension theory of metric spaces. In the second part we discuss applications to groups and we present some computations of the asymptotic dimension and some finite dimensionality results. Our survey is in no way complete. We do not discuss the many now-existing generalizations of asymptotic dimension, nor do we present any applications of asymptotic dimension to coarse geometry and to the Novikov-type conjectures; we omit entirely cohomological asymptotic dimension theory.

I. Asymptotic dimension of spaces

2. Basic facts of classical dimension theory

Unless stated otherwise, all topological spaces considered in this paper are paracompact.

2.1. Definitions

The definition of the covering dimension dim of a topological space X is due to Lebesgue: $\dim X \leqslant n$ *if and only if for every open cover $\mathcal{V}$ of X there is an open cover $\mathcal{U}$ of X refining $\mathcal{V}$ with multiplicity* $\leqslant n + 1$.

We will use the words *order* and *multiplicity* of a cover interchangeably to mean the largest number of elements of the cover meeting any point of the space. Given a cover $\mathcal{V}$ of a topological space, we say that the cover $\mathcal{U}$ *refines* $\mathcal{V}$ if every $U \in \mathcal{U}$ is contained in some element $V \in \mathcal{V}$.

As usual, we define $\dim X = n$ if it is true that $\dim X \leqslant n$ but it is not true that $\dim X \leqslant n - 1$.

Recall that a closed subset C of a topological space X is a *separator* between disjoint subsets $A, B \subset X$ if $X \setminus C = U \cup V$, where U and V are open subsets in X, $U \cap V = \emptyset$, $A \subset U$, $B \subset V$. A closed subset C of a topological space X is a *cut* between disjoint subsets A and $B \subset X$ if every continuum (compact connected space) $T \subset X$ that intersects both A and B also intersects C. The definition of large inductive dimension is due to Brouwer and Poincaré: $\operatorname{Ind} X \leqslant n$ *if for every pair of closed disjoint sets A and $B \subset X$ there is a separator C with* $\operatorname{Ind} C \leqslant n - 1$ *where* $\operatorname{Ind} X = -1$ *if and only if* $X = \emptyset$. By replacing separators with cuts we obtain the definition of Brouwer's *Dimensiongrad*, $\operatorname{Dg}(X)$. For compact Hausdorff spaces there are the inequalities:

$$\dim X \leqslant \operatorname{Dg}(X) \leqslant \operatorname{Ind} X.$$

The definition of the small inductive dimension is due to Menger and Urysohn: $\operatorname{ind} X = -1$ if and only if $X = \emptyset$; $\operatorname{ind} X \leqslant n$ if for every point $x \in X$ and every neighborhood U of x there is a smaller neighborhood V, $x \in V \subset U$ with $\operatorname{ind}(\partial V) \leqslant n-1$.

Theorem 1. *For a compact metric space X the following conditions are equivalent.*

(1) $\dim X \leqslant n$;
(2) $\operatorname{Ind} X \leqslant n$;
(3) $\operatorname{ind} X \leqslant n$;
(4) $\operatorname{Dg}(X) \leqslant n$;
(5) *Every continuous map $f : A \to S^n$ of a closed subset $A \subset X$ to the n-sphere has a continuous extension $\bar{f} : X \to S^n$*;
(6) *For every $\varepsilon > 0$ there is an ε-map $\phi : X \to K^n$ to an n-dimensional polyhedron*;
(7) *X is the limit of an inverse sequence, $X = \lim_{\leftarrow} K_i$, of n-dimensional polyhedra K_i*;
(8) *For every $\varepsilon > 0$ there is an ε-cover $\mathcal{U}$ of X which can be decomposed into $n+1$ disjoint families $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$*;
(9) *For every $m \geqslant n+1$ and every $\varepsilon > 0$ there exist m discrete families of open sets $\mathcal{U}_1, \ldots, \mathcal{U}_m$ of mesh $\leqslant \varepsilon$ such that the union of every $n+1$ subfamilies is a cover of X*;
(10) *X can be presented as the union of $n+1$ 0-dimensional subsets*;
(11) *X admits a light map $f : X \to I^n$ onto the n cube.*

The equivalence of (4) to all other was proven in [56]. The equivalence of (9) to the rest is proven in [79]. All other equivalences are well-known facts and can be found in any textbook (see, for example [54]). One can extend these equivalences (probably with the exception of (4)) to all separable metric spaces by replacing (where it is appropriate) an arbitrary ε by an arbitrary open cover of X.

We now recall all the terminology used in the above theorem: a map $\phi : X \to Y$ of a metric space is an *ε-map* if $\operatorname{diam}(\phi^{-1}(y)) < \varepsilon$ for all $y \in Y$; a cover $\mathcal{U}$ is an *ε-cover* if $\operatorname{diam} U < \varepsilon$ for all $U \in \mathcal{U}$; a family $\mathcal{V}$ of subsets of a space X is *disjoint* if $V \cap V' = \emptyset$ for all $V, V' \in \mathcal{V}$, $V \neq V'$; a map $f : X \to Y$ is called *light* if $\dim f^{-1}(y) = 0$ for all $y \in Y$.

Note that for every closed subset $Y \subset X$, $\dim Y \leqslant \dim X$. This is also true for every subset of a metric space and is not generally true for every subset of compact (non-metrizable) spaces [54].

2.2. Union theorems

Theorem 2. *Let X be a separable metric space, then*:

(1) $\dim X = \max\{\dim A, \dim(X \setminus A)\}$ *for every closed subset $A \subset X$.*
(2) (*Countable Union Theorem*) *Let $X = \bigcup_{i=1}^{\infty} X_i$ be the countable union of closed subsets, then* $\dim X = \sup_i\{\dim X_i\}$.
(3) (*Menger–Urysohn sum Formula*) $\dim(X \cup Y) \leqslant \dim X + \dim Y + 1$ *for arbitrary sets.*

2.3. Dimension and mappings

Theorem 3 *(Hurewicz Mapping Theorem 1). Let $f : X \to Y$ be a map between compact spaces. Then*

$$\dim X \leqslant \dim Y + \dim f,$$

where $\dim f = \sup\{\dim f^{-1}(y) \mid y \in Y\}$.

The Hurewicz Mapping Theorem 1 implies that a light map of a compactum cannot lower the dimension. The Hurewicz Mapping Theorem 2 implies that a map with finite preimages cannot raise the dimension:

Theorem 4 *(Hurewicz Mapping Theorem 2). Let* $f: X \to Y$ *be a surjective map between compact spaces with* $|f^{-1}(y)| \leqslant k$. *Then*

$$\dim Y \leqslant \dim X + k - 1.$$

We note that these theorems hold for all metric spaces and closed mappings.

2.4. Dimension of the product

Most of the theorems on the dimension of product are proven using a cohomological approach.

Theorem 5. *For all normal spaces* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

Note that when one of the factors is compact, this theorem follows from the Hurewicz Mapping Theorem 1.

Theorem 6 *(Morita Theorem). For all normal spaces* X, $\dim(X \times [0,1]) = \dim X + 1$.

The corresponding equality does not hold for 2-dimensional subsets $X \subset \mathbb{R}^3$ if one replaces the interval [0, 1] by a continuum [45], although the following does hold:

Theorem 7. *For a compact space* X *and a continuum* Y *there is the inequality*: $\dim(X \times Y) \geqslant \dim X + 1$.

Theorem 8. *For a compact metric space* X *either* $\dim(X \times X) = 2\dim X$ *or* $\dim(X \times X) = 2\dim X - 1$.

This theorem divides all compacta in two types: *regular* when $\dim(X \times X) = 2\dim X$ and *Boltjanskii* when $\dim(X \times X) = 2\dim X - 1$. We call compact metric spaces X of the second type *Boltjanskii compacta* in honor of the person who first exhibited them [14]. We note that for all $n \in \mathbb{N}$, $\dim X^n = n\dim X$ for regular compacta and $\dim X^n = n\dim X - n + 1$ for Boltjanskii compacta. There is a theory of Bockstein that allows one to determine the type of a compactum by means of its cohomological dimension with respect to different coefficient groups [40]. This theory allows one to compute the dimension of the product. The Bockstein theory combined with the Realization Theorem of [41] implies

Theorem 9. *For any natural numbers* m, n, *and any* k *with* $\max\{m, n\} < k \leqslant m + n$, *there are compact metric spaces* X *and* Y *with the dimensions* $\dim X = m$, $\dim Y = n$, $\dim(X \times Y) = k$.

The cohomological dimension theory allows one to improve the union theorem and the Hurewicz Mapping Theorem 1:

Theorem 10. *Suppose that* $X \cup Y$ *is a compactum, then*

(1) $\dim(X \cup Y) \leqslant \dim(X \times Y) + 1$ *if* $X \cup Y$ *is regular*;
(2) $\dim(X \cup Y) \leqslant \dim(X \times Y) + 2$ *if* $X \cup Y$ *is Boltjanskii.*

There are examples where the inequality (2) is sharp.

Theorem 11. *Let* $f: X \to Y$ *be a map between compacta. Then*

(1) $\dim X \leqslant \sup\{\dim(Y \times f^{-1}(y)) \mid y \in Y\}$ *if* X *is regular*;
(2) $\dim X \leqslant \sup\{\dim(Y \times f^{-1}(y)) \mid y \in Y\} + 1$ *if* X *is Boltjanskii.*

We note that in both theorems, statement (2) is not always an improvement on the classic results.

2.5. Embedding theorems

We denote the Stone–Čech compactification of a space X by βX.

Theorem 12. $\dim X = \dim \beta X$.

This together with the Mardešić Factorization theorem implies the existence of compact metric spaces of dimension n that contain a topological copy of every separable metric space of dimension n. Such compacta are called *universal*.

Theorem 13 *(Mardešić Factorization Theorem). For every continuous map $f : X \to Y$ of a compact Hausdorff space to a metric space there are maps $g : X \to X'$ and $f' : X' \to Y$ such that X' is metrizable, $g(X) = X'$, $\dim(X') \leqslant \dim(X)$ and $f = f' \circ g$.*

There are "nice" universal compacta, called Menger compacta μ^n, which are characterized by the following [12]:

Theorem 14 *(Bestvina Criterion). A compact metric space X is homeomorphic to the Menger compactum μ^n if and only if it is n-dimensional, $n-1$-connected and locally $n-1$-connected, and it satisfies the disjoint n-disc property, DDP^n, i.e., every two maps of the n-disc $f, g : D^n \to X$ can be approximated by maps with disjoint images.*

In non-compact case there are universal spaces ν^n in dimension n called Nöbeling spaces, defined as the set of all points in $\mathbb{R}^{2n+1}$ with at most n rational coordinates. An analogous topological characterization of the Nöbeling space ν^n was given by Nagórko [76] (see also [69] for a different treatment). Since by its construction ν^n is a subset of $\mathbb{R}^{2n+1}$, we obtain

Theorem 15 *(Nöbeling–Pontryagin Theorem). Every n-dimensional compact metric space can be embedded into $\mathbb{R}^{2n+1}$.*

Theorem 16. *(See [46,91].) Every compact n-dimensional metric space of Boltjanskii type admits an embedding into $\mathbb{R}^{2n}$.*

In these two theorems as well as in the case of a Menger space, every map of an n-dimensional compactum to μ^n, $\mathbb{R}^{2n+1}$ (Theorem 15) or to $\mathbb{R}^{2n}$ (Theorem 16) can be approximated by embeddings.

We recall that a *dendrite* is a 1-dimensional, 1-connected, locally 1-connected compact metric space.

Theorem 17. *(See [19].) Every compact metric space X of dimension $\dim X \leqslant n$ can be embedded into the product $\prod_{i=1}^{n} T_i \times [0,1]$ of n dendrites and an interval.*

A similar result was obtained in [92].

2.6. Infinite-dimensional spaces

A space X is called *strongly infinite-dimensional* if it admits an essential map onto the Hilbert cube $I^\infty = \prod_{i=1}^{\infty}[0,1]$. We recall that a map $f : X \to I^\infty$ is called *essential* if every projection onto a finite-dimensional face $\pi : I^\infty \to I^n$ is essential. A map $g : X \to I^n$ is essential if it cannot be deformed to a map $g' : X \to \partial I^n$ by a deformation fixed on $g^{-1}(\partial I^n)$. An infinite-dimensional space is called *weakly infinite-dimensional* if it is not strongly infinite-dimensional. A space X is called *countable-dimensional* if it can be presented as a countable union of 0-dimensional subspaces. A space X has *Property C* if for every sequence of open covers $\mathcal{V}_1, \dots, \mathcal{V}_k, \dots$ of X there are disjoint families of open subsets $\mathcal{U}_1, \dots, \mathcal{U}_k, \dots$ such that each $\mathcal{U}_i$ is a refinement of $\mathcal{V}_i$ and $\bigcup_i \mathcal{U}_i$ is a cover of X.

Theorem 18. *Among compact metric spaces there are inclusions*:

$$\{\textit{Countable dim}\} \subset \{\textit{Property C}\} \subset \{\textit{Weakly infinite dim}\}.$$

Due to examples of R. Pol [81] and P. Borst [17], both inclusions are strict. Most of the theorems for finite-dimensional compacta can be extended to compacta with Property C, but usually not to the class of strongly infinite-dimensional compacta. For example, the Alexandroff Theorem stating that the covering dimension $\dim X$ coincides with the cohomological dimension $\dim_{\mathbb{Z}} X = \sup\{n \mid \check{H}^n(X, A) \neq 0,\ A \subset_{Cl} X\}$ for finite-dimensional compacta can be extended to compacta with Property C [2], but cannot be extended to strongly infinite-dimensional compacta [40].

3. Definitions of asdim

3.1. Equivalent definitions

Definition. Let X be a metric space. We say that the *asymptotic dimension* of X does not exceed n and write $\operatorname{asdim} X \leqslant n$ provided for every uniformly bounded open cover $\mathcal{V}$ of X there is a uniformly bounded open cover $\mathcal{U}$ of X of multiplicity $\leqslant n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$. We write $\operatorname{asdim} X = n$ if it is true that $\operatorname{asdim} X \leqslant n$ and $\operatorname{asdim} X \nleqslant n-1$.

Note that the asymptotic dimension can be viewed as somehow dual to Lebesgue covering dimension.

Often we will need to consider very large positive constants and we remind ourselves that they are large by writing $r < \infty$ instead of $r > 0$. On the other hand, writing $\varepsilon > 0$ is supposed to mean that ε is a small positive constant.

