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Finite type invariants of order 3 for a spatial handcuff graph

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ABSTRACT

We express a basis for the vector space of finite type invariants of order less than or equal to three for an embedded handcuff graph in a 3-sphere in terms of the linking number, the Conway polynomial, and the Jones polynomial of the sublinks of the handcuff graph.

1. Introduction

Generalizing the finite type invariant, or Vassiliev invariant for a knot or link [1,4,30] Stanford [27,28] defined a finite type invariant for an embedded graph in a 3-sphere S^3. Then the first author [10] has given a basis for the space of the finite type invariants for the theta curve of order $\leqslant 3$, and Koike [17] has given a basis for the space of order 4. See also [26]. In this paper, we consider an embedded handcuff graph in S^3, which consists of two vertices P_1, P_2, and three edges e_1, e_2, e_3, with orientation as shown in Fig. 1. We give a basis for the space of the finite type invariants for the spatial handcuff graph of order $\leqslant 3$. Our method is similar to one adopted in [9,11–14,24], where bases of finite type invariants of knots or links of small dimension are given.

The value of an order n finite type invariant of a spatial handcuff graph with n singular points depends only on the corresponding n-chord diagram (Proposition 4.1), and every such value determines the space of finite type invariant of order $\leqslant n$, $\mathcal{V}_n$. However, by the generalized Reidemeister moves for a spatial handcuff graph (Fig. 2) some n-chord diagrams share the same value, which cause the relations (FI), (4T), (VE), (RV) as shown in Fig. 7. Let $\mathcal{D}_n$ be the space of all n-chord diagrams, and $\mathcal{A}_n$ the quotient space of $\mathcal{D}_n$ modulo these relations. Then there is a natural monomorphism $\mathcal{V}_n/\mathcal{V}_{n-1} \to \mathcal{A}_n^*$, which sends a finite type invariant of order n to the linear function on the space on $\mathcal{A}_n$. The main result of this paper is to give a basis for $\mathcal{V}_n/\mathcal{V}_{n-1}$, $n \leqslant 3$ (Theorems 7.1, 7.2, 8.1). Our method is similar to [10], where we have given a basis of the space of finite type invariant for a theta curve of order $\leqslant 3$. First, we give a spanning set for $\mathcal{A}_n$, $n \leqslant 3$ (Section 6), which gives an upper bound for the dimension of $\mathcal{V}_n/\mathcal{V}_{n-1}$. Next, we give a certain set of finite type invariants whose number is the same as the obtained upper bound for $\dim \mathcal{V}_n/\mathcal{V}_{n-1}$; these invariants are derived from the Conway polynomials or the Jones polynomials of sublinks of a spatial handcuff graph (Proposition 3.1). Lastly, we show they are linearly independent.

Notice that the above monomorphism $\mathcal{V}_n/\mathcal{V}_{n-1} \to \mathcal{A}_n^*$ is actually surjective [25] (Proposition 5.1); for the knot case, such a theorem is known as the Kontsevich theorem [1,5,18,19,30]. However, we do not use this fact. Koike [17] and Sugita [29]

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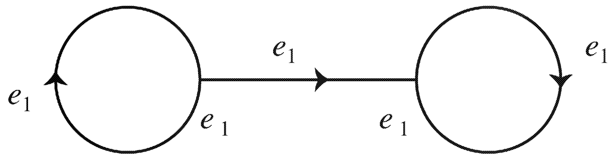


Fig. 1. The oriented handcuff graph.

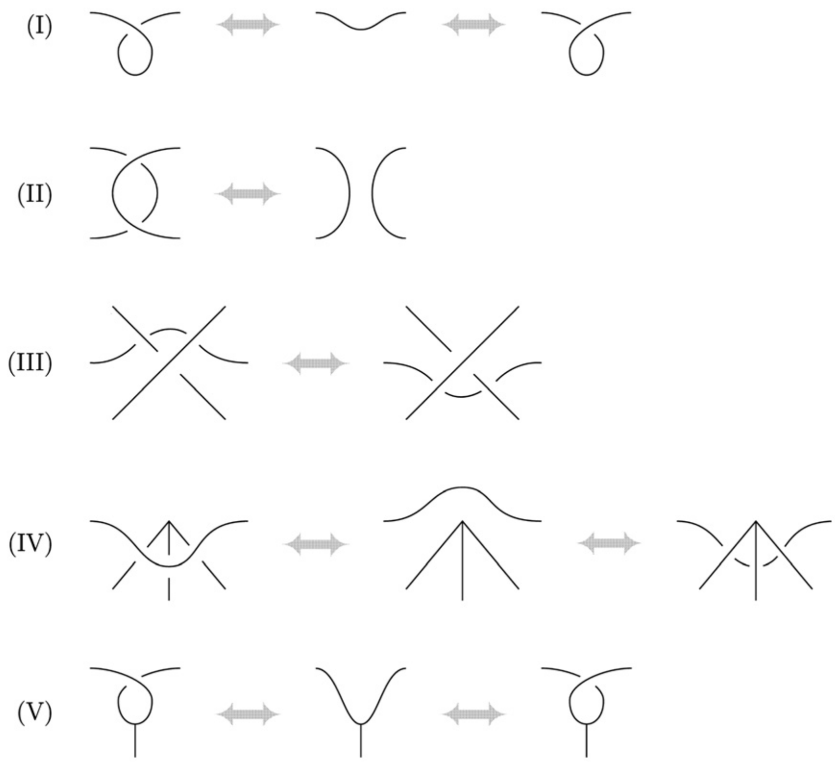


Fig. 2. Generalized Reidemeister moves for a spatial trivalent graph.

have given a basis of $\mathcal{A}_n$ for the theta curve of order $\leqslant 4$ and the spatial handcuff graph of order $\leqslant 3$, respectively, which eventually gave a basis of $\mathcal{V}_n/\mathcal{V}_{n-1}$.

This paper is organized as follows: In Sections 2–5, we briefly explain a finite type invariant for a spatial handcuff graph, including the space of chord diagrams. In Section 6, we give a spanning set for $\mathcal{A}_n$, $n \leqslant 3$. In Sections 7 and 8, we give a basis for $\mathcal{V}_n/\mathcal{V}_{n-1}$, $n \leqslant 3$, which allows us to give a basis of $\mathcal{A}_n$ (Corollary 8.2). Then comparing the result of the space of finite type invariant of order $\leqslant 3$ for an ordered 2 component oriented links in [14], a finite type invariant of order $\leqslant 3$ for a spatial handcuff graph is determined by the link consisting of the two loops, and is not affected by the connecting edge (Corollary 8.3).

