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Applied Mathematics Letters 13 (2000) 27–30

Applied Mathematics Letters

www.elsevier.nl/locate/aml

The Oscillation of Perturbed Functional Differential Equations

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(Received October 1999; accepted November 1999)

Abstract—We provide new oscillation criteria for the perturbed functional differential equations. This solves some open problems of [1]. An application to an equation arising in nonlinear neural networks is illustrated.

Keywords—Oscillation, Perturbed equation, Nonlinear neural networks.

1. INTRODUCTION

In this paper, we shall study the oscillatory behavior of perturbed functional differential equations of the form

$$\delta x^{(n)}(t) + H(t, x[g(t)]) = P(t, x[g(t)]), \tag{E,δ}$$

where $n \geq 1$, $\delta = \pm 1$, $g : [t_0, \infty) \to \mathbb{R} = (-\infty, \infty)$, $H, P : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous, $t_0 \geq 0$, and $\lim_{t\to\infty} g(t) = \infty$.

We shall assume that there exist continuous functions $a, p, q : [t_0, \infty) \to [0, \infty)$ and positive constants λ and μ, $\gamma := \mu - \lambda > 1$ such that

$$H(t, x) \operatorname{sgn} x \leq a(t)|x|^{\lambda+1}, \qquad \text{for } x \neq 0, \quad t \geq t_0, \tag{1}$$

$$P(t, x) \operatorname{sgn} x \geq p(t)|x|^{\mu} + q(t)|x|^{\lambda}, \qquad \text{for } x \neq 0, \quad t \geq t_0, \tag{2}$$

and

$$Q(t) := q(t) - \beta a^{\gamma/(\gamma-1)}(t) p^{1/(1-\gamma)}(t) \geq 0, \qquad \text{for } t \geq t_0, \tag{3}$$

and $Q(t) \not\equiv 0$ on any ray of the form $[t^*, \infty)$ for some $t^* \geq t_0$, where $\beta = (\gamma - 1)\gamma^{\gamma/(1-\gamma)}$.

As usual, a nontrivial solution of equation (E,δ) is called oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Equation (E,δ) is called oscillatory if all of its solutions are oscillatory.

 Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0893-9659(00)00091-4

The oscillatory behavior of equation (E,1) with n even and $g(t) = t$ has been introduced and discussed by Kartsatos [1–3]. In [1], Kartsatos raised some open problems regarding the oscillation of equation (E,δ) without assuming either

$$\lim_{t\to\infty} \frac{H(t,u[g(t)])}{P(t,u[g(t)])} = 0 \tag{$*$}$$

or

$$\lim_{t\to\infty} \sup_{|u|\le K} \frac{H(t,u)}{P(t,u)} = 0 \tag{$**$}$$

(see Problems IX and X).

The purpose of this paper is to provide sufficient conditions for the oscillation of equation (E,δ) without necessarily requiring assumptions (*) or (**).

2. MAIN RESULTS

We need the following lemma.

LEMMA 1. *(See [4].) If A and B are nonnegative, then*

$$A^\lambda - \lambda A B^{\lambda-1} + (\lambda - 1)B^\lambda \ge 0, \qquad \lambda > 1,$$

and equality holds if and only if $A = B$.

THEOREM 1. *Let n be even, and conditions (1)–(3) hold. If the equation*

$$x^{(n)}(t) + Q(t)|x[\sigma(t)]|^\lambda \operatorname{sgn} x[\sigma(t)] = 0 \tag{4}$$

is oscillatory, where $\sigma(t) = \min\{t, g(t)\}$ and $\lim_{t\to\infty} \sigma(t) = \infty$, then equation (E,$-1$) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (E,-1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \ge t_0$. Using conditions (1) and (3) in equation (E,-1), we have

$$\begin{aligned} 0 &\ge x^{(n)}(t) - a(t)x^{\lambda+1}[g(t)] + p(t)x^\mu[g(t)] + q(t)x^\lambda[g(t)] \\ &= x^{(n)}(t) + q(t)x^\lambda[g(t)] - x^\lambda[g(t)]\left[a(t)x[g(t)] - p(t)x^\gamma[g(t)]\right], \qquad \text{for } t \ge t_0. \end{aligned} \tag{5}$$