Let $r < \infty$ be given and let X be a metric space. We will say that a family $\mathcal{U}$ of subsets of X is r*-disjoint* if $d(U, U') > r$ for every $U \neq U'$ in $\mathcal{U}$. Here, $d(U, U')$ is defined to be $\inf\{d(x, x') \mid x \in U,\ x' \in U'\}$. The r*-multiplicity* of a family $\mathcal{U}$ of subsets of X is defined to be the largest n so that no ball $B_r(x) \subset X$ meets more than n of the sets from $\mathcal{U}$; more succinctly, the r-multiplicity of $\mathcal{U}$ is $\sup_{x\in X} \operatorname{Card}\{U \in \mathcal{U} \mid U \cap B_r(x) \neq \emptyset\}$. Recall that the *Lebesgue number* of a cover $\mathcal{U}$ of X is the largest number λ so that if $A \subset X$ and $\operatorname{diam}(A) \leqslant \lambda$ then there is some $U \in \mathcal{U}$ so that $A \subset U$.

Let K be a countable simplicial complex. There are two natural metrics one can place on is geometric realization $|K|$: the uniform metric and the geodesic metric. The primary metric we consider on $|K|$ is the uniform metric, defined by embedding K into ℓ^2 by mapping each vertex to an element of an orthonormal basis for ℓ^2 and giving it the metric it inherits as a subspace. A map $\varphi : X \to Y$ between metric spaces is uniformly cobounded if for every $R > 0$, $\operatorname{diam}(\varphi^{-1}(B_R(y)))$ is uniformly bounded.

Theorem 19. *Let X be a metric space. The following conditions are equivalent.*

(1) $\operatorname{asdim} X \leqslant n$;
(2) *For every $r < \infty$ there exist r-disjoint families $\mathcal{U}^0, \dots, \mathcal{U}^n$ of uniformly bounded subsets of X such that $\bigcup_i \mathcal{U}^i$ is a cover of X;*
(3) *For every $d < \infty$ there exists a uniformly bounded cover $\mathcal{V}$ of X with d-multiplicity $\leqslant n+1$;*
(4) *For every $\lambda < \infty$ there is a uniformly bounded cover $\mathcal{W}$ of X with Lebesgue number $> \lambda$ and multiplicity $\leqslant n+1$; and*
(5) *For every $\varepsilon > 0$ there is a uniformly cobounded, ε-Lipschitz map $\varphi : X \to K$ to a uniform simplicial complex of dimension n.*

The proof can be found in [8].

We conclude this section with an example.

Example. $\operatorname{asdim} T \leqslant 1$ for all trees T in the edge-length metric.

Proof. Fix some vertex x_0 to be the root of the tree. Let $r < \infty$ be given and take concentric annuli centered at x_0 of thickness r as follows: $A_k = \{x \in T \mid d(x, x_0) \in [kr, (k+1)r)\}$. Although alternating the annuli (odd k, even k) yields r-disjoint sets, these sets clearly do not have uniformly bounded diameter. We have to further subdivide each annulus.

Fix $k > 1$. Define $x \sim y$ in A_k if the geodesics $[x_0, x]$ and $[x_0, y]$ in T contain the same point z with $d(x_0, z) = r(k - \frac{1}{2})$. Clearly in a tree this forms an equivalence relation. The equivalence classes are $3r$ bounded and elements from distinct classes are at least r apart. So, define $\mathcal{U}$ to be equivalence classes corresponding to even k (along with A_0

itself) and $\mathcal{V}$ to be equivalence classes corresponding to odd k. These two families cover T and consist of uniformly bounded, r-disjoint sets. Thus, asdim $T \leqslant 1$. □

3.2. Large-scale invariance of asdim

Let X and Y be metric spaces. A map $f : X \to Y$ between metric spaces is a *coarse embedding* if there exist non-decreasing functions ρ_1 and ρ_2, $\rho_i : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ such that $\rho_i \to \infty$ and for every $x, x' \in X$

$$\rho_1\big(d_X(x,x')\big) \leqslant d_Y\big(f(x), f(x')\big) \leqslant \rho_2\big(d_X(x,x')\big).$$

Such a map is often called a *coarsely uniform embedding* or just a *uniform embedding*. The metric spaces X and Y are *coarsely equivalent* if there is a coarse embedding $f : X \to Y$ so that there is some R such that $Y \subset N_R(f(X))$. Equivalently, metric spaces are coarsely equivalent if there exist coarse embeddings $f : X \to Y$ and $g : Y \to X$, whose compositions (in both ways) are K-close to the identity, for some $K > 0$.

A coarse embedding $f : X \to Y$ is a *quasi-isometric embedding* if it admits linear ρ_i. The two spaces X and Y are quasi-isometric if there is a quasi-isometry $f : X \to Y$ and a constant C so that $Y \subset N_C(f(X))$.

A metric space (X, d) is called *geodesic* if for every two points $x, y \in X$ there is an isometric embedding of the interval $\xi : [0, a] \to X$ with $a = d(x, y)$, $\xi(0) = x$, and $\xi(a) = y$. We note that a coarse embedding of a geodesic metric space always admit a linear function ρ_2. This implies that a coarse equivalence between geodesic metric spaces is in fact a quasi-isometry.

A metric space (X, d) is called *c-discrete* if $d(x, x') \geqslant c$ for all $x, x' \in X$, $x \neq x'$.

Proposition 20. *Every metric space (X, d) is coarsely equivalent to a 1-discrete metric space.*

Proof. Let $S \subset X$ be a maximal 1-discrete subset. Then the inclusion is a coarse equivalence. □

Corollary 21. *Every geodesic metric space is quasi-isometric to a connected graph.*

Proof. Let $S \subset X$ be a maximal 1-discrete subset in X. Connect all points $s \neq s'$ in S with $d(s, s') \leqslant 4$ by an edge. Let ρ be a simplicial metric on the graph (so that every edge has length one). Clearly, $\rho(s, s') \geqslant \frac{1}{3} d_X(s's')$. We show that $\rho(s, s') \leqslant d_X(s's') + 2$. Let $x_0, x_1, \ldots, x_n$ be points on a geodesic segment joining s and s' such that $s = x_0$, $s' = x_n$, and $d_X(x_{i-1}, x_i) = 1$ for $i < n$. Let $s_i \in S$ be such that $d_X(s_i, x_i) \leqslant 1$. Then s_i is joined by an edge with s_{i+1} for all i. Hence $\rho(s, s') \leqslant n + 2 \leqslant d_X(s, s') + 2$. □

Proposition 22. *Let $f : X \to Y$ be a coarse equivalence. Then* asdim $X =$ asdim Y.

Proof. If $\mathcal{U}^0, \ldots, \mathcal{U}^n$ are r-disjoint, D-bounded families covering X then the families $f(\mathcal{U}^i)$ are $\rho_1(r)$-disjoint and $\rho_2(D)$-bounded. Since $N_R(f(X))$ contains Y we see that taking families $N_R(f(\mathcal{U}^i))$ will cover Y and be $(2R + \rho_2(D))$-bounded and $(\rho_1(r) - 2R)$-disjoint. Since $\rho_i \to \infty$, r can be chosen large enough for $\rho_1(r) - 2R$ to be as large as one likes. Therefore, asdim $Y \leqslant$ asdim X.

The same proof applied to a coarse inverse for f proves that asdim $X \leqslant$ asdim Y. □

Example. asdim $\mathbb{R} =$ asdim $\mathbb{Z} = 1$.

Proposition 23. *Let X be a metric space and $Y \subset X$. Then* asdim $Y \leqslant$ asdim X.

Proof. Let $R < \infty$ be given and take a cover $\mathcal{U}$ of X by uniformly bounded sets with R-multiplicity $\leqslant n + 1$. Clearly the restriction of this cover to Y yields a cover whose elements have uniformly bounded diameter and at most $n + 1$ of them can meet any ball of radius R in Y. Thus, asdim $Y \leqslant$ asdim X. □

4. Union theorems

Before proceeding further, we need to establish a basic result: a union theorem for asymptotic dimension. It should be noted that here asymptotic dimension varies slightly from covering dimension. For example, the finite union theorem for covering dimension says $\dim(X \cup Y) \leqslant \dim X + \dim Y + 1$, and that inequality is sharp. Also, the standard countable union theorem for covering dimension is $\dim(\bigcup_i C_i) \leqslant \max_i\{\dim C_i\}$ where the C_i are closed subsets of X. Notice that there can be no direct analog of this theorem for asymptotic dimension since every finitely generated group is a countable set of points, and as we shall see, finitely generated groups can have arbitrary (even infinite) asymptotic dimension. We follow the notation and development of [5].

Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of X. Define the r-saturated union of $\mathcal{V}$ with $\mathcal{U}$ by

$$\mathcal{V}\bigcup_r \mathcal{U} = \{N_r(V;\mathcal{U}) \mid V \in \mathcal{V}\} \cup \{U \in \mathcal{U} \mid d(U,\mathcal{V}) > r\},$$

where $N_r(V;\mathcal{U}) = V \cup \bigcup_{d(U,V)\leqslant r} U$.

It is easy to verify the following proposition.

Proposition 24. *Let $\mathcal{U}$ be an r-disjoint, R-bounded family of subsets of X with $R \geqslant r$. Let $\mathcal{V}$ be a $5R$-disjoint, D bounded family of subsets of X. Then $\mathcal{V}\bigcup_r \mathcal{U}$ is r-disjoint and $(D + 2(r + R))$-bounded.*

Let X be a metric space. We will say that the family $\{X_\alpha\}$ of subsets of X satisfies the inequality $\operatorname{asdim} X_\alpha \leqslant n$ *uniformly* if for every $r < \infty$ one can find a constant R so that for every α there exist r-disjoint families $\mathcal{U}^0_\alpha, \dots, \mathcal{U}^n_\alpha$ of R-bounded subsets of X_α covering X_α. A typical example of such a family is a family of isometric subsets of a metric space. Another example is any family containing finitely many sets.

Theorem 25 *(Union Theorem). Let $X = \bigcup_\alpha X_\alpha$ be a metric space where the family $\{X_\alpha\}$ satisfies the inequality* $\operatorname{asdim} X_\alpha \leqslant n$ *uniformly. Suppose further that for every r there is a $Y_r \subset X$ with* $\operatorname{asdim} Y_r \leqslant n$ *so that $d(X_\alpha - Y_r, X_{\alpha'} - Y_r) \geqslant r$ whenever $X_\alpha \neq X_{\alpha'}$. Then* $\operatorname{asdim} X \leqslant n$.

Before proving this theorem, we state a corollary: the finite union theorem for asymptotic dimension.

Corollary 26 *(Finite Union Theorem). Let X be a metric space with $A, B \subset X$. Then* $\operatorname{asdim}(A \cup B) \leqslant \max\{\operatorname{asdim} A, \operatorname{asdim} B\}$.

Proof. Apply the union theorem to the family $\{A, B\}$ with $B = Y_r$ for every r. □

Proof of Union Theorem. Let $r < \infty$ be given and take r-disjoint, R-bounded families $\mathcal{U}^i_\alpha$ $(i = 0, \dots, n)$ of subsets of X_α so that $\bigcup_i \mathcal{U}^i_\alpha$ covers X_α. We may assume $R \geqslant r$. Take $Y = Y_{5R}$ as in the statement of the theorem and cover Y by families $\mathcal{V}^0, \dots, \mathcal{V}^n$ that are D-bounded and $5R$-disjoint. Let $\bar{\mathcal{U}}^i_\alpha$ denote the restriction of $\mathcal{U}^i_\alpha$ to the set $X_\alpha - Y$. For each i, take $\mathcal{W}^i_\alpha = \mathcal{V}^i \bigcup_r \bar{\mathcal{U}}^i_\alpha$. By the proposition $\mathcal{W}^i$ consists of uniformly bounded sets and is r-disjoint. Finally, put $\mathcal{W}^i = \{W \in \mathcal{W}^i_\alpha \mid \alpha\}$. Observe that each $\mathcal{W}^i$ is r-disjoint and uniformly bounded. Also, it is easy to check that $\bigcup_i \mathcal{W}^i$ covers X. □

5. Connection with covering dimension—Higson corona

Let $\varphi : X \to \mathbb{R}$ be a function defined on a metric space X. For every $x \in X$ and every $r > 0$ let $V_r(x) = \sup\{|\varphi(y) - \varphi(x)|\colon y \in N_r(x)\}$. A function φ is called *slowly oscillating* if for every $r > 0$ we have $V_r(x) \to 0$ as $x \to \infty$. (This means that for every $\varepsilon > 0$ there exists a compact subspace $K \subset X$ such that $|V_r(x)| < \varepsilon$ for all $x \in X \setminus K$.) Let $\bar{X}$ be the compactification of the proper metric space X corresponding to the family of all continuous bounded slowly oscillating functions. The *Higson corona* of X is the remainder $\nu X = \bar{X} \setminus X$ of this compactification.

It is known that the Higson corona is a functor from the category of proper metric space and coarse maps into the category of compact Hausdorff spaces. In particular, if $X \subset Y$, then $\nu X \subset \nu Y$.

For any subset A of X we denote by A' its trace on νX, i.e. the intersection of the closure of A in $\bar{X}$ with νX. Obviously, the set A' coincides with the Higson corona νA.

Dranishnikov, Keesling and Uspenskij [44] proved the inequality

$$\dim \nu X \leqslant \operatorname{asdim} X,$$

for any proper metric space X. It was shown there that $\dim \nu X \geqslant \operatorname{asdim} X$ for a large class of spaces, in particular for $X = \mathbb{R}^n$. Later Dranishnikov proved [32] that the equality $\dim \nu X = \operatorname{asdim} X$ holds provided $\operatorname{asdim} X < \infty$. The question of whether there is a metric space X with $\operatorname{asdim} X = \infty$ and $\dim \nu X < \infty$ is still open.

6. Inductive approach

The notion of asymptotic inductive dimension asInd was introduced in [33].

Let X be a proper metric space. A subset $W \subset X$ is called an *asymptotic neighborhood* of a subset $A \subset X$ if $\lim_{r\to\infty} d(A \setminus N_r(x_0), X \setminus W) = \infty$. Two sets A_1, A_2 in a metric space are *asymptotically disjoint* if $\lim_{r\to\infty} d(A_1 \setminus N_r(x_0), A_2 \setminus N_r(x_0)) = \infty$. In other words, two sets are asymptotically disjoint if the traces A_1', A_2' on νX are disjoint.

A subset C of a metric space X is an *asymptotic separator* between asymptotically disjoint subsets $A_1, A_2 \subset X$ if the trace C' is a separator in νX between A_1' and A_2'.

We recall the definition of the asymptotic Dimensiongrad in the sense of Brouwer, asDg, from [50].