2. Spatial handcuff graph

Two spatial handcuff graphs Φ and Φ' are equivalent if there is an orientation preserving homeomorphism h of S^3 such that $h(\Phi) = \Phi'$ and $h|_\Phi : \Phi \to \Phi'$ is orientation preserving. If two spatial handcuff graphs are equivalent, then they are related by a finite sequence of the five moves (I)–(V) on diagrams as shown in Fig. 2; see [16,31]. They are called generalized Reidemeister moves for spatial graphs.

3. Finite type invariant

A *singular spatial handcuff graph* is the image of an oriented handcuff graph under an immersion into S^3 whose only singularities are transverse double points. We assume that a double point on a singular spatial handcuff graph is a rigid (or flat) vertex.

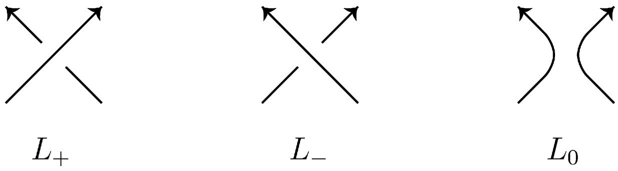


Fig. 3. A skein triple.

Let v be an invariant of a spatial handcuff graph in S^3, which takes values in the rational numbers $\mathbf{Q}$. We may extend it to a singular spatial handcuff graph via the *Vassiliev skein relation*:

$$v(\Phi_x) = v(\Phi_{x_+}) - v(\Phi_{x_-}), \tag{3.1}$$

where Φ_x is a spatial handcuff graph with x a double point and Φ_{x_+}, Φ_{x_-} are ones obtained from Φ_x by replacing x by a positive crossing x_+ and a negative crossing x_-, respectively. Then v is a *finite type invariant* of order $\leqslant n$ if $v(\Phi) = 0$ for an arbitrary singular spatial handcuff graph Φ that has more than n double points. If v is of order $\leqslant n$ but not of order $\leqslant n-1$, then v is called a finite type invariant of order n.

Denote by $\mathcal{V}_n$ the vector space consisting of all finite type invariants for a spatial handcuff graph of order $\leqslant n$. There is a filtration:

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_n \subset \cdots. \tag{3.2}$$

Note that the finite type invariants form an algebra, that is, the product of a finite type invariant of order m and one of n is a finite type invariant of order $m+n$; cf. [13, Section 4].

The *Conway polynomial* $\nabla(L) \in \mathbf{Z}[z]$ [7], and the *Jones polynomial* $V(L;t) \in \mathbf{Z}[t^{\pm 1/2}]$ [8] are invariants for an unordered oriented link L, which are defined by the following formulas:

$$\nabla(O) = 1; \tag{3.3}$$

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0); \tag{3.4}$$

$$V(O;t) = 1; \tag{3.5}$$

$$t^{-1}V(L_+;t) - tV(L_-;t) = \left(t^{1/2} - t^{-1/2}\right)V(L_0;t); \tag{3.6}$$

where O is the unknot and L_+, L_-, L_0 are three links that are identical except near one point where they are as in Fig. 3. We denote by $a_n(L)$ the coefficient of z^n of the Conway polynomial $\nabla(L)$.

For a spatial handcuff graph Φ having two loops e_1 and e_2, we use the following notations:

- $\lambda(\Phi)^k$ denotes the kth power of the linking number of e_1 and e_2; $\lambda(\Phi)^k = \mathrm{lk}(e_1, e_2)^k$,
- $a_2[i](\Phi) = a_2(e_i)$ $(i = 1, 2)$,
- $a_3(\Phi) = a_3(e_1 \cup e_2)$,
- $V^{(3)}[i](\Phi)$ denotes the 3rd derivative of the Jones polynomial of e_i evaluated at $t = 1$ $(i = 1, 2)$; $V^{(3)}[i](\Phi) = V^{(3)}(e_i; 1)$,

which we also simply denote by λ^k, $a_2[i]$, a_3, $V^{(3)}[i]$, respectively. Then they are finite type invariants for a spatial handcuff graph.

Proposition 3.1.

$$\lambda(\Phi)^k \in \mathcal{V}_k, \tag{3.7}$$

$$a_2[i](\Phi) \in \mathcal{V}_2, \tag{3.8}$$

$$a_3(\Phi), V^{(3)}[i](\Phi) \in \mathcal{V}_3. \tag{3.9}$$

In fact, according to Stanford [27], a finite type invariant of a 2-component link $e_1 \cup e_2$ or a knot e_i $(i = 1, 2)$ is a finite type invariant for Φ. Bar-Natan [1] has shown that the coefficient of the Conway polynomial of a link L is a finite type invariant for a link L. Birman and Lin [4] proved that the Jones polynomial of a knot can be interpreted as an infinite sequence of finite type knot invariants, and Stanford [28] generalized this for a link. See also [13,14].

Remark 3.2. For a 2-component link L, $a_1(L) = \mathrm{lk}(L)$; see [15, p. 13]. For a knot K, $V^{(2)}(K;1) = -6a_2(K)$; see [23,24].

The following is an immediate consequence of Eq. (3.1); cf. [2, (10d)], [28, Section 5].

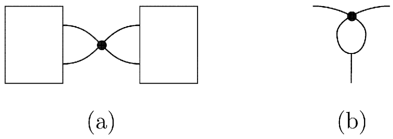


Fig. 4. Singular spatial graphs with trivial finite type invariant.

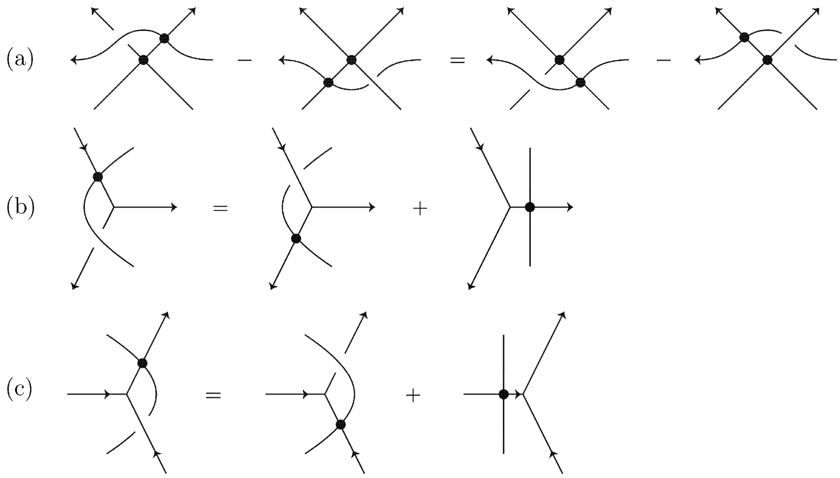


Fig. 5. Relations of a finite type invariant for singular spatial graphs.

Proposition 3.3. *The value of a finite type invariant of each singular spatial handcuff graph in Fig.* 4 *is zero*; *in Fig.* 4(a) *the squares contain the whole diagram away from the singular crossing.*

Besides this, a finite type invariant satisfies other important equations.

Proposition 3.4. *A finite type invariant satisfies the relations as shown in Fig.* 5*, where the equations signify the numerical equality of values of the invariant on these singular spatial handcuff graphs.*

The relation given in Fig. 5(a) is known as the 4-*term relation* in the case of a knot. The proof is the same as the knot case; see [4], [6, 1.1.3]. The relations given in Figs. 5(b) and (c), which are called the *vertex relations*, are obtained from the generalized Reidemeister move (V) in Fig. 2.

4. Chord diagram of a singular spatial handcuff graph

Let H be an oriented handcuff graph as in Fig. 1. Consider a singular spatial handcuff graph with n double points as the mapping $\Phi : H \to S^3$. Then join all the pairs of the preimages of every double point of $\Phi(H)$ with n dashed arcs. The resulting configuration C is called the *chord diagram of order* n, or n*-chord diagram*, of the singular spatial handcuff graph $\Phi(H)$; see [10, Section 4] for a theta curve. We say that $\Phi(H)$ respects the chord diagram C. Since a handcuff graph is trivalent, Proposition 1.1 in [28] implies the following, which generalizes the case of knots; see Lemma 1 in [2], Proposition 1 in [3]:

Proposition 4.1. *Two singular spatial handcuff graphs with n double points become equivalent after an appropriate series of crossing changes if and only if they respect the same chord diagram of order n.*

In particular, any spatial handcuff graph becomes trivial after an appropriate series of crossing changes. Thus if v is a finite type invariant of order zero, then $v(\Phi) = v(U)$ for any spatial handcuff graph Φ, where U denotes the unknotted spatial handcuff graph, that is, U is a planar handcuff graph in S^3. Namely, we have:

Proposition 4.2. *A finite type invariant of order zero for a spatial handcuff graph is a constant map.*

The singular spatial handcuff graphs as shown in Figs. 4(a), (b) respect the chord diagrams shown in Figs. 6(a), (b), respectively, where the squares contain the whole chord diagram away from the chord shown. We call such chord diagrams *inadmissible*. A chord diagram is called *admissible* if it is not inadmissible.

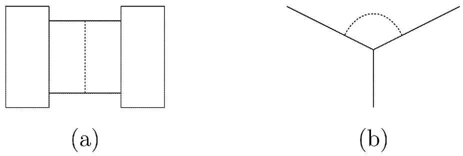


Fig. 6. Inadmissible chord diagrams.

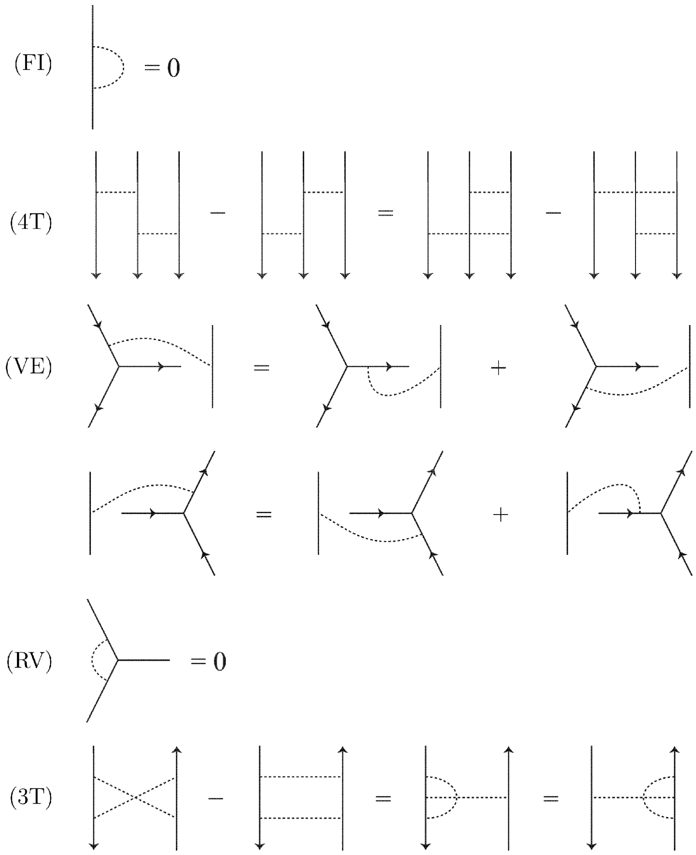


Fig. 7. Relations for chord diagrams.

Let us suppose we have made a list of the distinct admissible chord diagrams of order j, α_i^j; $1 \leqslant i \leqslant r_j$, $j = 2, 3, \dots,$ and chosen, for each α_i^j, a singular spatial handcuff graph M_i^j respecting it. By Proposition 4.1, using a resolution tree, we can calculate the value of a finite type invariant of a singular spatial handcuff graph; cf. [3, Proposition 2], [4, Proof of Theorem 2.4]:

Proposition 4.3. *Let v be a finite type invariant of order $\leqslant m$, and Φ^n a singular spatial handcuff graph respecting an admissible chord diagram of order n, α_p^n, $n \leqslant m$. Then*

$$v(\Phi^n) \equiv v(M_p^n),$$

where "$\equiv$" means equality up to a $\mathbf{Z}$-linear combination of $v(M_i^j)$, $1 \leqslant i \leqslant r_j$, $n+1 \leqslant j \leqslant m$. In particular, if $m = n$, then "$\equiv$" is "$=$", and so the v-value of a singular spatial handcuff graph with n double points depends only on its chord diagram.

5. Space of chord diagrams

We denote by $\mathcal{D}$ the $\mathbf{Q}$-linear space spanned by chord diagrams for a handcuff graph, which is naturally graded by the number of chords. We denote by $\mathcal{D}_n$ the subspace of $\mathcal{D}$ that is spanned by the chord diagrams of order n. We consider the four kinds of relations in $\mathcal{D}$ and $\mathcal{D}_n$ as shown in Fig. 7; (FI) the *framing independence relation*, (4T) the 4-*term relation*, (VE) the *vertex-edge relation*, and (RV) the relation induced from the generalized Reidemeister move (V). There is also the

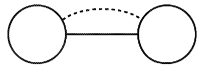


Fig. 8. The admissible chord diagram of order one, α_1.

relation (3T), which is implied from the relations (4T) and (FI). We consider the quotient space of $\mathcal{D}_n$ modulo the relations (FI), (4T), (VE), (RV), which we denote by $\mathcal{A}_n$.