Let X be a metric space and $r > 0$. A finite sequence $x_1, \dots, x_k$ in X is an *r-chain* between subsets $A_1, A_2 \subset X$ if $x_1 \in A_1$, $x_k \in A_2$ and $d(x_i, x_{i+1}) < \lambda$ for every $i = 1, \dots, k-1$. We say that a subset C of a metric space X is an *asymptotic cut* between the asymptotically disjoint subsets $A_1, A_2 \subset X$ if for every $r > 0$ there is a $\lambda = \lambda(r) > 0$ such that every r-chain between A_1 and A_2 intersects $N_\lambda(C)$.

By definition, $\operatorname{asInd} X = \operatorname{asDg}(X) = -1$ if and only if X is bounded. Suppose we have defined the class of all proper metric spaces Y with $\operatorname{asInd} Y \leqslant n-1$ (respectively with $\operatorname{asDg}(Y) \leqslant n-1$). Then $\operatorname{asInd} X \leqslant n$ (respectively $\operatorname{asDg}(Y) \leqslant n$) if and only if for every asymptotically disjoint subsets $A_1, A_2 \subset X$ there exists an asymptotic separator (respectively asymptotic cut) C between A_1 and A_2 with $\operatorname{asInd} C \leqslant n-1$ (respectively $\operatorname{asDg}(C) \leqslant n-1$). The dimension functions asInd and asDg are called the *asymptotic inductive dimension* and *asymptotic Brouwer inductive dimension* respectively.

Proposition 27. $\operatorname{asDg}(X) \leqslant \operatorname{asInd} X$, *for every proper metric space* X.

Proof. We use induction on $n = \operatorname{asInd} X$. The inequality holds for $n = -1$. Let $n \geqslant 0$. We show that $\operatorname{asDg}(X) \leqslant n$. Let A_1, A_2 be an asymptotically disjoint sets. Let C be an asymptotic separator between them with $\operatorname{asInd} C \leqslant n-1$. By the induction assumption $\operatorname{asDg}(X) \leqslant n-1$. We show that C is an asymptotic cut. Assume the contrary. Then there is an $r > 0$, a sequence $\lambda_n \to \infty$, and a sequence of r-paths $p_n = \{x_1^n, \dots, x_{k_n}^n\}$ from A_1 to A_2 that miss the λ_n-neighborhood of C. We may assume that all paths p_n are disjoint. For every n and i we glue to X the interval $[x_i^n, x_{i+1}^n]$ of length $d(x_i^n, x_{i+1}^n)$ to obtain a proper metric space Y that contains X isometrically and lies in the r-neighborhood of X. Hence the inclusion $X \subset Y$ is a coarse equivalence. Hence the induced inclusion $\nu X \to \nu Y$ is a homeomorphism. Each r-path p_n is represented in Y by a continuous path $\bar{p}_n : [0,1] \to Y$. Let $K_n = im(\bar{p}_n)$. Since $\bigcup K_n$ and C diverge, $\nu(\bigcup_n K) \cap C' = \emptyset$.

In the hyperspace $\exp_c \bar{Y}$ of connected closed subsets of Y taken with the Vietoris topology the sequence K_n has a limit point K. Clearly, $K \subset \nu Y = \nu X$ is a continuum joining A_1' with A_2'. Hence $K \cap C' \neq \emptyset$. Note that $K \subset \nu(\bigcup_n K_n)$, which is a contradiction. □

It is unknown if asInd = asDg for proper metric spaces.

The following theorem was proven in [50]:

Theorem 28. *For all proper metric spaces* X *with* $0 < \operatorname{asdim} X < \infty$ *we have*

$$\operatorname{asdim} X = \operatorname{asInd} X.$$

This theorem is a very important step in the original proof of the exact formula of the asymptotic dimension of the free product $\operatorname{asdim} A * B$ of groups [9].

Notice that there is a small problem with coincidence of asdim and asInd in dimension 0. This leads to philosophical discussions of whether bounded metric spaces should be defined to have asdim $= -1$ or 0. Observe that in the world of finitely generated groups, $\operatorname{asdim} \Gamma = 0$ if and only if Γ is finite. On the other hand, there are metric spaces, for instance $2^n \subset \mathbb{R}$ that are unbounded yet have asymptotic dimension 0.

7. Hurewicz-type mapping theorem

In [7] the authors prove an asymptotic analog of the Hurewicz theorem, cf. Theorem 3. In particular, the following theorem is proved:

Theorem 29. *Let $f : X \to Y$ be a Lipschitz map from a geodesic metric space X to a metric space Y. Suppose that for every $R > 0$ the set family $\{f^{-1}(B_R(y))\}_{y\in Y}$ satisfies the inequality* asdim $\leqslant n$ *uniformly. Then* $\operatorname{asdim} X \leqslant \operatorname{asdim} Y + n$.

The proof of the theorem uses mapping cylinders and is too technical to include in this survey. The main application of this theorem is to the case where X is a Cayley graph of a finitely generated group.

Later, Brodskiy, Dydak, Levin and Mitra [24] generalized Bell and Dranishnikov's result to the following theorem. We follow their development. First, we give a definition.

Let $f : X \to Y$ be a map of metric spaces. Define

$$\operatorname{asdim} f = \sup\bigl\{\operatorname{asdim} A \bigm| A \subset X \text{ and } \operatorname{asdim}\bigl(f(A)\bigr) = 0\bigr\}.$$

Theorem 30. *Let $f : X \to Y$ be a large-scale uniform* (*bornologous in Roe's terminology*) *function between metric spaces. Then*

$$\operatorname{asdim} X \leqslant \operatorname{asdim} Y + \operatorname{asdim} f.$$

The main idea of the proof is to reformulate the definition of asymptotic dimension in terms of a double-parameter family that allows the use of the so-called Kolmogorov Trick. The following is just a rephrasing of the definition of asymptotic dimension.

Assertion. *Let X be a metric space with* $\operatorname{asdim} X \leqslant n$. *Then there is a function $D_X : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $r > 0$ there is a cover $\mathcal{U}$ of X that can be expressed as $\bigcup_{i=1}^{n+1} \mathcal{U}_i$ so that the $\mathcal{U}_i$ are r-disjoint and $D_X(r)$-bounded.*

Definition. The function D_X defined above is called an *n-dimensional control function for X*.

Let $k \geqslant n + 1 \geqslant 1$. An *$(n, k)$-dimensional control function for X* is a function $D_X : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $r > 0$ there is a cover $\mathcal{U}$ of X that can be expressed as $\bigcup_{i=0}^{k} \mathcal{U}_i$ so that the $\mathcal{U}_i$ are r-disjoint and $D_X(r)$-bounded and such that $\bigcup_{i\in T} \mathcal{U}_i$ covers X for all subsets $T \subset \{0, 1, \ldots, k\}$ with $|T| \geqslant n + 1$. Notice that this condition can be rephrased by saying that each $x \in X$ is in at least $k - n$ of the $\mathcal{U}_i$.

Lemma 31. *Let D_X^{n+1} be an n-dimensional control function of X. Define $\{D_X^i\}_i \geqslant n + 1$ inductively by*

$$D_X^{i+1}(r) = D_X^i(3r) + 2r.$$

Then, each D_X^k is an (n, k)-dimensional control function of X for all $k \geqslant n + 1$.

Proof. We proceed inductively, with the case $k = n + 1$ being trivially true.

Suppose the result to be true for some $k \geqslant n + 1$. Let $\mathcal{U} = \bigcup_{i=1}^{k} \mathcal{U}_i$ be a $3r$-disjoint, $D_X^k(3r)$-bounded family so that any $n + 1$ of the $\mathcal{U}_i$ cover X. Define $\mathcal{U}_i'$ to be the r-neighborhoods of elements of $\mathcal{U}_i$ for $i \leqslant k$. Notice that the elements of $\mathcal{U}_i'$ are $D_X^k(3r) + 2k$-bounded and r-disjoint.

Define $\mathcal{U}_{k+1}'$ to be the collection of all sets of the form $\bigcap_{s\in S} A_s \setminus \bigcup_{i\notin S} \mathcal{U}_i'$, where S is a subset of $\{1, \ldots, k\}$ consisting of exactly $k - n$ elements and $A_s \in \mathcal{U}_s$. Observe that each element of $\mathcal{U}_{k+1}'$ is contained in some $\mathcal{U}_j$ so the families $\mathcal{U}_i$ are $(D_X^k(3r) + 2r)$-bounded.

Next, we must show that the elements of the collection $\{\mathcal{U}_i\}_{i=1}^{k+1}$ are r-disjoint. Obviously all that needs to be shown is that the elements of $\mathcal{U}'_{k+1}$ are r-disjoint. Let A and B be two elements of $\mathcal{U}'_{k+1}$, say $A = \bigcap_{s\in S} A_s \setminus \bigcup_{i\notin S} \mathcal{U}'_i$ and $B = \bigcap_{t\in T} B_t \setminus \bigcup_{i\notin T} \mathcal{U}'_i$ with $S \neq T$. Suppose that $a \in A$ and $b \in B$ with $d(a,b) < r$. Then there is an $s \in S \setminus T$ such that $a \in A_s$. But, then there is a $U \in \mathcal{U}'_s$ containing b, a contradiction.

Finally, suppose that $x \in X$ belongs to exactly $k-n$ sets $\bigcup \mathcal{U}'_i$, $i \leqslant k$, and let $S = \{i \leqslant k \mid x \in \bigcup \mathcal{U}'_i\}$. If $x \notin \bigcup U'_{k+1}$, then $x \in \bigcup \mathcal{U}'_j$ for some $j \notin S$, a contradiction. Thus, each x belongs to at least $k+1-n$ elements of $\{\mathcal{U}'_i\}_{i=1}^{k+1}$. □

Next we prove a product theorem for asymptotic dimension. The easiest proof (intuitively) involves embedding into uniform complexes. The product theorem also follows from the asymptotic Hurewicz Theorem. The following proof comes from [24] and is a nice illustration of the Kolmogorov Trick.

Theorem 32. *Let X and Y be metric spaces. Then*

$$\operatorname{asdim}(X \times Y) \leqslant \operatorname{asdim} X + \operatorname{asdim} Y.$$

Proof. Put $\operatorname{asdim} X = m$ and $\operatorname{asdim} Y = n$ and $k = m+n+1$. Let D_X be an (m,k)-dimension control function for X and D_Y be an (n,k)-dimension control function for Y. Take r-disjoint, $D_X(r)$-bounded families $\{\mathcal{U}_i\}_{i=1}^k$ so that any $n+1$ families cover X and r-disjoint, $D_Y(r)$-bounded families $\{\mathcal{V}_i\}_{i=1}^k$ so that any $m+1$ of the families cover Y. Then the family $\{\mathcal{U}_i \times \mathcal{V}_i\}_{i=1}^k$ covers $X \times Y$, is uniformly bounded and $\sqrt{r}$ disjoint. □

The inequality in this theorem can be strict [27]. It can be strict even when one of the factors is the reals $\mathbb{R}$. In [37] an example of a metric space (uniform simplicial complex) X is constructed with the properties $\operatorname{asdim} X = 2$ and $\operatorname{asdim}(X \times \mathbb{R}) = 2$. Thus there is no Morita-type theorem for the asymptotic dimension.

Next we move to the proof of the Hurewicz Theorem.

Definition. Let $f : X \to Y$ be a function between metric spaces. We say that $A \subset X$ is (r_X, R_Y)-*bounded* if $d_Y(f(x), f(x')) \leqslant R_Y$ whenever $d_X(x,x') \leqslant r_X$.

Definition. Let $f : X \to Y$ be a function of metric spaces. Let $m \geqslant 0$. We say that $D_f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is an *m-dimensional control function of f* if for any $r_X > 0$ and $R_Y > 0$ and any $A \subset X$ with $\operatorname{diam}(f(A)) \leqslant R_Y$, A can be expressed as the union of $m+1$ sets whose r_X-components are $D_f(r_X, R_Y)$-bounded.

Proposition 33. *Suppose that $f : X \to Y$ is a function between metric spaces and that $m \geqslant 0$. If* $\operatorname{asdim} f \leqslant m$ *then f has an m-dimensional control function D_f.*

Proof. Fix non-negative r_X and R_Y. Suppose that for each n there is a $y_n \in Y$ such that $A_n = f^{-1}(B(y_n, R_Y))$ cannot be expressed as a union of $m+1$ sets whose r_X-components are n-bounded. Then, the set $C = \bigcup_{n=1}^{\infty} B(y_n, R_Y)$ cannot be bounded. If C were bounded, then $\operatorname{asdim} f^{-1}(C) \leqslant m$. By passing to a subsequence, we may arrange $y_n \to \infty$ and $\operatorname{asdim} C = 0$, a contradiction. □

Definition. Let $k \geqslant M+1 \geqslant 1$. An *$(m,k)$-dimensional control function of $f : X \to Y$* is a function $D_f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $r_X > 0$ and all $R_Y > 0$ any (∞, R_Y)-bounded subset $A \subset X$ can be expressed as the union of k sets $\{A_i\}_{i=1}^k$ whose r_X-components are $D_f(r_X, R_Y)$-bounded and so that any $x \in A$ belongs to at least $k-m$ of the A_i.

Proposition 34. *Let $f : X \to Y$ be a function between metric spaces and $m \geqslant 0$. Suppose that there is an m-dimensional control function D_f^{m+1} of f. Define $D_f^k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ inductively by $D_f^k(r_X, R_Y) = D_f^{k-1}(3r_X, R_Y) + 2r_X$ for each $k \geqslant m$. Then each D_f^k is an (m,k)-dimensional control function of f.*

The proof is similar to that of Lemma 31.

Theorem 35. *Let $k = m+n+1$ and suppose $f : X \to Y$ is a large-scale uniform function of metric spaces with* $\operatorname{asdim} Y \leqslant n$. *If there is an (m,k)-dimensional control function D_f of f then* $\operatorname{asdim} X \leqslant m+n$.

The proof is a bit technical, so we give the idea; the interested reader is referred to the original article [24].

Sketch of proof. Given r we want to find a constant $D_X(r)$ so that X can be written as the union $X=\bigcup_{j=1}^{k} D^j$ with each r-component $D_X(r)$-bounded. This clearly implies the conclusion of the theorem.