There is a natural map $\mathcal{V}_n \to \mathcal{A}_n^*$ sending every finite type invariant of order n to the corresponding linear function on $\mathcal{A}_n$, which induces a monomorphism $\mathcal{V}_n/\mathcal{V}_{n-1} \to \mathcal{A}_n^*$. Moreover, this map is surjective, that is, for a spatial handcuff graph, the "Kontsevich Theorem" also holds; Jun Murakami and Ohtsuki [25] have shown the following (over the field of complex numbers **C**, and for any spatial trivalent graph). However, we do not use that $\mathcal{V}_n/\mathcal{V}_{n-1} \to \mathcal{A}_n^*$ is surjective.

Proposition 5.1. *The space $\mathcal{V}_n/\mathcal{V}_{n-1}$ is isomorphic to the space $\mathcal{A}_n^*$ of linear functions on chord diagrams of order n modulo the relations* (FI), (4T), (VE), (RV).

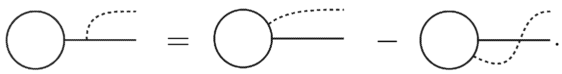
For the space $\mathcal{A}_n$, we give some lemmas, which we will use in Section 6. Let α be a chord diagram of order n. If the number of the endpoints of chords on the edge e_i is k_i ($i = 1, 2, 3$), then we call α a chord diagram of type (k_1, k_2, k_3), whence $2n = k_1 + k_2 + k_3$.

Lemma 5.2. *Let α be an n-chord diagram of type (k_1, k_2, k_3) with $k_3 > 0$. Then α is a linear combination of the chord diagrams of type $(k_1 + j, k_2 + k_3 - j, 0)$ in $\mathcal{A}_n$ with $0 \leqslant j \leqslant k_3$.*

Proof. Using the relation (VE), α is a linear combination of the chord diagrams of type $(k_1 + 1, k_2, k_3 - 1)$ (or of type $(k_1, k_2 + 1, k_3 - 1)$). So by induction we obtain the result. □

Lemma 5.3. *Let α be an n-chord diagram of type (k_1, k_2, k_3). If either $k_1 = 0$, $k_3 > 0$ or $k_2 = 0$, $k_3 > 0$, then $\alpha = 0$ in $\mathcal{A}_n$.*

Proof. Suppose that $k_1 = 0$, $k_3 > 0$. Using the relation (VE), we have:

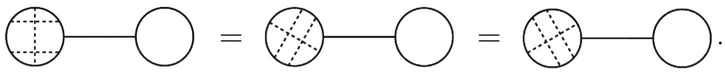
$\alpha =$  (5.1)

Since the last two chord diagrams are the same, we have $\alpha = 0$ in $\mathcal{A}_n$. □

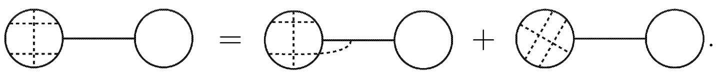
For an order n chord diagram α of type $(2n, 0, 0)$ (resp. $(0, 2n, 0)$), we denote by $R_1(\alpha)$ (resp. $R_2(\alpha)$) the chord diagram for a circle obtained from α by deleting the edges e_2 and e_3 (resp. e_1 and e_3).

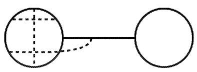
Lemma 5.4. *Let $\alpha, \alpha' \in \mathcal{D}_n$ be of type $(2n, 0, 0)$ (resp. $(0, 2n, 0)$). If $R_1(\alpha) = R_1(\alpha')$ (resp. $R_2(\alpha) = R_2(\alpha')$), then $\alpha = \alpha'$ in $\mathcal{A}_n$.*

Proof. For example, this lemma claims that the following equality holds:

 (5.2)

Let us prove the first equality of Eq. (5.2). By the relation (VE) we have

 (5.3)

Since  $= 0$ by Lemma 5.3, we obtain the result. □

6. Chord diagrams of order $\leqslant 3$

In this section, we give a spanning set for each of $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$.

There is only one admissible chord diagram of order one α_1 as shown in Fig. 8, and so $\mathcal{A}_1$ is spanned by α_1.

Lemma 6.1. *The space $\mathcal{A}_2$ is spanned by the chord diagrams β_1, β_2, β_3 as shown in Fig.* 9.

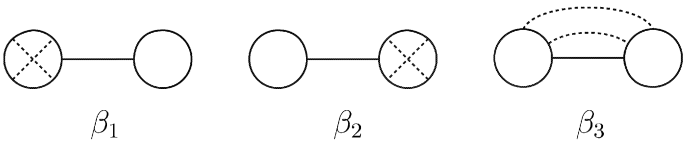


Fig. 9. Spanning set for $\mathcal{A}_2$.

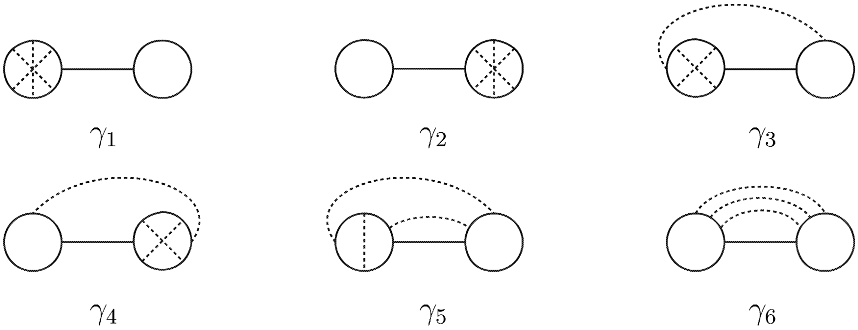


Fig. 10. Spanning set for $\mathcal{A}_3$.

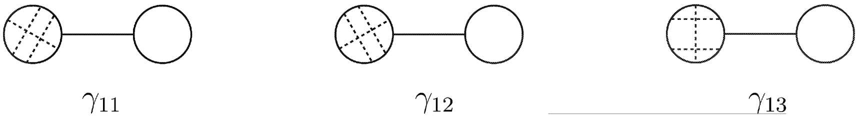
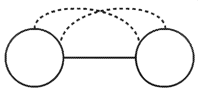
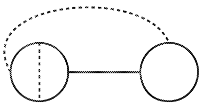


Fig. 11. Admissible chord diagrams of type $(6, 0, 0)$.

Proof. By Lemma 5.2 the chord diagram of type (i, j, k) with $k > 0$ is a linear combination of the chord diagrams of type $(i', j', 0)$ in $\mathcal{A}_2$. So $\mathcal{A}_2$ is spanned by the chord diagrams β_1, β_2, β_3 together with $\beta_4 =$ . However, by the relations (3T) and (VE) we have:

$\beta_4 = \beta_3 +$  $= \beta_3,$ (6.1)

completing the proof. □

Lemma 6.2. *The space $\mathcal{A}_3$ is spanned by the chord diagrams γ_i, $i = 1, \ldots, 6$, as shown in Fig.* 10.

The proof of Lemma 6.2 is divided into Sublemmas 6.3–6.8.

Sublemma 6.3. *In addition to γ_1, there are* 3 *admissible chord diagrams of type* $(6, 0, 0)$, γ_{11}, γ_{12}, γ_{13} *as shown in Fig.* 11, *which satisfy in $\mathcal{A}_3$*:

$$\gamma_{11} = \gamma_{12} = \gamma_{13}; \tag{6.2}$$
$$\gamma_1 = 2\gamma_{13}. \tag{6.3}$$

Proof. Eq. (6.2) follows from Lemma 5.4. Using the relation (3T), we have

$$\gamma_1 = \gamma_{11} + \gamma_{12} = 2\gamma_{13}, \tag{6.4}$$

completing the proof. □

Similarly, we have:

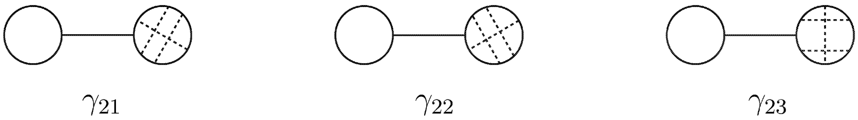


Fig. 12. Admissible chord diagrams of type $(0, 6, 0)$.

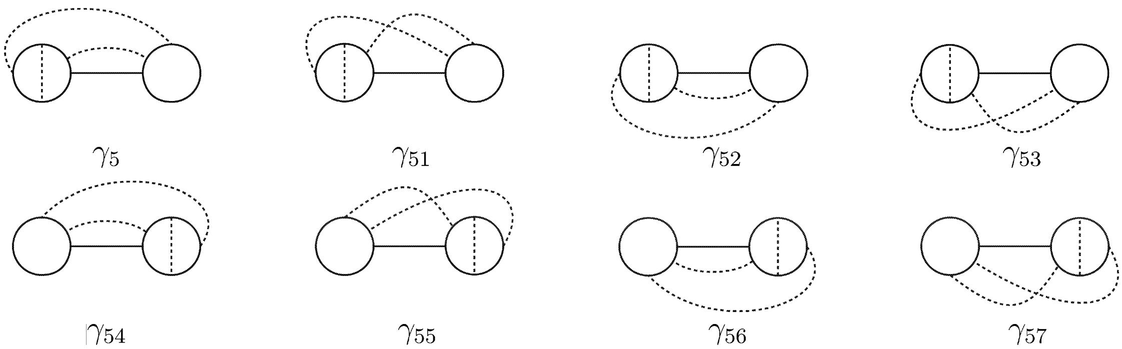


Fig. 13. Admissible chord diagrams of type $(4, 2, 0)$ or $(2, 4, 0)$.

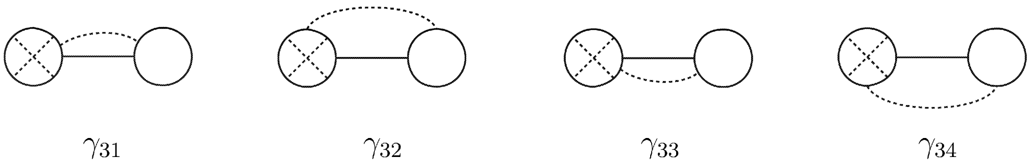


Fig. 14. Admissible chord diagrams of type $(5, 1, 0)$.

Sublemma 6.4. *In addition to* γ_2*, there are* 3 *admissible chord diagrams of type* $(0, 6, 0)$, γ_{21}, γ_{22}, γ_{23} *as shown in Fig.* 12*, which satisfy in* $\mathcal{A}_3$:

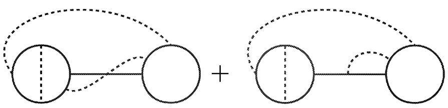
$$\gamma_{21} = \gamma_{22} = \gamma_{23}; \tag{6.5}$$

$$\gamma_2 = 2\gamma_{23}. \tag{6.6}$$

Sublemma 6.5. *In addition to* γ_5*, there are* 7 *admissible chord diagrams of type* $(4, 2, 0)$ *or type* $(2, 4, 0)$, γ_{5i}, $i = 1, \dots, 7$*, as shown in Fig.* 13*, which satisfy in* $\mathcal{A}_3$:

$$\gamma_{5i} = \gamma_5 \quad (i = 1, \dots, 7). \tag{6.7}$$

Proof. Using the relations (VE) and (RV), we have

$\gamma_5 =$  . (6.8)

Similarly, we can prove $\gamma_5 = \gamma_{51} = \gamma_{52}$. Next, using the relation (3T), we have $\gamma_5 = \gamma_{54}$. Similarly, we can prove $\gamma_{51} = \gamma_{57}$, $\gamma_{52} = \gamma_{56}$, and $\gamma_{53} = \gamma_{55}$. This completes the proof. □

Sublemma 6.6. *In addition to* γ_3*, there are* 4 *admissible chord diagrams of type* $(5, 1, 0)$, γ_{3i}, $i = 1, \dots, 4$*, as shown in Fig.* 14*, which satisfy in* $\mathcal{A}_3$:

$$\gamma_{3i} = \gamma_3 \quad (i = 1, \dots, 4). \tag{6.9}$$

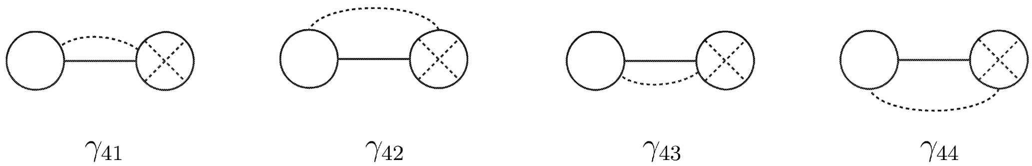


Fig. 15. Admissible chord diagrams of type $(1, 5, 0)$.