We want to apply the Kolmogorov Trick, so we take a cover of Y by sets A_i $(i=1,\dots,k)$, with conditions on boundedness of components. For each i write A_i as a union of k sets $\{U_i^j\}$ with certain boundedness conditions on components in such a way that every point of y belongs to at least m sets. For each i cover the set $f^{-1}(A_i)$ by k sets $\{B_i^j\}_{j=1}^k$ with conditions on boundedness of components so that every point in $f^{-1}(A_i)$ is in at least m sets. Then, apply the Kolmogorov Trick to $D^j=\bigcup_i(B_i^j\cap f^{-1}(U_i^j))$. □

Corollary 36. *Suppose $f:X\to Y$ is a function of metric spaces and $m\geqslant 0$. Then,* asdim $f\leqslant m$ *if and only if f has an m-dimensional control function.*

Proof. The "only if" part is Proposition 33. The "if" part follows from Theorem 35 with $n=0$. □

Theorem 37. *Suppose that f is a large-scale uniform function $f:X\to Y$ between metric spaces. Then*

$$\operatorname{asdim} X\leqslant \operatorname{asdim} Y+\operatorname{asdim} f.$$

Proof. Put $n=\operatorname{asdim} Y$ and $m=\operatorname{asdim} f$. Then by Theorem 35 it suffices to show that there is an (m,k)-dimensional control function D_f for f with $k=m+n+1$. By the previous corollary, f has an m-dimensional control function. Finally, by Proposition 34 there is an (m,k)-dimensional control function, and the proof is complete. □

8. Coarse embedding

The goal of this section is to see that a metric space with finite asymptotic dimension admits a coarse embedding into Hilbert space. This result is of particular interest in connection with the Novikov higher signature conjecture. Guoliang Yu showed in [94] that finitely generated groups that admit a coarse embedding into Hilbert space satisfy this conjecture.

In [95], Yu defined a property called "Property A", which can be thought of as a generalization of amenability for discrete spaces with bounded geometry. For groups, this is equivalent to the exactness of the reduced C^*-algebra by a result of Ozawa [80], so it has come to be known as exactness of the group. This property is implied by finite asymptotic dimension and is enough to imply a coarse embedding into Hilbert space.

Theorem 38. *Let X be a metric space with finite asymptotic dimension. Then X admits a coarse embedding into Hilbert space.*

Proof. We follow Roe's proof in [84]. By the fifth characterization of asymptotic dimension in Theorem 19, for every $k>0$ we can find uniformly cobounded 2^{-k}-Lipschitz maps $\phi_k:X\to L_k$ where L is a finite-dimensional simplicial complex with the metric induced from its inclusion in Hilbert space. Let x_0 be some fixed basepoint in X. Then define $\Phi:X\to\bigoplus_{k=1}^{\infty}\mathcal{H}$ by $\Phi(x)=\{\phi_k(x)-\phi_k(x_0)\}$.

Then, since ϕ_k is 2^{-k}-Lipschitz, we see that

$$\left\|\Phi(x)-\Phi(x')\right\|^2=\sum_{k=1}^{\infty}\left\|\phi_k(x)-\phi_k(x')\right\|^2\leqslant\|x-x'\|^2\sum_{k=1}^{\infty}2^{-2k}.$$

On the other hand, since each ϕ_k is uniformly cobounded, this means there is an R_k so that $\operatorname{diam}(\phi_K^{-1}(\sigma))\leqslant R_k$ for each simplex $\sigma\in L_k$. So if $\|x-x'\|^2>R_\ell$ for some ℓ then $\phi_k(x)$ and $\phi_k(x')$ are orthogonal unit vectors for $k\leqslant\ell$. Thus, $\|\phi_k(x)-\phi_k(x')\|^2=2$ for all $k\leqslant\ell$ and so $\|\Phi(x)-\Phi(x')\|^2\geqslant 2\ell$. □

Note that there cannot be an asymptotic analog of the Nöbeling–Pontryagin Theorem (Theorem 15). For example, the binary tree T has asdim $T=1$, but it does not admit a coarse embedding in Euclidean space of any dimension. The obstacle is volume growth. Nevertheless there is the following analog of Theorem 17 [34].

Theorem 39. *Every proper metric space X with* asdim $X \leqslant n$ *admits a coarse embedding in $n+1$ locally finite trees.*

In [50] a universal metric space for the class of metric spaces of bounded geometry and asymptotic dimension $\leqslant n$ is constructed. It is not an asymptotic analog of the Menger space μ^n since it does not have bounded geometry. Moreover, it is proven in [50] that there is no proper universal space for asymptotic dimension n. Macro-scale analogs of Nöbeling spaces have been constructed that are universal for asymptotic dimension and coarse embeddings [11].

9. Hyperbolic spaces

Let X be a metric space. For $x, y, z \in X$ we define the *Gromov product*

$$(x|y)_z := \frac{1}{2}\big(|zx| + |zy| - |xy|\big).$$

Let $\delta \geqslant 0$. A triple $(a_0, a_1, a_2) \in \mathbb{R}^3$ is called a *δ-triple*, if $a_\mu \geqslant \min\{a_{\mu+1}, a_{\mu+2}\} - \delta$ for $\mu = 0, 1, 2$, where the indices are taken modulo 3.

The space X is called *hyperbolic* if there is $\delta > 0$ such that for every $o, x, y, z \in X$ the triple $((x|y)_o, (y|z)_o, (x|z)_o)$ is a δ-triple.

Note that if X satisfies the δ-inequality for one individual base point $o \in X$, then it satisfies the 2δ-inequality for any other base point $o' \in X$, see, for example [60]. Thus, to check hyperbolicity, one has to check this inequality only at one point.

Let X be a hyperbolic space and $o \in X$ be a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity,* if $\lim_{i,j\to\infty}(x_i|x_j)_o = \infty$. Two sequences $\{x_i\}$, $\{x_i'\}$ that converge to infinity are *equivalent* if $\lim_{i\to\infty}(x_i|x_i')_o = \infty$. Using the δ-inequality, one easily sees that this defines an equivalence relation for sequences in X converging to infinity. The *boundary at infinity* $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging to infinity.

A hyperbolic space Y is said to be *visual*, if for some base point $o \in Y$ there is a positive constant D such that for every $y \in Y$ there is $\xi \in \partial_\infty Y$ with $|oy| \leqslant (y|\xi)_o + D$ (one easily sees that this property is independent of the choice of o). Here $(y|\xi)_o = \inf \liminf_{i\to\infty}(y|x_i)_o$, where the infimum is taken over all sequences $\{x_i\} \in \xi$. For hyperbolic geodesic spaces this property is a rough version of the property that every segment $oy \subset Y$ can be extended to a geodesic ray beyond the end point y.

Most of our interest is in geodesic metric spaces. A geodesic metric space is hyperbolic if there is $\delta > 0$ such that every geodesic triangle is δ-thin [21], which means that every side of the triangle is contained in a δ-neighborhood of the other two.

The following is straightforward.

Proposition 40. *If a geodesic metric space is quasi-isometric to hyperbolic space then it is hyperbolic.*

In the second part of this paper, we prove that the asymptotic dimension of a finitely generated hyperbolic group is finite. In fact, this applies to more general hyperbolic spaces, see, for example [10,15,85]. We note that in the case of hyperbolic groups asdim $\Gamma = \dim \partial_\infty \Gamma + 1$ [27,26].

The fundamental group $\pi_1(X, x_0)$ of a metric space X is generated by a set of free loops S if it is generated by the set of based loops of the form $\phi = pf\bar{p}$, $f \in S$, where p is a path from x_0 to $f(0) = f(1)$.

The fundamental group $\pi_1(X)$ of a metric space X is *uniformly generated* if there is $L > 0$ such that $\pi_1(X)$ is generated by free loops of length $\leqslant L$.

The following proposition can be extracted from [57].

Proposition 41. *The fundamental group of a hyperbolic space X is uniformly generated.*

In the case of hyperbolic spaces the embedding theorem of Section 8 can be improved to the following [28]

Theorem 42. *Every visual hyperbolic space X admits a quasi-isometric embedding into the product of $n+1$ copies of the binary metric tree where $n = \dim \partial_\infty X$ is the topological dimension of the boundary at infinity.*

10. Spaces of dimension 0 and 1

Recall that spaces with asymptotic dimension 0 are those spaces that can be presented as a union of uniformly bounded, r-disjoint sets for each (large) r. Notice that the collection of asymptotically zero-dimensional metric spaces includes all compacta, but is clearly not limited to such things. On the other hand, if X is assumed to be geodesic, then $\operatorname{asdim} X = 0$ implies that X is compact. For asymptotically 1-dimensional spaces we have the following result.

Theorem 43. *Let X be a geodesic metric space with* $\operatorname{asdim} X = 1$ *whose fundamental group is uniformly generated. Then X is quasi-isometric to an infinite tree.*

Proof. Since $\operatorname{asdim} X = 1$, there is a $1/2L$-Lipschitz map $p : X \to K$ to a uniform 1-dimensional simplicial complex which is a quasi-isometry. Thus, p sends every loop of length $\leqslant L$ to a null-homotopic loop. Therefore, $p_* : \pi_1(X) \to \pi_1(K)$ is the zero homomorphism. Since every geodesic space is path connected and locally path connected, by the Lifting Criterion there is a lift $\tilde{p} : X \to \tilde{K}$ of p to the universal cover $u : \tilde{K} \to K$. Since u is a local isometry, $\tilde{p}$ is locally Lipschitz. Since X is geodesic $\tilde{p}$ is globally Lipschitz. Clearly, it is a quasi-isometry onto the image $T = \tilde{p}(X)$. Note that T is a tree as a connected subcomplex of a tree $\tilde{K}$. □

This result first appeared as [57, Theorem 0.1]. There, the proof appeals to Manning's bottle-neck property, [72]. The main application of this result is the following

Theorem 44. *(See [57].) Let S be a compact oriented surface of genus $g \geqslant 2$ and with one boundary component. Let $C(S)$ be the curve graph of S. Then* $\operatorname{asdim} C(S) > 1$.

Proof. In this case $C(S)$ is one-ended by a result of Schleimer [88]. Hence it cannot be quasi-isometric to a tree. Since $C(S)$ is hyperbolic [75], we obtain a contradiction with Theorems 48 and Proposition 46. □

Here we recall that curve graph of a surface S is the graph whose vertices are isotopy classes of essential, nonperipheral, simple closed curves in S, with two distinct vertices joined by an edge if the corresponding classes can be represented by disjoint curves. Bell and Fujiwara proved that $\operatorname{asdim} C(S) < \infty$ for all surfaces S [10].

11. Linear control

Let X be a metric space with $\operatorname{asdim} X \leqslant n$. If there is a $C > 0$ so that for every D, there is a cover $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ of X by D-disjoint sets with $\operatorname{mesh}(\mathcal{U}) < CD$, then we say that $\operatorname{asdim} X \leqslant n$ with *linear control*. This property was defined in [50], where it was called the *Higson property*. This is related to the *Assouad–Nagata dimension*, which is defined as follows.

Definition. For a metric space X, the *Assouad–Nagata dimension AN*-dim X is the infimum of all integers n such that there is a $C > 0$ so that for any $D > 0$, X can be covered by a CD-bounded cover with D-multiplicity $\leqslant n + 1$ [1].

Using the Assouad–Nagata dimension U. Lang and T. Schlichenmaier gave the following refinement of Theorem 44 [67]:

Theorem 45. *If for a metric space AN*-dim$(X, d) \leqslant n$ *then for sufficiently small ε, (X, d^{ε}) admits a bi-Lipschitz embedding in the product of $n + 1$ locally finite trees.*

The Assouad–Nagata dimension is a way of simultaneously considering dimension at all scales. In [22] the Assouad–Nagata dimension was characterized in terms of extension of Lipschitz maps to the unit n-sphere S^n. When it is applied to discrete spaces like finitely generated groups the small scales are not important and it defines a quasi-isometry invariant. Many of the results contained in this survey have corresponding results in the theory of Assouad–Nagata dimension (e.g., in [24] a Hurewicz-type theorem for Assouad–Nagata dimension is proved).

Although a thorough discussion of Assouad–Nagata dimension is beyond the scope of this survey, we do want to draw attention to some curious features of linear control.

Proposition 46. *(See [50, Proposition 4.1].) Every proper metric space X is coarsely equivalent to a proper space Y with* $\operatorname{asdim} Y \leqslant n$ *with linear control.*

The proof of this proposition in [50] uses universal spaces for asymptotic dimension. Each metric space admits a coarsely uniform embedding into such a universal space, and the universal space has asymptotic dimension $\leqslant n$ with linear control. Since linear control passes to subsets, the (coarsely) embedded copy of X has asdim $\leqslant n$ with linear control. An alternative proof is given in [22] where it was shown that Y can be taken to be hyperbolic. In case of groups it was shown in [24] that there is a proper left-invariant metric for which $\operatorname{asdim} G = AN\text{-}\dim G$.

This is an illustration of a difference between quasi-isometry and coarse equivalence. Whereas quasi-isometry will preserve the property of linear control, coarse equivalence will not.

In particular, the Morita type theorem holds for Assouad–Nagata dimension of cocompact spaces [49]:

Theorem 47. $AN\text{-}\dim(X \times \mathbb{R}) = AN\text{-}\dim X + 1$.

In [23] it was shown that the lamplighter group $G = \mathbb{Z}_2 \wr \mathbb{Z}^2$ has $\operatorname{asdim} G = 2$ and $AN\text{-}\dim G = \infty$.

In [51] it was proven that the dimension of asymptotic cone of a metric space does not exceed its Assouad–Nagata dimension:

Theorem 48. $\dim(\operatorname{cone}_\omega X) \leqslant AN\text{-}\dim X$ *for all ultrafilters* $\omega \in \beta\mathbf{N}$.

We recall the definition of the asymptotic cone $\operatorname{cone}_\omega(X)$ of a metric space with base point $x_0 \in X$ with respect to a non-principal ultrafilter ω on $\mathbf{N}$ [61,84]. On the sequences of points $\{x_n\}$ with $\|x_n\| \leqslant Cn$ for some C, we define an equivalence relation

$$\{x_n\} \sim \{y_n\} \quad \Leftrightarrow \quad \lim_\omega d(x_n, y_n)/n = 0.$$

We denote by $[\{x_n\}]$ the equivalence class of $\{x_n\}$. The space $\operatorname{cone}_\omega(X)$ is the set of equivalence classes $[\{x_n\}]$ with the metric $d_\omega([\{x_n\}], [\{y_n\}]) = \lim_\omega d(x_n, y_n)/n$. We note that the space $\operatorname{cone}_\omega(X)$ does not depend on the choice of the base point.

12. Dimension of general coarse structures

So far we have only considered the asymptotic dimension of metric spaces. John Roe has shown that what is really important for this large-scale version of dimension is the so-called coarse structure of the space [83,84]. This should be thought of as the analog of the situation one encounters beginning a study of topology. The first examples one sees in topology are metric spaces, but one soon realizes that the abstract notion of a topology is really what makes the theory work. This section follows the development given in [84]. An alternative approach to the coarse structures is given in [52] and some of the basic properties of asymptotic dimension in the coarse sense are developed in [59].