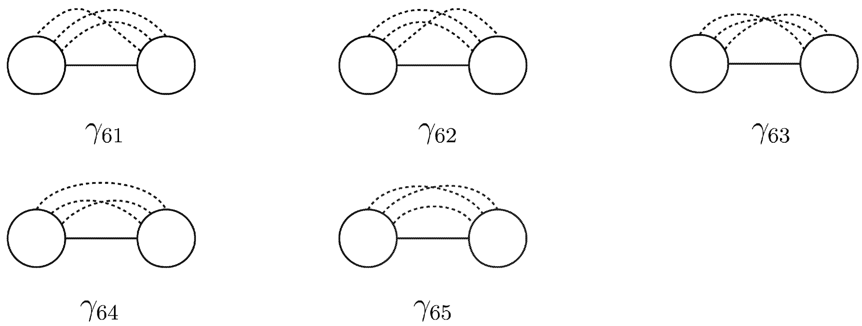
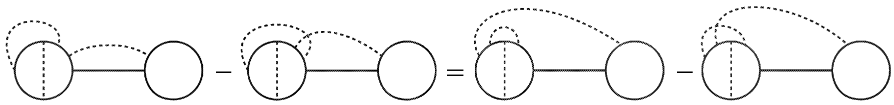


Fig. 16. Admissible chord diagrams of type $(3, 3, 0)$.

Proof. Using the relation (4T), we have

$\gamma_5 - \gamma_{51} =$ 

$$= \gamma_{31} - \gamma_{32} = \gamma_{34} - \gamma_3. \tag{6.10}$$

Since $\gamma_5 = \gamma_{51}$ by Sublemma 6.5, we obtain $\gamma_{31} = \gamma_{32}$, $\gamma_{34} = \gamma_3$. Similarly, we obtain $\gamma_{33} = \gamma_{34}$, $\gamma_{32} = \gamma_3$. This completes the proof. □

Similarly, we have:

Sublemma 6.7. *In addition to* γ_4, *there are* 4 *admissible chord diagrams of type* $(1, 5, 0)$, γ_{4i}, $i = 1, \dots, 4$, *as shown in Fig.* 15, *which satisfy in* $\mathcal{A}_3$:

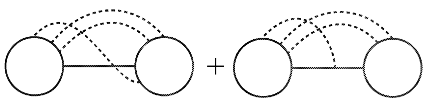
$$\gamma_{4i} = \gamma_4 \quad (i = 1, \dots, 4). \tag{6.11}$$

Sublemma 6.8. *In addition to* γ_6, *there are* 5 *admissible chord diagrams of type* $(3, 3, 0)$, γ_{6i}, $i = 1, \dots, 6$, *as shown in Fig.* 16, *which satisfy in* $\mathcal{A}_3$:

$$\gamma_{61} = \gamma_{62} = \gamma_6; \tag{6.12}$$

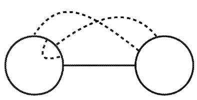
$$\gamma_{63} = \gamma_{64} = \gamma_{65} = \gamma_5 + \gamma_6. \tag{6.13}$$

Proof. By the relation (VE) and (RV), we have

$\gamma_{61} =$  $= \gamma_6.$ (6.14)

Similarly, we have $\gamma_{62} = \gamma_6$, obtaining Eq. (6.12).

Next, by the relation (3T), we have

$\gamma_{63} = \gamma_{61} +$  $= \gamma_{61} + \gamma_{53} = \gamma_6 + \gamma_5,$ (6.15)

where we use Sublemmas 6.5 and 6.8. Similarly, we obtain $\gamma_{64} = \gamma_{65} = \gamma_6 + \gamma_5$. This completes the proof. □

Proof of Lemma 6.2. By Lemma 5.2, $\mathcal{A}_3$ is spanned by the chord diagrams of type $(k, 6 - k, 0)$ with $k = 0, 1, \dots, 6$. Then by Sublemmas 6.3–6.8 the result follows. □

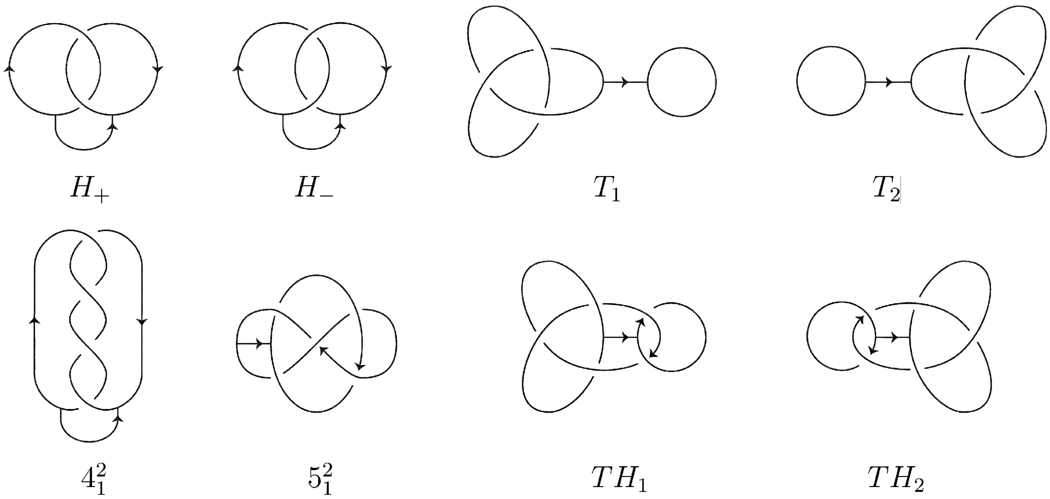


Fig. 17. Spatial handcuff graphs.

Table 1
Values of finite type invariants.

Handcuff graphs	U	H_+	H_-	T_1	T_2	$T_1!$	$T_2!$	4_1^2	5_1^2	TH_1	TH_2
λ	0	1	−1	0	0	0	0	2	0	1	1
$a_2[1]$	0	0	0	1	0	1	0	0	0	1	0
$a_2[2]$	0	0	0	0	1	0	1	0	0	0	1
λ^2	0	1	1	0	0	0	0	4	0	1	1
$V^{(3)}[1]/18$	0	0	0	−1	0	3	0	0	0	−1	0
$V^{(3)}[2]/18$	0	0	0	0	−1	0	3	0	0	0	−1
λ^3	0	1	−1	0	0	0	0	8	0	1	1
a_3	0	0	0	0	0	0	0	0	−1	1	1
$\lambda a_2[1]$	0	0	0	0	0	0	0	0	0	1	0
$\lambda a_2[2]$	0	0	0	0	0	0	0	0	0	0	1

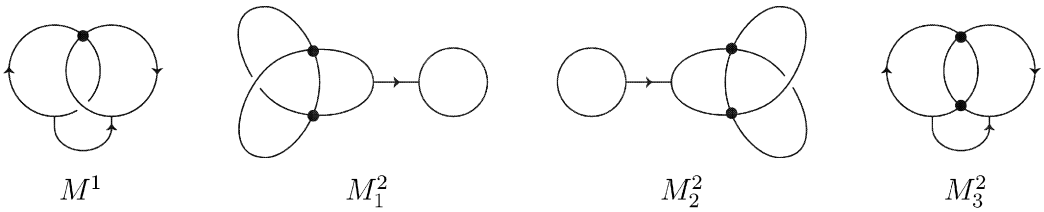


Fig. 18. Singular handcuff graphs of order 1 or 2.