To begin, we define the abstract notion of a coarse structure. Let X be a set. If $E \subset X \times X$, then the *inverse* of E, denoted E^{-1} is the set $\{(x, x') \mid (x', x) \in E\}$. If E' and E'' are subsets of $X \times X$, then the product is denoted $E' \circ E''$ and is defined to be

$$E' \circ E'' = \{(x', x'') \mid \exists x \in X \text{ such that } \exists (x', x) \in E' \text{ and } \exists (x, x'') \in E''\}.$$

If $E \subset X \times X$ and $K \subset X$, define $E[K] = \{x' \mid \exists x \in K,\ (x', x) \in E\}$. When $K = x$ we use the notation $E_x = E[\{x\}]$ and $E^x = E^{-1}[\{x\}]$.

Definition. A *coarse structure* on a set X is a collection $\mathcal{E}$ of sets (called *controlled sets* or *entourages* for the coarse structure) that contains the diagonal and is closed under the formation of subsets, inverses, products and finite unions. A set equipped with a coarse structure is called a *coarse space.*

A subset $D \subset X$ of a coarse space is *bounded* if $D \times D$ is controlled. A family of sets $\{D_i\}$ is *uniformly bounded* if $\bigcup_i (D_i \times D_i)$ is controlled.

Recall the following characterization of asymptotic dimension for metric spaces:

(1) $\operatorname{asdim} X = 0$ on r-scale if and only if there is a cover of X by uniformly bounded, r-disjoint sets.
(2) $\operatorname{asdim} X \leqslant n$ if and only if, for every $r < \infty$, X can be written as a union of at most $n+1$ sets that are 0-dimensional on r-scale.

To translate this to the coarse category, we need a notion of disjointness.

Definition. Let X be a coarse space and let U be a controlled set. We say that $D \subset X$ is *U-disconnected* if it can be written as a disjoint union $D = \bigsqcup_{i=1}^{\infty} D_i$ such that

(1) $\{D_i\}$ is uniformly bounded, i.e., $\bigcup (D_i \times D_i) = W$ is controlled (in this case, we call the family $\{D_i\}$ *W-bounded*), and
(2) when $i \neq j$, $D_i \times D_j$ is disjoint from U (in this case, we call the family $\{D_i\}$ *U-disjoint*).

Definition. Let $(X, \mathcal{E})$ be a coarse space. Then:

(1) $\operatorname{asdim} X = 0$ if it is U-disconnected for every controlled set U.
(2) $\operatorname{asdim} X \leqslant n$ if for every controlled set U, X can be written as the union of at most $n+1$ U-disconnected subsets.

Example. Let (X, d) be a metric space and let $\mathcal{E}$ be the collection of all $E \subset X \times X$ for which $\sup\{d(x, x') \mid (x, x') \in E\}$ is finite. Then, $\mathcal{E}$ is a coarse structure called the *bounded coarse structure associated to* (X, d).

It is straightforward to prove the following proposition.

Proposition 49. *Let (X, d) be a metric space with bounded coarse structure $\mathcal{E}$. Then* $\operatorname{asdim}(X, d) = \operatorname{asdim}(X, \mathcal{E})$.

A coarse structure on a topological space is *consistent with the topology* if the bounded sets for this structure are exactly those that are relatively compact. Suppose $\mathcal{E}$ is a coarse structure that is consistent with the topology on a locally compact space X. We say that $f : X \to \mathbf{C}$ is a *Higson function*, denoted $f \in C_h(X, \mathcal{E})$ if for every $E \in \mathcal{E}$ and every $\varepsilon > 0$, there is a compact set $K \subset X$ such that $|f(x) - f(y)| < \varepsilon$ whenever $(x, y) \in E \setminus (K \times K)$. Then by the Gelfand–Naimark theorem there is a compactification $h_{\mathcal{E}} X$ of X called the *Higson compactification*. The *Higson corona* is defined by $\nu_{\mathcal{E}} X = h_{\mathcal{E}} X \setminus X$.

We consider a proper metric space (X, d) with basepoint x_0 and define $\|x\| = d(x, x_0)$.

Definition. We define the *sublinear coarse structure*, denoted $\mathcal{E}_L$, on X as follows:

$$\mathcal{E}_L = \left\{ E \subset X \times X \colon \lim_{x \to \infty} \frac{\sup_{y \in E_x} d(y, x)}{\|x\|} = 0 = \lim_{x \to \infty} \frac{\sup_{y \in E^x} d(x, y)}{\|x\|} \right\}.$$

By the statement $\lim_{x \to \infty} \frac{\sup_{y \in E_x} d(y,x)}{\|x\|} = 0$, we mean that for each $\varepsilon > 0$, there is a compact subset K of X containing x_0 such that

$$\frac{\sup_{y \in E_x} d(y, x)}{\|x\|} \leqslant \varepsilon$$

for all $x \notin K$. Equivalently we could say for each $\varepsilon > 0$ there is an $r \geqslant 0$ such that $\frac{\sup_{y \in E_x} d(y,x)}{\|x\|} \leqslant \varepsilon$ for all x with $d(x_0, x) > r$. It would perhaps be better to think of this as $\lim_{\|x\| \to \infty}$. We leave to the reader to check that $\mathcal{E}_L$ is indeed a coarse structure and it does not depend on the choice of basepoint. The Higson corona for the sublinear coarse structure on X will be denoted by $\nu_L X$.

The sublinear coarse structure is useful for the Assouad–Nagata dimension [49].

Theorem 50. *For a cocompact connected proper metric space,*

$$AN\text{-}\dim X = \dim \nu_L X$$

provided $AN\text{-}\dim X < \infty$.

A metric space X is *cocompact* if there is a compact set $C \subset X$ such that $Iso(X)(C) = X$ where $Iso(X)$ is the group of isometries of X.

II. Asymptotic dimension of groups

13. Metrics on groups

A *norm* on a group G is a unary operation $\|\cdot\|$ satisfying

(1) $\|g\| = 0$ if and only if $g = e$;
(2) $\|g\| = \|g^{-1}\|$ for all $g \in G$; and
(3) $\|gh\| \leqslant \|g\| + \|h\|$.

Let Γ be a finitely generated group. To any finite generating set $S = S^{-1}$, one can assign a norm $\|\cdot\|_S$ defined by setting $\|g\|_S$ equal to the length of the shortest S-word presenting the element g.

One can now define the *left-invariant word metric associated to* S by $d_S(g,h) = \|g^{-1}h\|_S$. When the generating set is understood we write $d(g,h)$ for $d_S(g,h)$ and $\|\cdot\|$ for $\|\cdot\|_S$. This metric is left-invariant, i.e., the action of G on itself by left multiplication is an isometry: $d(ag,ah) = \|g^{-1}a^{-1}ah\| = \|g^{-1}h\| = d(g,h)$. The word metric turns the group Γ into a discrete metric space with bounded geometry. Since closed balls are finite—and hence compact—a finitely generated group in the word metric is proper.

We define the asymptotic dimension of a finitely generated group Γ by $\operatorname{asdim}\Gamma = \operatorname{asdim}(\Gamma, d_S)$, where S is any finite, symmetric generating set for Γ.

Corollary 51. *Let Γ be a finitely generated group. Then* $\operatorname{asdim}\Gamma$ *is an invariant of the choice of generating set, i.e., it is a group property.*

Proof. Let S and S' be finite generating sets for Γ. We have to show that (Γ, d_S) and $(\Gamma, d_{S'})$ are coarsely equivalent. In fact, they are Lipschitz equivalent, as we now show.

Let $\lambda_1 = \max\{\|s\|_{S'} \mid s \in S\}$ and $\lambda_2 = \max\{\|s'\|_S \mid s' \in S'\}$. It follows that $\lambda_1^{-1}\|\gamma\|_{S'} \leqslant \|\gamma\|_S \leqslant \lambda_2\|\gamma\|_{S'}$. Take $\lambda = \max\{\lambda_1, \lambda_2\}$. Then $\lambda^{-1}d_{S'}(g,h) \leqslant d_S(g,h) \leqslant \lambda d_{S'}(g,h)$. □

This proof shows that from the large-scale point of view two word metrics on a finitely generated group Γ are indistinguishable.

Alternatively, we could define $\operatorname{asdim}\Gamma$ to be $\operatorname{asdim}(Cay(\Gamma, S))$ where $Cay(\Gamma, S)$ denotes the Cayley graph of Γ with respect to the generating set S. It is easy to see that Γ with the word metric associated to S and the Cayley graph $Cay(\Gamma, S)$ with its edge-length metric are quasi-isometric.

13.1. Coarse equivalence on groups

For finitely generated groups, the primary notion of large-scale equivalence is that of quasi-isometry. Since finitely generated groups are quasi-isometric to geodesic metric spaces (Cayley graphs), a coarse equivalence between them is a quasi-isometry. A coarse equivalence is practically useful when dealing with countable but not finitely generated groups. We note that an alternative (equivalent) approach was implemented by means of extension of the notion of quasi-isometry to all countable groups in [87].

A fundamental result in geometric group theory is the following.

Theorem 52 *(Švarc–Milnor Lemma). Let X be a proper geodesic metric space and Γ be a group acting properly cocompactly by isometries on X. Then Γ is finitely generated and* (*in the word metric*) *Γ and X are quasi-isometric. In particular, they are coarsely equivalent.*

Corollary 53. *The following are easy consequences of the Švarc–Milnor Lemma*:

(1) *Let Γ be a finitely generated group and let $\Gamma' \subset \Gamma$ be a finite index subgroup. Then Γ and Γ' are quasi-isometric.*
(2) *Let*

$$1 \to K \to \Gamma \xrightarrow{\phi} H \to 1$$

be an exact sequence with K finite and H finitely generated. Then Γ is finitely generated and quasi-isometric to H.

Since asymptotic dimension is an invariant of quasi-isometry, we immediately obtain:

Corollary 54. *Let Γ be a finitely generated group.*

(1) *Let $\Gamma' \subset \Gamma$ be a finite index subgroup. Then* $\operatorname{asdim} \Gamma = \operatorname{asdim} \Gamma'$.
(2) *Let*

$$1 \to K \to \Gamma \to H \to 1$$

be an exact sequence with K finite and H finitely generated. Then $\operatorname{asdim} \Gamma = \operatorname{asdim} H$ (*cf. Theorem* 63).

Two groups are said to be *commensurable* if they have isomorphic finite-index subgroups. The previous corollary immediately implies the following result.

Corollary 55. *Let Γ and Γ' be commensurable with Γ finitely generated. Then* $\operatorname{asdim} \Gamma = \operatorname{asdim} \Gamma'$.

Another application of the Švarc–Milnor Theorem is the following, cf. [61].

Corollary 56. *Let M be a compact Riemannian manifold with universal cover $\tilde{M}$ and* (*finitely generated*) *fundamental group π. Then* $\operatorname{asdim} \tilde{M} = \operatorname{asdim} \pi$.

13.2. Asymptotic dimension of general groups

An immediate problem that one encounters when dealing exclusively with finitely generated groups is that a finitely generated group can have a subgroup that is not finitely generated. Although it is tempting to define the asymptotic dimension of the subgroup to be the asymptotic dimension of the metric subspace of the finitely generated group, the asymptotic dimension of the subgroup should be defined in terms of its abstract group structure, not in terms of a particular homomorphic embedding into a larger, finitely generated group.

It is for this reason that we make the following convention. When dealing with a countable, possibly non-finitely generated group G we define the asymptotic dimension $\operatorname{asdim} G$ to be the asymptotic dimension of the group endowed with some left-invariant, proper metric. It remains to show that the choice of left-invariant proper metric does not affect the asymptotic dimension. This is guaranteed in view of the following [87], [89, Proposition 1].

Proposition 57. *Let Γ be a countable group. Then any two left-invariant proper metrics on Γ are coarsely equivalent.*

Corollary 58. *Let Γ be a finitely generated group and let $\Gamma' \subset \Gamma$. Then,* $\operatorname{asdim} \Gamma' \leqslant \operatorname{asdim} \Gamma$.

Proof. Since all left-invariant proper metrics on Γ' are coarsely equivalent, we need only consider the asymptotic dimension of Γ' as a subspace of Γ, where Γ is endowed with the left-invariant word metric associated to a finite generating set. The fact that $\operatorname{asdim} \Gamma' \leqslant \operatorname{asdim} \Gamma$ as a subspace is an easy consequence of the definition. □

Moreover, the following holds [48]:

Theorem 59. *Let G be a countable group. Then* $\operatorname{asdim} G = \sup\{\operatorname{asdim} F \mid F \subset G$ *is finitely generated*$\}$.

This leads to the following

Definition. Let G be a (possibly uncountable) group

$$\operatorname{asdim} G = \sup\{\operatorname{asdim} F \mid F \subset G \text{ is finitely generated}\}.$$

Notice that with this definition, it is immediate that $\operatorname{asdim} H \leqslant \operatorname{asdim} G$ for any subgroup $H \subset G$.

There is a remark in [48] that the definition of asymptotic dimension of arbitrary groups given above coincides with the asymptotic dimension of the coarse space $(G, \mathcal{E})$ where $E \in \mathcal{E}$ if and only if $\{x^{-1}y \mid (x, y) \in E\}$ is finite.

It should also be noted that for an uncountable group, the asymptotic dimension as defined here does not necessarily agree with the asymptotic dimension the group of equipped with a left-invariant proper metric.

Taking groups with proper left-invariant metrics can give rise to interesting examples, as pointed out by [89].

Example. Endow $\mathbb{Q}$ with a left-invariant proper metric. Then $\operatorname{asdim} \mathbb{Q} = 0$. On the other hand, as a metric subspace of $\mathbb{R}$, $\mathbb{Q}$ is coarsely equivalent to $\mathbb{R}$ so $\operatorname{asdim} \mathbb{Q} = 1$. The problem is that the metric $\mathbb{Q}$ inherits from $\mathbb{R}$ is not a proper metric.

13.3. Groups with asdim 0 and 1

In the first section we proved stated that geodesic spaces with asymptotic dimension 0 are compact. This easily implies the following:

Proposition 60. *Let Γ be a finitely generated group. Then* $\operatorname{asdim} \Gamma = 0$ *if and only if Γ is finite.*

The following theorem was proven independently by several authors [58,64].

Theorem 61. *Every finitely presented group Γ with* $\operatorname{asdim} \Gamma = 1$ *is virtually free.*

Proof. Since Γ is finitely presented, we may assume that the 2-skeleton K^2 of $K(\Gamma, 1)$ is finite. Let X be the universal cover of K^2 with a lifted metric. By the Švarc–Milnor Lemma, X is quasi-isometric to Γ. By Theorem 43, X (and hence Γ) is quasi-isometric to a tree. By Stallings' theorem Γ is virtually free. □

This theorem does not hold for finitely presented groups. For instance, $\mathbb{Z}_2 \wr \mathbb{Z}$ is not virtually free and $\operatorname{asdim}(Z_2 \wr \mathbb{Z}) = 1$ [58].

14. The Hurewicz-type theorem for groups

The Hurewicz-type theorem (Theorem 30) took various forms in [7]:

Theorem 62. *Let Γ be a finitely generated group acting by isometries on the geodesic metric space X. Let $x_0 \in X$ and suppose that for every R, the set $\{g \in \Gamma \mid d(g(x_0), x_0) \leqslant R\}$ has* $\operatorname{asdim} \leqslant n$. *Then* $\operatorname{asdim} \Gamma \leqslant \operatorname{asdim} X + n$.

Theorem 63 *(Extension Theorem). Let*

$$1 \to K \to G \to H \to 1$$

be an exact sequence with G finitely generated. Then

$$\operatorname{asdim} G \leqslant \operatorname{asdim} H + \operatorname{asdim} K.$$

We remark that Dranishnikov and Smith have extended both of these versions of the Hurewicz-type theorem to all groups. Also they have shown that for a short exact sequence of Abelian groups,

$$0 \to B \to A \to C \to 0$$

we have the equality

$$\operatorname{asdim} A = \operatorname{asdim} B + \operatorname{asdim} C$$

using Theorem 69 below.

The extension theorem above (Theorem 63) was used by Bell and Fujiwara [10] in their upper bound estimates for mapping class groups of surfaces with genus at most 2:

Theorem 64. *Let $S_{g,p}$ be an orientable surface of genus $g \leqslant 2$ with p punctures. Suppose that $3g - 3 + p > 1$. Then,*

$$\operatorname{asdim} MCG(S_{g,p}) = cd\big(MCG(S_{g,p})\big),$$

where MCG denotes the mapping class group and $cd(\cdot)$ is the cohomological dimension.

Since the braid group B_n is isomorphic to the mapping class group of a disk with n punctures, we see that a copy of B_n sits inside $MCG(S_{0,n+1})$, the mapping class group of the sphere with $n + 1$ punctures. Applying the previous result, we obtain the following.

Corollary 65. *Let B_n be the braid group on n strands. If $n \geqslant 3$,* $\operatorname{asdim} B_n \leqslant n - 2$.

Presently, we discuss other applications of the Extension Theorem.

15. Polycyclic groups

Definition. A group G is called *polycyclic* if there exists a sequence of subgroups $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that $G_i \lhd G_{i+1}$ and G_{i+1}/G_i is cyclic.

The *Hirsch length* of a polycyclic group, $h(G)$, is defined to be the number of factors G_{i+1}/G_i isomorphic to $\mathbb{Z}$.

Theorem 66. *Let Γ be a finitely generated polycyclic group. Then* $\operatorname{asdim} \Gamma = h(\Gamma)$.

Proof. The proof of the inequality $\operatorname{asdim} \Gamma = h(\Gamma)$ is given in [7]. The inequality $\operatorname{asdim} \Gamma \geqslant h(\Gamma)$ was proven in [48]. We repeat the first. Denote the sequence of subgroups satisfying the polycyclic condition by: $\{1\} = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$. Then, by Theorem 63, we have

$$\begin{aligned}
\operatorname{asdim} \Gamma &\leqslant \operatorname{asdim} \Gamma_n/\Gamma_{n-1} + \Gamma_{n-1} \\
&\leqslant \operatorname{asdim} \Gamma_n/\Gamma_{n-1} + \operatorname{asdim} \Gamma_{n-1}/\Gamma_{n-2} + \operatorname{asdim} \Gamma_{n-2} \\
&\vdots \\
&\leqslant \operatorname{asdim} \Gamma_n/\Gamma_{n-1} + \cdots + \operatorname{asdim} \Gamma_1/\Gamma_0 + \operatorname{asdim} \Gamma_0.
\end{aligned}$$

Since $\operatorname{asdim} \Gamma_{i+1}/\Gamma_i$ is only positive if Γ_{i+1}/Γ_i is isomorphic to $\mathbb{Z}$, and since in this case, $\operatorname{asdim} \Gamma_{i+1}/\Gamma_i = 1$, we conclude $\operatorname{asdim} \Gamma \leqslant h(\Gamma)$. □

Corollary 67. *(See [20].) For polycyclic groups the Hirsch length $h(\Gamma)$ is a quasi-isometry invariant.*

This result was extended to solvable groups [87,93].

Since every finitely generated nilpotent group is polycyclic we immediately obtain the following result.

Corollary 68. *Let Γ be a finitely generated nilpotent group. Then* $\operatorname{asdim} \Gamma = h(\Gamma)$.

More generally, let G be a solvable group with commutator series:

$$1 = G_0 \subset G_1 \subset \cdots \subset G_n = G,$$

so that $G_i = [G_{i+1}, G_{i+1}]$. Then, the Hirsch length is defined to be

$$h(G) = \sum \dim_{\mathbb{Q}}(G_{i+1}/G_i \otimes \mathbb{Q}).$$

Theorem 69. *(See [48, Theorem 3.2].) For an Abelian group A,* $\operatorname{asdim} A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$.

Corollary 70. *(See [48, Theorem 3.4].) For a solvable group G,* $\operatorname{asdim} G \leqslant h(G)$.

Example. Consider the integer Heisenberg group, H, i.e., all matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

with the usual multiplication. It is easy to find a central series for H:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

So its Hirsch length is 3 (there are three copies of $\mathbb{Z}$ in the factor groups). Thus, by Corollary 68, $\operatorname{asdim} H \leqslant 3$.

Modifying the previous example, one could instead take the real Heisenberg group, H. It is the simplest nilpotent Lie group. With this example in mind, one could extend Corollary 68 to nilpotent Lie groups N by defining the Hirsch length $h(N)$ as the sum of the number of factors in Γ_{i+1}/Γ_i isomorphic to $\mathbb{R}$ for the central series $\{\Gamma_i\}$ of N. We take an equivariant metric on N and on the quotients. Then the projection $\Gamma_{i+1} \to \Gamma_{i+1}/\Gamma_i$ is 1-Lipschitz and Γ_{i+1}/Γ_i is coarsely isomorphic to $\mathbb{R}^{n_i}$. Then we have

Corollary 71. *Let N be a nilpotent Lie group endowed with an equivariant metric. Then* $\operatorname{asdim} N = h(N)$.

Since $h(N) = \dim N$ for simply connected N, we obtain

Corollary 72. *(See [29, Theorem 3.5].) For a simply connected nilpotent Lie group N endowed with an equivariant metric* $\operatorname{asdim} N = \dim N$.

Corollary 72 is the main step in the proof of the following theorem which can be derived directly by application of Theorem 62 to the Iwasawa decomposition of a Lie group.

Theorem 73. *(See [29].) For a connected Lie group G and its maximal compact subgroup K there is a formula* $\operatorname{asdim} G/K = \dim G/K$ *where G/K is endowed with a G-invariant metric.*

16. Groups acting on trees

16.1. Bass–Serre theory

We briefly discuss the Bass–Serre theory describing the correspondence between groups acting on trees and generalizations of amalgamated products. This treatment follows [86]. For generalizations of the theory, see [21] and [4].

Let Y be a non-empty connected graph with vertex set $V(Y)$ and (directed) edge set $E(Y)$. If $y \in E(Y)$ is an edge, we denote by $\bar{y}$ the edge y with opposite orientation. The vertex $t(y)$ will denote the terminal vertex of the edge y and the vertex $i(y)$ will denote the initial vertex of y. We wish to define a structure called a graph of groups, which can be thought of as a recipe for building groups in a geometric way. For each $P \in V(Y)$ let G_P be a group and to each

$y \in E(Y)$ assign a group G_y and two injective homomorphisms $\phi_y : G_y \to G_{t(y)}$ and $\phi_{\bar{y}} : G_y \to G_{i(y)}$. Together, the groups, homomorphisms and graph form the graph of groups (G, Y).

We want to define a group called the fundamental group of the graph of groups associated to (G, Y).

First, we define an auxiliary group $F(G, Y)$ associated to the graph of groups (G, Y). In terms of generators and relations, $F(G, Y)$ can be described as the group generated by all elements of the vertex groups G_P along with all edges $y \in E(Y)$ of the graph Y. The relations are (1) the relations amongst the groups G_P, (2) the relation that $\bar{y} = y^{-1}$ and (3) an interaction between edge groups and vertex groups described by $y\phi_y(a)y^{-1} = \phi_{\bar{y}}(a)$, if $y \in E(Y)$ and $a \in G_y$.

More succinctly, $F(G, Y)$ can be described as the quotient of the free product $*_{P \in V(Y)} G_P * \langle y \in E(Y) \rangle$ by the normal subgroup generated by elements of the form $y\phi_y(a)y^{-1}(\phi_{\bar{y}}(a))^{-1}$, where $y \in E(Y)$ and $a \in G_y$.

Let c be a path in Y starting at a vertex P_0. Let $y_1, y_2, \ldots, y_n$ denote the edges of the path in Y with $t(y_i) = P_i$. The *length* of c is $\ell(c) = n$, its initial vertex is $i(c) = P_0$ and its terminal vertex is $t(c) = P_n$. A word of type c in $F(G, Y)$ is a pair (c, μ) where c is a path as above and μ is a sequence $r_0, r_1, \ldots, r_n$ with $r_i \in G_{P_i}$. The associated element of the auxiliary group is $|c, \mu| = r_0 y_1 r_1 \ldots y_n r_n \in F(G, Y)$.

There are two equivalent description of the fundamental group of (G, Y), but we describe only the one in terms of based loops in $F(G, Y)$. Let P_0 be a fixed vertex of Y. The *fundamental group* of (G, Y) is the subgroup $\pi_1(G, Y, P_0) \subset F(G, Y)$ consisting of words associated to loops c in Y based at P_0, i.e., paths with $i(c) = t(c) = P_0$.

Example. We give three basic examples which we will refer to later.

(1) If all vertex groups are trivial, then $\pi_1(G, Y, P_0) \cong \pi_1(Y, P_0)$.
(2) Suppose Y is a graph with two vertices P and Q and one edge y connecting them. Then $\pi_1(G, Y, P) = G_P \underset{G_y}{*} G_Q$.
(3) Suppose Y is a graph with one vertex and one edge. Then $\pi_1(G, Y, P) = G_P \underset{G_y}{*}$, the HNN-extension.

Having constructed the fundamental group, π, of the graph of groups, we now describe the construction of the Bass–Serre tree $\tilde{Y}$ on which the group π acts by isometries.

Let T be a maximal tree in Y and let π_P denote the canonical image of G_P in π, obtained via conjugation by the path c, where c is the unique path in T from the basepoint P_0 to the vertex P. Similarly, let π_y denote the image of $\phi_y(G_{t(y)})$ in $\pi_{t(y)}$. Then, set

$$V(\tilde{Y}) = \coprod_{P \in V(Y)} \pi/\pi_P$$

and

$$E(\tilde{Y}) = \coprod_{y \in E(Y)} \pi/\pi_y.$$

For a more explicit description of the edges, observe that the vertices $x\pi_{i(y)}$ and $xy\pi_{t(y)}$ are connected by an edge for all $y \in E(Y)$ and all $x \in \pi$. Obviously the stabilizer of the vertices are conjugates of the corresponding vertex groups, and the stabilizer of the edge connecting $x\pi_{i(y)}$ and $xy\pi_{t(y)}$ is $xy\pi_y y^{-1}x^{-1}$, a conjugate of the image of the edge group. This obviously stabilizes the second vertex, and it stabilizes the first vertex since $y\pi_y y^{-1} = \pi_{\bar{y}} \subset \pi_{i(y)}$. It is known (see [86]) that the action of left multiplication on $\tilde{Y}$ is isometric.

Now we will assume that the graph Y is finite and that the groups associated to the edges and vertices are finitely generated with some fixed set of generators chosen for each group. We let S denote the disjoint union of the generating sets for the groups, and require that $S = S^{-1}$. By the norm $\|x\|$ of an element $x \in G_P$ we mean the minimal number of generators in the fixed generating set required to present the element x. We endow each of the groups G_P with the word metric given by $\mathrm{dist}(x, y) = \|x^{-1}y\|$. We extend this metric to the group $F(G, Y)$ and hence to the subgroup $\pi_1(G, Y, P_0)$ in the natural way, by adjoining to S the collection $\{y, y^{-1} \mid y \in E(Y)\}$.

The principal result in the theory is the following [86]:

Theorem 74. *To every fundamental group of a graph of groups there corresponds a tree and an action of the fundamental group on the tree by isometries. To every isometric action of a group on a tree, there corresponds a graph of groups construction.*

16.2. Asdim of amalgams and HNN-extensions

We want to apply the Hurewicz-type theorem to groups acting on trees by isometries. To do this, we have to know the structure of the so-called R-stabilizers, i.e., the set $W_R(x) = \{\gamma \in \Gamma \mid d(\gamma . x, x) \leqslant R\}$. The following description is straightforward to prove. It appears in [6].

Proposition 75. *Let Y be a non-empty, finite, connected graph, and (G, Y) the associated graph of finitely generated groups. Let P_0 be a fixed vertex of Y, then under the action of π on $\tilde{Y}$, the R-stabilizer $W_R(1 . P_0)$ is precisely the set of elements of type c in $F(G, Y)$ with $i(c) = P_0$, and $l(c) \leqslant R$.*

Using the union theorem, we estimate the asymptotic dimension of the R-stabilizers in terms of the asymptotic dimension of the vertex stabilizers.

Lemma 76. *Let Γ act on a tree with compact quotient and finitely generated stabilizers satisfying* $\operatorname{asdim} \Gamma_x \leqslant n$ *for all vertices x, then* $\operatorname{asdim} W_R(x_0) \leqslant n$ *for all R.*

Now, by the Hurewicz-type theorem, we have the following theorem.

Theorem 77. *(See [6].) Let π denote the fundamental group of a finite graph of groups with finitely generated vertex groups, V_σ. Then*

$$\operatorname{asdim} \pi \leqslant \max_{\sigma}\{\operatorname{asdim} V_\sigma\} + 1.$$

Corollary 78. *Let A and B be finitely generated groups with* $\operatorname{asdim} A \leqslant n$ *and* $\operatorname{asdim} B \leqslant n$. *Then* $\operatorname{asdim}(A *_C B) \leqslant n + 1$.