7. Finite type invariants of order $\leqslant 2$

First, we give a table of spatial handcuff graphs in Fig. 17 (cf. [21,22]), and their invariants in Table 1, where $T_1!$, $T_2!$ are mirror images of T_1, T_2, respectively. We will use them in Sections 7 and 8.

Let M^1, M_i^2 $(i = 1, 2, 3)$ be the singular spatial handcuff graphs as shown in Fig. 18 respecting the order one or two chord diagrams α_1, β_i, respectively.

Using the Vassiliev skein relation (3.1), we have:

$$\begin{aligned}&[\, v(M^1) \quad v(M_1^2) \quad v(M_2^2) \quad v(M_3^2)\,]\\ &\quad = [\, v(U) \quad v(H_+) \quad v(H_-) \quad v(T_1) \quad v(T_2)\,] \begin{bmatrix} -1 & -1 & -1 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\end{aligned} \tag{7.1}$$

where H_+, H_-, T_1, T_2 are spatial handcuff graphs as shown in Fig. 17.

Theorem 7.1.

(i) *Let v be a finite type invariant of order $\leqslant 1$ for a spatial handcuff graph. Then*

$$v(\Phi) = A + B\lambda(\Phi), \tag{7.2}$$

where $A = v(U)$ and $B = v(M^1)$.

(ii) *The space $\mathcal{V}_1/\mathcal{V}_0$ has a basis $\{\lambda\}$.*

Proof. Since there is only one admissible chord diagram of order one, α_1 (Fig. 8), by Proposition 4.3, we have

$$v(\Phi) = v(U) + pv\bigl(M^1\bigr) \tag{7.3}$$

for some integer p. Then by Eq. (7.1), we have

$$v(\Phi) = (1-p)v(U) + pv(H_+), \tag{7.4}$$

and so we have

$$\lambda(\Phi) = (1-p)\lambda(U) + p\lambda(H_+) = p. \tag{7.5}$$

This completes the proof. □

Theorem 7.2.

(i) *Let v be a finite type invariant of order $\leqslant 2$ for a spatial handcuff graph. Then*

$$v(\Phi) = A + B\lambda(\Phi) + \sum_{i=1,2} C_i a_2[i](\Phi) + C_3\lambda(\Phi)^2, \tag{7.6}$$

where

$$A = v(U), \qquad B = v\bigl(M^1\bigr) - \frac{1}{2}v\bigl(M_2^2\bigr), \qquad C_i = v\bigl(M_i^2\bigr) \quad (i = 1, 2), \qquad C_3 = \frac{1}{2}v\bigl(M_3^2\bigr).$$

(ii) *The space $\mathcal{V}_2/\mathcal{V}_1$ has a basis $\{a_2[1], a_2[2], \lambda^2\}$.*

Proof. From Proposition 4.3 and Lemma 6.1, we have

$$v(\Phi) = v(U) + \begin{bmatrix} v(M^1) & v(M_1^2) & v(M_2^2) & v(M_3^2) \end{bmatrix} \begin{bmatrix} p \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{7.7}$$

for some integers p, q_i. Then

$$\begin{bmatrix} \lambda(\Phi) \\ a_2[1](\Phi) \\ a_2[2](\Phi) \\ \lambda(\Phi)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} p \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \tag{7.8}$$

and so we have

$$v(\Phi) = v(U) + \begin{bmatrix} v(M^1) & v(M_1^2) & v(M_2^2) & v(M_3^2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \lambda(\Phi) \\ a_2[1](\Phi) \\ a_2[2](\Phi) \\ \lambda(\Phi)^2 \end{bmatrix}, \tag{7.9}$$

obtaining (i). Since Eq. (7.8) assures that $a_2[1]$, $a_2[2]$, λ^2 are linearly independent in $\mathcal{V}_2/\mathcal{V}_1$, we obtain (ii). □

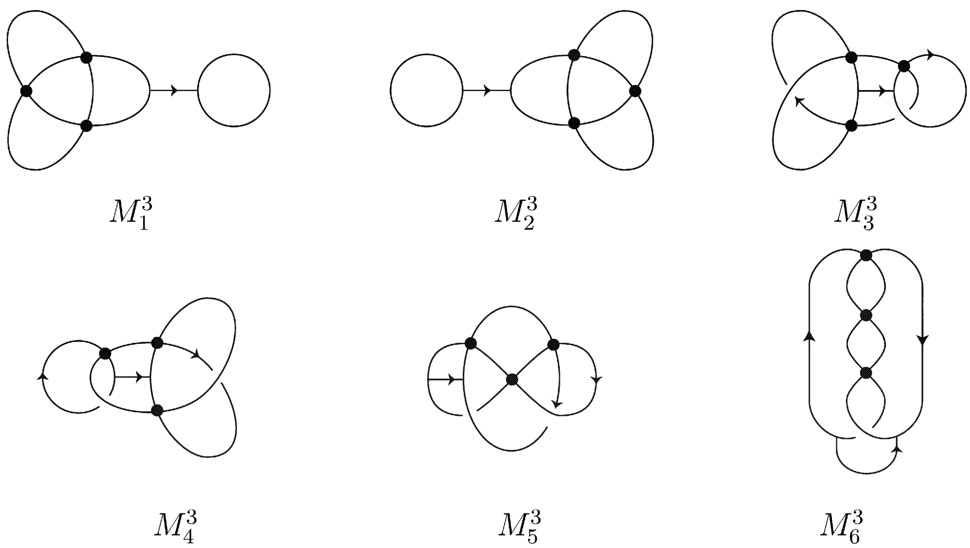


Fig. 19. Singular handcuff graphs of order 3.

8. Finite type invariants of order $\leqslant 3$

Let M_i^3 $(i = 1, \ldots, 6)$ be the singular spatial handcuff graph as shown in Fig. 19 respecting the chord diagram γ_i of order 3.

Using the Vassiliev skein relation (3.1), we have:

$$[\, v(M_1^3) \quad v(M_2^3) \quad v(M_3^3) \quad v(M_4^3) \quad v(M_5^3) \quad v(M_6^3)\,] = \boldsymbol{h}Z, \tag{8.1}$$

where

$$\boldsymbol{h} = \begin{bmatrix} v(U) \\ v(H_+) \\ v(H_-) \\ v(T_1) \\ v(T_2) \\ v(T_1!) \\ v(T_2!) \\ v(4_1^2) \\ v(5_1^2) \\ v(TH_1) \\ v(TH_2) \end{bmatrix}^T, \qquad Z = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{8.2}$$

with H_+, H_-, T_1, T_2, 4_1^2, 5_1^2, TH_1, TH_2 being spatial handcuff graphs as shown in Fig. 17. Here X^T means the transpose of X.

Theorem 8.1.

(i) *Let v be a finite type invariant of order $\leqslant 3$ for a spatial handcuff graph. Then*

$$\begin{aligned} v(\Phi) = {} & A + B\lambda(\Phi) + \sum_{i=1,2} C_i a_2[i](\Phi) + C_3\lambda(\Phi)^2 \\ & + \sum_{i=1,2} D_i V^{(3)}[i](\Phi)/18 + D_3\lambda(\Phi)^3 + D_4 a_3(\Phi) + \sum_{i=5,6} D_i\lambda(\Phi)a_2[i-4](\Phi), \end{aligned} \tag{8.3}$$

where

$$A = v(U); \tag{8.4}$$

$$B = v\big(M^1\big) - \frac{1}{2}v\big(M_2^2\big) - \frac{1}{6}v\big(M_6^3\big); \tag{8.5}$$

$$C_i = v\left(M_i^2\right) - \frac{1}{4} v\left(M_1^3\right) \quad (i = 1, 2); \tag{8.6}$$

$$C_3 = \frac{1}{2} v\left(M_3^2\right); \tag{8.7}$$

$$D_i = -\frac{1}{4} v\left(M_1^3\right) \quad (i = 1, 2); \tag{8.8}$$

$$D_3 = -\frac{1}{6} v\left(M_6^3\right); \tag{8.9}$$

$$D_4 = v\left(M_5^3\right); \tag{8.10}$$

$$D_i = -\frac{1}{4} v\left(M_{i-2}^3\right) - v\left(M_5^3\right) \quad (i = 5, 6). \tag{8.11}$$

(ii) *The space $\mathcal{V}_3/\mathcal{V}_2$ has a basis $\{V^{(3)}[1], V^{(3)}[2], \lambda^3, a_3, \lambda a_2[1], \lambda a_2[2]\}$.*

Proof. From Proposition 4.3 and Lemmas 6.1 and 6.2, we have

$$v(\Phi) = v(U) + \boldsymbol{m}\boldsymbol{x}, \tag{8.12}$$

where

$$\boldsymbol{m} = [\, v(M_1) \quad v(M_1^2) \quad v(M_2^2) \quad v(M_3^2) \quad v(M_1^3) \quad \cdots \quad v(M_6^3) \,], \tag{8.13}$$

$$\boldsymbol{x} = [\, p \quad q_1 \quad q_2 \quad q_3 \quad r_1 \quad \cdots \quad r_6 \,]^T \tag{8.14}$$

with p, q_i, r_j rational numbers; notice Eqs. (6.3) and (6.6). Then using Table 1 and Eq. (8.2), we have:

$$I = Y\boldsymbol{x}, \tag{8.15}$$

where

$$I = \begin{bmatrix} \lambda(\Phi) \\ a_2[1](\Phi) \\ a_2[2](\Phi) \\ \lambda(\Phi)^2 \\ V^{(3)}[1](\Phi)/18 \\ V^{(3)}[2](\Phi)/18 \\ \lambda(\Phi)^3 \\ a_3(\Phi) \\ \lambda a_2[1](\Phi) \\ \lambda a_2[2](\Phi) \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{8.16}$$

and so we have

$$v(\Phi) = v(U) + \boldsymbol{m}Y^{-1}I, \tag{8.17}$$

with

$$Y^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/4 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ -1/6 & 0 & 0 & 0 & 0 & 0 & -1/6 & 0 & 0 & 0 \end{bmatrix}, \tag{8.18}$$

obtaining (i). Since Eqs. (8.15) and (8.16) assure that $V^{(3)}[1]$, $V^{(3)}[2]$, λ^3, a_3, $\lambda a_2[1]$, $\lambda a_2[2]$ are linearly independent in $\mathcal{V}_3/\mathcal{V}_2$, we obtain (ii). $\square$

Theorems 7.1, 7.2, 8.1 imply the following, which is the main theorem of [29]:

Corollary 8.2. *The spaces $\mathcal{A}_1$, $\mathcal{A}_2$,$\mathcal{A}_3$ have the bases as follows*:

$$\mathcal{A}_1 = \langle \alpha_1 \rangle; \tag{8.19}$$

$$\mathcal{A}_2 = \langle \beta_1, \beta_2, \beta_3 \rangle; \tag{8.20}$$

$$\mathcal{A}_3 = \langle \gamma_1, \dots, \gamma_6 \rangle. \tag{8.21}$$

Given a spatial handcuff graph Φ, let $\mathrm{Dt}(\Phi)$ be the 2-component link obtained from Φ by deleting the connecting edge e_3. Then Dt is a morphism from the object of spatial handcuff graphs to the object of ordered 2-component oriented links; see Example 2.4 in [27]. We define a linear map $\Delta_n : \mathcal{V}_n(S^1 \sqcup S^1) \to \mathcal{V}_n$ by $\Delta_n(v) = v \circ \mathrm{Dt}$, where $\mathcal{V}_n(S^1 \sqcup S^1)$ is the space of finite type invariant of order $\leqslant n$ for an ordered 2-component oriented link. Then from Theorem 5.2 in [14] we have the following corollary; see also Theorem 3.1 in [12]. Note that $V^{(3)}(K;1) = (3/4)P_0^{(3)}(K;1)$ for a knot K, where $P_0^{(3)}(K;1)$ is the third derivative of the 0th coefficient polynomial of the HOMFLYPT polynomial of K at $t=1$; see [20], [13, (5.9)].

Corollary 8.3. *If $n \leqslant 3$, then Δ_n is an isomorphism.*

This means that a finite type invariant of order $\leqslant 3$ for a spatial handcuff graph is determined by the link consisting of the two loops, and is not affected by the connecting edge e_3. So we are interested in the following question.

Question 8.4. Is Δ_n an isomorphism for $n \geqslant 4$?

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