Example. Since $\mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, we see that $\operatorname{asdim}(\mathrm{SL}(2, \mathbb{Z}) \leqslant 1$. Since it is an infinite group, $\operatorname{asdim} \mathrm{SL}(2, \mathbb{Z}) = 1$.

We knew that $\operatorname{asdim} \mathrm{SL}(2, Z) = 1$ already, since it is quasi-isometric to a tree, by the Švarc–Milnor Lemma, but the point of this example is to apply the estimate for amalgamated products.

Next, we give an example to show that this upper bound is sharp in the case of the amalgamated product. We work out the asymptotic dimension of the free product in a later section.

Example. Using the van Kampen Theorem one can obtain the fundamental group of the closed orientable surface of genus 2 as an amalgamated product of two free groups. Observe that $\operatorname{asdim} \pi_1(M_2) = 2$, and $\operatorname{asdim} F = 1$ for a free group.

The fact that limit groups have finite asymptotic dimension was pointed out to the first author by Bestvina to be an easy consequence of the example above and deep results in the theory of limit groups. The class of limit groups consists of those groups which naturally arise in the study of solutions to equations in finitely generated groups. One definition is the following: A finitely presented group L is a *limit group* if for each finite subset $L_0 \subset L$ there is a homomorphism to a free group which is injective on L_0. For more information the reader is referred to [13] and the references therein.

Proposition 79. *Let L be a limit group. Then* $\operatorname{asdim} L < \infty$.

Proof. Construct L (say with height h) via fundamental groups of graphs of groups where vertices have height $(h - 1)$ and height 0 groups are free groups, free Abelian groups and surface groups. □

Finally, we consider the HNN-extension of a group. Recall that this corresponds to a fundamental group of a loop of groups.

Corollary 80. *Let $A*_C$ denote an HNN-extension of the finitely generated group A. Then* $\operatorname{asdim} A*_C \leqslant \operatorname{asdim} A + 1$.

An easy consequence of this example is that one relator groups have finite asymptotic dimension.

Proposition 81. *Let $\Gamma = \langle S \mid r_1 r_2 \dots r_n = 1\rangle$ be a finitely generated group with one defining relator. Then* $\operatorname{asdim} \Gamma < n + 1$.

Proof. A result of Moldavanskii from 1967 (see [70] for example) states that a finitely generated one-relator group is an HNN-extension of a finitely presented group with shorter defining relator or is cyclic. In the example above we showed that taking HNN-extensions preserved finite asymptotic dimension and clearly, all cyclic groups have asymptotic dimension $\leqslant 1$. We conclude that $\operatorname{asdim} \Gamma \leqslant n + 1$. □

This proof first appeared in [8] and then it was rediscovered in [74].

17. A formula for the asdim of a free product

Previously we saw that $\operatorname{asdim} A *_C B \leqslant \max\{\operatorname{asdim} A, \operatorname{asdim} B\} + 1$. This inequality was improved in [38] to the following

Theorem 82. $\operatorname{asdim} A *_C B \leqslant \max\{\operatorname{asdim} A, \operatorname{asdim} B, \operatorname{asdim} C + 1\}$.

In the following example, the inequality is strict. Let F_2 denote a free group on two letters. Then $\Gamma = F_2 *_{\mathbb{Z}} F_2$ is a free group, so $\operatorname{asdim} \Gamma = 1 < \operatorname{asdim} \mathbb{Z} + 1 = 2$. Here the inclusion map $\mathbb{Z} \to \langle a, b \mid\rangle$ is given by $n \mapsto a^n$.

In this section we focus on free products only; we show that the formula of Theorem 82 is exact for the asymptotic dimension of a free product (amalgamated over e). We begin with two motivating examples.

Example. Since it is finite, $\operatorname{asdim} \mathbb{Z}_2 = 0$. On the other hand, $\mathbb{Z}_2 * \mathbb{Z}_2$ is infinite, so $\operatorname{asdim} \mathbb{Z}_2 * \mathbb{Z}_2 = 1$.

Example. The free group $\mathbb{F}_2$ on two letters has $\operatorname{asdim} \mathbb{F}_2 = 1$, yet it is a free product of two copies of the asymptotically 1-dimensional group $\mathbb{Z}$.

Theorem 83. *Let A and B be finitely generated groups with* $\operatorname{asdim} A = n$ *and* $\operatorname{asdim} B \leqslant n$. *Then,* $\operatorname{asdim} A * B = \max\{n, 1\}$.

Instead of giving the full proof, which can be found in [9], we simply give the ideas behind the proof.

Let Γ denote the group $A * B$ and let X be the Bass–Serre tree on which it acts, say by $\pi : \Gamma \to X$ defined by $\pi(\gamma) = \gamma A$.

Our first observation is that if W_1 and W_2 are disjoint bounded sets in X then the sets $\pi^{-1}(W_i)$ are asymptotically disjoint in Γ.

Given $\varepsilon > 0$ we want to construct an ε-Lipschitz, uniformly cobounded map to a uniform n-dimensional complex. For a basepoint $x_0 \in X$ we take a cover $\mathcal{W} = \{W_i\}_i$ of $\Gamma . x_0$ by uniformly bounded disjoint sets. Take these sets so that very large neighborhoods of these sets have multiplicity 2.

Each of the sets $\pi^{-1}(W_i)$ has asymptotic dimension n, which is the same as the asymptotic dimension of A. Also, any two of these sets are asymptotically disjoint. Each set $\pi^{-1}(W_i)$ admits an ε-Lipschitz, uniformly cobounded map to an n-dimensional complex. Given disjoint sets W_i and W_j whose large neighborhoods meet, we can find an asymptotic separator for these sets whose asymptotic dimension is $\leqslant n - 1$. Note that here we use the fact that $\operatorname{asInd} X \leqslant \operatorname{asdim} X$. Thus, there is an ε-Lipschitz uniformly cobounded map from each asymptotic separator to an $n - 1$-complex. We define the mapping from Γ to an n-dimensional complex by using uniform mapping cylinders from the inclusion of the asymptotic separator for W_i and W_j into each of W_i and W_j. By tweaking the size of the neighborhoods, one can make the mapping defined this way ε-Lipschitz and it will be uniformly cobounded.

Example. $\operatorname{asdim} \mathbb{Z}_2 * \mathbb{Z}_3 = 1$.

18. Coxeter groups

Let S be a finite set. A Coxeter matrix is a symmetric function $M : S \times S \to \{1, 2, 3, \ldots\} \cup \{\infty\}$ with $m(s, s) = 1$ and $m(s, s') = m(s', s) \geqslant 2$ if $s \neq s'$. The corresponding Coxeter group $\mathcal{W}(M)$ is the group with presentation

$$\mathcal{W}(M) = \langle S \mid (ss')^{m(s,s')} = 1 \rangle$$

where $m(s, s') = \infty$ means no relation. The associated Artin group $\mathcal{A}(M)$ is the group with presentation

$$\mathcal{A}(M) = \langle S \mid (ss')^{m(s,s')} = (s's)^{m(s,s')} \rangle.$$

Theorem 84. *(See [43].) Every Coxeter group has finite asymptotic dimension.*

The proof is based on a remarkable embedding theorem of Januszkiewicz [43,63]:

Theorem 85. *Every Coxeter group Γ can be isometrically embedded in a finite product of trees $\prod T_i$ with the ℓ_1 metric on it in such a way that the image of Γ under this embedding in contained in the set of vertices of $\prod T_i$.*

It should be noted that, except in specific cases, the asymptotic dimension of a Coxeter group is unknown. It follows from [37] that the asymptotic dimension of a Coxeter group can be estimated from below as its virtual cohomological dimension: $\operatorname{asdim} W(M) \geqslant vcd(W(M))$. The upper bound is given by the number of trees in Januszkiewicz's embedding theorem which equals the number of generators $|S|$. It was noted in [47] that this number can be lowered to the chromatic number of the nerve $N(M)$ of the Coxeter group. Recently in [38] it was shown that $\operatorname{asdim} W(M) \leqslant \dim N(M) + 1$ for right-angled Coxeter groups. A Coxeter group is called *right-angled* if all non-diagonal entries in its Coxeter matrix are 2 (or ∞).

An Artin group with associated Coxeter matrix M, $\mathcal{A} = \mathcal{A}(M)$, is said to be of finite type if $\mathcal{W}(M)$ is finite. It is said to be of affine type if $\mathcal{W}(M)$ acts as a proper, cocompact group of isometries on some Euclidean space with the elements of S acting as affine reflections.

The following approach to finding an upper bound for the asdim of (certain) Artin groups was suggested by Robert Bell.

In [30], Charney and Crisp observe that each of the Artin groups $\mathcal{A}(A_n)$, $\mathcal{A}(B_n)$ of finite type and the Artin groups $\mathcal{A}(\tilde{A}_{n-1})$ and $\mathcal{A}(\tilde{C}_{n-1})$ of affine type is a central extension of a finite index subgroup of $MCG(S_{0,n+2})$ when $n \geqslant 3$. Combining this with the fact that the centers of the Artin groups of finite type are infinite cyclic and the centers of those of affine type are trivial gives the following corollary.

Corollary 86. *Let $n \geqslant 3$. Then if $\mathcal{A}$ is an Artin group of finite type A_n or $B_n = C_n$, we have* $\operatorname{asdim} \mathcal{A} \leqslant n$; *if $\mathcal{A}$ is an Artin group of affine type $\tilde{A}_{n-1}$ or $\tilde{C}_{n-1}$,* $\operatorname{asdim} \mathcal{A} = n - 1$.

This follows easily from the formula for $\operatorname{asdim} MCG(S_{g,p})$ from [10].

19. Hyperbolic groups

The goal of this section will be to see that a δ-hyperbolic finitely generated group has finite asymptotic dimension. This result was announced in Gromov's book [61] and an explicit proof of more general results appear in [84] and [85].

Theorem 87. *Every finitely generated hyperbolic group has finite asymptotic dimension.*

Proof. (Cf. example in Section 3.1.) Let $Y = \operatorname{Cay}(\Gamma, S)$ where S is a symmetric finite generating set. Then Y is a geodesic metric space with bounded geometry and Y is quasi-isometric to Γ in the word metric d_S. We denote by e the vertex of Y corresponding to the identity of Γ.

Let $r \gg \delta$ be given and define concentric annuli

$$A^n = \{x \in Y \mid 2(n-1)r \leqslant d(x,e) \leqslant 2nr\}$$

of thickness $2r$ and shells,

$$S^n = \{x \in Y \mid d(x,e) = 2nr\}.$$

Let $\{s_i^n\}$ be an enumeration of the elements of S^n.

For $n = 1, 2$ define $U^n = A^n$. For $n \geqslant 3$ define families U_i^n as follows:

$$U_i^n = \{x \in A^n \mid \exists \text{ geodesic } [e,x] \text{ containing } s_i^{n-2} \in S^{n-2}\}.$$

It is easy to check that $\operatorname{diam}(U_i^n) \leqslant 4r$. Next, take $B = B(x, r/2)$. Clearly, B can meet at most 2 of the annuli. Suppose $y_i \in U_i^n \cap B$. Then since Y is δ-hyperbolic, $d(s_i, s) \leqslant \delta$ where $s \in S^{n-2}$ is on a geodesic $[e,x]$. We conclude that the number of U_i^n meeting B is bounded by the cardinality number of geodesics in a δ-ball in Y. This number is $|S|^{\lceil \delta \rceil}$, where $\lceil \delta \rceil$ denotes the least integer greater than or equal to δ. Thus, the $r/2$-multiplicity of $\{U_i^n\}$ is no more than $2|S|^{\lceil \delta \rceil}$. Hence, $\operatorname{asdim} \Gamma < \infty$. □

We note that this theorem follows from Theorem 47 where the sharp upper bound for the asymptotic dimension is given. It also follows from the work of Bonk and Schramm [15], who provide a roughly quasi-isometric embedding of a hyperbolic metric space with bounded growth at some scale into a convex subset of hyperbolic space.

A different proof of this theorem was obtained by Buyalo and Lebedeva [26,27]. For hyperbolic groups they established the equality

$$\operatorname{asdim} \Gamma = \dim \partial_\infty \Gamma + 1.$$

Lebedeva found a formula for asymptotic dimension of product of hyperbolic groups [68]

$$\operatorname{asdim}(\Gamma_1 \times \Gamma_2) = \dim(\partial_\infty \Gamma_1 \times \partial_\infty \Gamma_2) + 2.$$

Her formula applied to examples of hyperbolic groups with Pontryagin surfaces as the boundaries [39] gives an example of Coxeter hyperbolic groups with

$$\operatorname{asdim}(\Gamma_1 \times \Gamma_2) < \operatorname{asdim} \Gamma_1 + \operatorname{asdim} \Gamma_2.$$

Metric spaces satisfying this condition were constructed in [27].

Also we remark that Theorem 87 was extended by Bell and Fujiwara [10] to finite asymptotic dimension of hyperbolic graphs with a certain type of geodesics.

20. Relatively hyperbolic groups

In [60], Gromov defined relative hyperbolicity. Since then it has been studied by many authors from many points of view.

The point of this section is an explanation of Osin's work [77]. The main theorem in that paper is the following.

Theorem 88. *Let G be a finitely generated group hyperbolic with respect to a (finite) collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Suppose that each of the groups H_λ has finite asymptotic dimension. Then* $\operatorname{asdim} G < \infty$.

There are several ways of defining relatively hyperbolic groups. We give the definition that Osin uses in his paper, see also [18,55,60].

Let G be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of G and X a subset of G. We say that X is a *relative generating set of G with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$* if G is generated by X and the union of the H_λ.

Let $F = (*_{\lambda \in \Lambda} H_\lambda) * F(X)$, where $F(X)$ is the free group on the set X. A *relative presentation* for G is a presentation of the form

$$\langle X, H_\lambda, \lambda \in \Lambda \mid \mathcal{R} \rangle.$$

We say that this presentation is *finite* and say that G is *finitely presented relative to the collection of subgroups* $\{H_\lambda\}_{\lambda\in\Lambda}$ if $\sharp X$ and $\sharp\mathcal{R}$ are finite.

Let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda}(H_\lambda \setminus \{1\})$. Given a word W in the alphabet $X \cup \mathcal{H}$ such that W represents 1 in G, there is an expression

$$W =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i$$

(with equality in the group F), where $R_i \in \mathcal{R}$ is a relation, and $f_i \in F$ for $i = 1, \dots, k$. The smallest possible k in such a presentation is called the relative area of W and is denoted by $Area^{rel}(W)$.

Definition. A group G is said to be *hyperbolic relative to a collection of subgroups* $\{H_\lambda\}_{\lambda\in\Lambda}$ if G is finitely presented relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ and there is a constant $L > 0$ such that for any word W in $X \cup \mathcal{H}$ representing the identity in G, we have $Area^{rel}(W) \leqslant L\|W\|$.

Osin showed in [78] that when G is generated by a finite set in the ordinary sense and finitely presented relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ then Λ is known to be finite and the subgroups H_λ are known to be finitely generated.

To prove Theorem 88, Osin applies the Hurewicz-type theorem. Let $\Gamma(G, X \cup \mathcal{H})$ denote the Cayley graph of G with respect to the generating set $X \cup \mathcal{H}$. Note that $\Gamma(G, X \cup \mathcal{H})$ is hyperbolic, see [78, Theorem 1.7]. It is not locally finite, but on the other hand, Osin proves that it has finite asymptotic dimension. Next, he proves that the asymptotic dimension of so-called relative balls $B(n)$ does not exceed the maximum of the asymptotic dimensions of the H_λ. Here, by a relative ball we mean a set of the form

$$B(n) = \{g \in G \mid |g|_{X\cup\mathcal{H}} \leqslant n\}.$$

In other words it is the ball of radius n in G with respect to the distance $d_{X\cup\mathcal{H}}$ centered at 1.

Proof of Theorem 88. The group G acts on $\Gamma(G, X \cup \mathcal{H})$ by left multiplication. The R-stabilizer at 1 coincides with the relative ball $B(R)$. By the Hurewicz-type theorem for groups, $\operatorname{asdim} G < \infty$. □

A notion related to relative hyperbolicity is that of weak relative hyperbolicity. A group G is said to be *weakly relatively hyperbolic* with respect to a collection $\{H_\lambda\}_{\lambda\in\Lambda}$ of subgroups if the Cayley graph $\Gamma(G, X\cup\mathcal{H})$ is hyperbolic, where X is a finite generating set for G modulo $\{H\}_{\lambda\in\Lambda}$ and $\mathcal{H}$ is defined as above.

Although relative hyperbolicity implies weak relative hyperbolicity, the converse does not hold. The natural question that arises is whether weak relative hyperbolicity is enough for finite asymptotic dimension in the sense of Theorem 88. Osin answers this question in the negative. In particular he proves the following theorem.

Theorem 89. *There exists a finitely presented boundedly generated group of infinite asymptotic dimension.*

Recall that a group is said to be *boundedly generated* if there are elements $x_1, \dots, x_n$ of G such that any $g \in G$ can be represented in the form $g = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$. So with respect to any generating set X and $\{H_\lambda\} = \langle x_\lambda\rangle$ we see that the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ has finite diameter. Thus as a corollary, we obtain the following.

Corollary 90. *There exists a finitely presented group G and a finite collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ such that*

(1) *each H_λ is cyclic* (*so* $\operatorname{asdim} H_\lambda \leqslant 1$);
(2) *the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ has finite diameter—so it is hyperbolic and G is weakly relatively hyperbolic with respect to $\{H_\lambda\}$; and*
(3) $\operatorname{asdim} G = \infty$.

21. Arithmetic groups

Let Γ be an arithmetic subgroup of a linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. L. Ji [65] generalized the result of Carlsson and Goldfarb, to show that Γ has finite asymptotic dimension.

We begin by recalling Carlsson and Goldfarb's theorem.

Theorem 91. *Let G be a connected Lie group with maximal compact subgroup K. Let $X = G/K$ be the associated homogeneous space endowed with a G-invariant Riemannian metric. Then* $\operatorname{asdim} X = \dim X$.

As a corollary, Ji observes the following result.

Corollary 92. *Let G be a connected Lie group and $\Gamma \subset G$ any finitely generated discrete subgroup. Then* $\operatorname{asdim} \Gamma < \infty$.

Proof. The group Γ acts properly isometrically on the proper metric space $X = G/K$. Observe that the map $\gamma \mapsto \gamma \,.\, x_0$ is a coarse equivalence between Γ and $\Gamma \,.\, x_0 \subset X$ for any choice of $x_0 \in X$ (cf. the Švarc–Milnor Lemma [31]). Thus $\operatorname{asdim} \Gamma \leqslant \operatorname{asdim} X$. □

Let $\mathbf{G}$ be a linear algebraic group defined over $\mathbb{Q}$, i.e., an algebraic subgroup of $GL(n, \mathbb{C})$ defined by polynomial equations with rational coefficients. A subgroup Γ of $\mathbf{G}(\mathbb{Q})$ is called *arithmetic* if Γ is commensurable with $\mathbf{G} \cap GL(n, \mathbb{Z})$.

The following finite dimensionality result essentially follows from the previous corollary. In addition, Ji gives a lower bound for the asymptotic dimension that can be achieved when the lattice is non-cocompact. Let ρ denote the $\mathbb{Q}$-rank of $\mathbf{G}$, i.e., the maximal dimension of $\mathbb{Q}$-split tori in $\mathbf{G}$.

Theorem 93. *Let $\mathbf{G}$ be a connected linear algebraic group defined over $\mathbb{Q}$. Let Γ be an arithmetic subgroup of $\mathbf{G}$, which is assumed to be a lattice in $G = \mathbf{G}(\mathbb{R})$. Then*

$$\dim X - \rho \leqslant \operatorname{asdim} \Gamma \leqslant \dim X.$$

Kleiner observed that the lower bound follows from a result of Borel and Serre [16] and the fact that $\operatorname{asdim} \Gamma \geqslant cd(\Gamma)$, where $cd(\cdot)$ denotes the cohomological dimension, see [61].

Later Lizhen Ji extended his result to S-arithmetic groups [66]:

Theorem 94. $\operatorname{asdim} \Gamma < \infty$ *for all S-arithmetic groups Γ.*

22. Buildings

In his thesis, D. Matsnev [73] applied the Hurewicz-type theorem for asymptotic dimension to show that affine buildings have finite asymptotic dimension. The method is to reduce the problem to computing the asymptotic dimension of a certain matrix group that is coarsely equivalent to a given affine building.

Throughout this section we let K field with a discrete valuation ν, i.e., a surjection $\nu : K^* \to \mathbb{Z}$ satisfying

$$\nu(x + y) \geqslant \min\{\nu(x), \nu(y)\}$$

for all $x, y \in K^*$. Let X denote the building associated to $SL(n, K)$, see [25, Chapter V.8] for a description of X.

We now describe the notion of distance on $SL(n, K)$. The "metric" on $SL(n, K)$ is actually not a metric at all, but rather a pseudometric. It inherits this pseudometric from a length function ℓ defined in terms of the discrete valuation ν as follows:

$$\ell(g) = -\min_{1 \leqslant i, j \leqslant n} \{\nu(g_{ij}), \nu(g^{ij})\}.$$

Here g_{ij} is the ijth entry of g and g^{ij} is the ijth entry of its inverse. The pseudometric is then defined to be

$$\operatorname{dist}(g, h) = \ell(g^{-1} h).$$

The following is a sketch of the proof of Matsnev's theorem:

Theorem 95. *Any affine building X has finite asymptotic dimension.*

The affine building X is coarsely equivalent to the group $G = SL(n, K)$, with the pseudometric described above so it suffices to compute $\operatorname{asdim} G$. If C denotes the maximal compact subgroup of G, then we can write $G = CB$, where B is the subgroup of upper triangular matrices. Since C is compact, G is coarsely equivalent to B, so it remains to compute $\operatorname{asdim} B$.

To this end, Matsnev defines a map $f : B \to A$ where A denotes the diagonal matrices. This map simply takes an upper-triangular matrix to the diagonal matrix obtained by replacing all off-diagonal entries by 0. It can be shown that this map is 1-Lipschitz and that the set $f^{-1}(B_R(a))$ has asymptotic dimension 0 uniformly (in a) for every R. Thus, by the Hurewicz-type theorem, we see that $\operatorname{asdim} B \leqslant \operatorname{asdim} A$. It is shown that A in the pseudometric is coarsely equivalent to a subgroup isomorphic to $\mathbb{Z}^{n-1}$. Although the metric on $\mathbb{Z}^{n-1}$ is not the standard one, it is Lipschitz equivalent to the standard one and so we conclude that $\operatorname{asdim} A = n - 1$. Since $A \subset B$, we conclude that $\operatorname{asdim} B = n - 1$ and therefore that $\operatorname{asdim} X = n - 1$.

Recently Jan Dymara and Thomas Schick extended Matsnev's result to general (Tits) buildings [53].

Theorem 96. *The asymptotic dimension of a building X equals the asymptotic dimension of the apartment.*

23. Infinite-dimensional groups

It is not at all difficult to find examples of finitely generated groups with infinite asymptotic dimension. Indeed, let Γ be a finitely generated group containing an isomorphic copy of $\mathbb{Z}^m$ for each m, then $\operatorname{asdim} Y = \infty$. Some attempts were made to start up a theory of asymptotically infinite-dimensional spaces by analogy with topology. In [32] an asymptotic Property C was defined as follows: A metric space X has asymptotic Property C if for any sequence of natural numbers $n_1 < n_2 < \cdots$ there is a finite sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=1}^n$ such that the union $\bigcup_{i=1}^n \mathcal{U}_i$ is a cover of X and each $\mathcal{U}_i$ is n_i-disjoint. T. Radul introduced a notion of transfinite asymptotic dimension trasdim and proved that X has Property C if and only if $\operatorname{trasdim}(X)$ is defined [82]. In view of Borst's theorem which states that the transfinite dimension is defined for weakly infinite-dimensional compacta and his recent example [17] this result shows a striking difference between asymptotic and topological dimension for infinite-dimensional spaces.

Another way to deal with asymptotically infinite-dimensional groups is to study the dimension function $ad_X(\lambda)$ of a metric space X. We define this function as follows:

Definition. Let X be a metric space and $\mathcal{U}$ a cover of X. Denote the multiplicity of $\mathcal{U}$ by $m(\mathcal{U})$, i.e., $m(\mathcal{U}) = \sup \operatorname{Card}\{U \in \mathcal{U} \mid x \in U\}$. Denote the Lebesgue number of $\mathcal{U}$ by $L(\mathcal{U})$, i.e., $L(\mathcal{U})$ is the largest number so that for any $A \subset X$ with $\operatorname{diam}(A) \leqslant L(\mathcal{U})$ there is a $U \in \mathcal{U}$ with $A \subset U$. Define the dimension function of X by:

$$ad_X(\lambda) = \min\{m(\mathcal{U}) \mid L(\mathcal{U}) \geqslant \lambda\} - 1.$$

It is easy to see that $ad_X(\lambda)$ is monotone and $\lim_{\lambda\to\infty} ad_X(\lambda) = \operatorname{asdim} X$.

In the section on coarse embeddings we proved that a metric space with finite asymptotic dimension admits a coarse embedding into Hilbert space. Although we proved this result directly, in that discussion we mentioned that what is really needed to prove embeddability into Hilbert space is Yu's Property A. This property can be defined in terms of anti-Čech approximations. Namely a metric space has Property A if it admits an anti-Čech approximation $\{\mathcal{U}_i\}$ such that the canonical projections to the nerves $p_{\mathcal{U}_i}$ are ε_i-Lipschitz with $\varepsilon_i \to 0$ where the nerves $Nerve(\mathcal{U}_i)$ are taken with the induced metric from $\ell_1(\mathcal{U}_i)$ [35].

In [32,36], Dranishnikov proved the following generalization of this embedding result.

Theorem 97. *A metric space X has Property A in each of the following cases*:

(a) *X is a discrete metric space with polynomial dimension growth*;
(b) *X has Property C*.

Corollary 98. *Let Γ be a finitely generated group whose dimension function grows polynomially. Then the Novikov conjecture holds for Γ.*

In order for these results to be interesting, we need examples of spaces with polynomial dimension growth. Our example (following Roe [84]) is the restricted wreath product of $\mathbb{Z}$ by $\mathbb{Z}$.

Let H be the set of finitely supported maps $\mathbb{Z} \to \mathbb{Z}$. Let u and v be the permutations of H defined by

$$uf(n) = f(n) + \delta_{n0}, \qquad vf(n) = f(n+1).$$

Let G be the group generated by u and v. Endow G with the word metric, i.e., define $d(g,h)$ to be the length of the shortest presentation of the element $g^{-1}h$ in the alphabet $\{u, v\}$. It is easy to see that the elements

$$u, \quad v^{-1}uv, \quad \dots, \quad v^{(-n-1)}u, \quad v^{n-1}$$

generate an isomorphic copy of $\mathbb{Z}^n$ for each n, so $\operatorname{asdim} G = \infty$. On the other hand, Dranishnikov proved the following theorem in [36].

Theorem 99. *Let N be a finitely generated nilpotent group and let G be a finitely generated group with $\operatorname{asdim} G < \infty$. Then the restricted wreath product $N \wr G$ has polynomial dimension growth.*

Another famous group with infinite asymptotic dimension is Thompson's group F. It has many incarnations, but the easiest description (combinatorially) is that it is the group whose presentation is

$$F = \langle x_0, x_1, x_2, \dots \mid x_{j+1} = x_i^{-1} x_j x_i, \text{ for } i < j \rangle.$$

Notice that for $i \geqslant 2$, $x_i = x_0^{1-i} x_1 x^{i-1}$, so F is finitely generated. The growth rate of the dimension function of Thompson's group F is unknown. The infinite dimensionality of the Thompson group is based on the fact that F contains $\mathbb{Z}^n$ as a subgroup for all n. An example of a torsion group with infinite asymptotic dimension is Grigorchuk's group [90]. We briefly recall a fact from [36] about the growth of the dimension function as it applies to groups.

We saw that a particular wreath product had polynomial dimension growth. Gromov's group [62] containing an expander has exponential dimension growth. The next proposition says that this is the fastest the function can grow.

Proposition 100. *(See [36, Proposition 2.1].) Let Γ be a finitely generated group. Then there is an $a > 0$ so that $ad_\Gamma(\lambda) \leqslant e^{a\lambda}$.*

Proof. There is an $a > 0$ for which $|B_\lambda(x)| \leqslant e^{a\lambda}$. Clearly the cover of Γ by all balls $B_\lambda(x)$ has Lebesgue number at least λ. Also, the sets are uniformly bounded and have multiplicity $\leqslant |B_\lambda(x)| \leqslant e^{a\lambda}$, as required. □

Also, the growth of the asymptotic dimension function is not a coarse invariant, but it is an invariant of quasi-isometries. Since any two word metrics on a group are quasi-isometric, the growth of the dimension function is a group invariant.

Acknowledgement

The authors would like to thank Yehuda Shalom for some valuable remarks.

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