Now we set

$$A = p^{1/\gamma}(t)x[g(t)] \qquad \text{and} \qquad B = \left(\frac{a(t)}{\gamma} p^{-1/\gamma}(t)\right)^{1/(\gamma-1)}$$

and apply Lemma 1, to get

$$0 \ge x^{(n)}(t) + q(t)x^\lambda[g(t)] - \beta a^{\gamma/(\gamma-1)}(t)p^{1/(1-\gamma)}(t)x^\lambda[g(t)]$$

or

$$x^{(n)}(t) + Q(t)x^\lambda[g(t)] \le 0, \qquad t \ge T_1 \ge t_0. \tag{6}$$

Since n is even, we see that $x(t)$ is an increasing function for $t \ge T_1$ and (6) reduces to

$$x^{(n)}(t) + Q(t)x^\lambda[\sigma(t)] \le 0, \qquad \text{for } t \ge T \ge T_1.$$

But, this in view of a result of [5] leads to a contradiction. ∎

REMARK 1. For $a(t) = p(t) = q(t)$, condition (3) holds for all $\lambda > 0$ and $\mu > \lambda + 1$. However, for this case conditions of type (*) and (**) are not valid.

In fact, this solves the open problems IX and X in [1] for equation (E,-1) when conditions (1)–(3) hold.

REMARK 2. From the proof of Theorem 1, we see that equation (E,-1) under assumptions (1)–(3) is reduced to an inequality of type (6). Now for any $n \ge 1$ and any $\lambda > 0$, one can apply the results of [6] to this inequality and obtain complete oscillation criteria for equation (E,-1), or make use of comparison results of [7] and compare the oscillatory and asymptotic behavior of equations of type (4) to that of (E,-1).

Next, we present the following oscillation criterion for equation (E,1), $\lambda = 1$ and n is odd. The other cases for any $n \ge 1$ and $\lambda > 0$ can be obtained similarly.

THEOREM 2. *Let $\lambda = 1$, n is odd, and conditions (1)–(3) hold. If*

$$\int^{\infty} s\sigma^{n-2}(s)g^{-\epsilon}(s)Q(s)\,ds = \infty, \qquad \text{for some } \epsilon > 0, \tag{7}$$

$$\int^{\infty} g^{n-1}(s)Q(s)\,ds = \infty, \tag{8}$$

and

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \frac{(g(s)-\rho(t))^j}{j!}\,\frac{(\rho(t)-s)^{n-j-1}}{(n-j-1)!}\,Q(s)\,ds > 1 \tag{9}$$

for some $j = 0, 1, \dots, n-1$, where $\sigma(t) = \min\{t, g(t)\} \to \infty$ as $t \to \infty$ and $\rho(t) = \min\{\max\{s, g(s)\} : s \geq t\}$, then equation (E,1) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (E,1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \geq t_0$. Using conditions (1) and (2) in equation (E,1), we obtain

$$\begin{aligned} 0 &\leq x^{(n)}(t) + a(t)x^{\lambda+1}[g(t)] - p(t)x^{\mu}[g(t)] - q(t)x^{\lambda}[g(t)] \\ &= x^{(n)}(t) - q(t)x^{\lambda}[g(t)] + x^{\lambda}[g(t)]\left[a(t)x[g(t)] - q(t)x^{\gamma}[g(t)]\right], \qquad \text{for } t \geq t_0. \end{aligned}$$

Now we let A and B be as in the proof of Theorem 1 and apply Lemma 1, to get

$$x^{(n)}(t) - Q(t)x^{\lambda}[g(t)] \geq 0, \qquad \text{for } t \geq T \geq t_0. \tag{10}$$

The rest of the proof follows by applying a result of [6]. ∎

As an application, we consider the following equation which arises in the study of nonlinear neural networks:

$$\begin{aligned} \delta\frac{dx(t)}{dt} &= -q(t)|x[g(t)]|^{\lambda}\operatorname{sgn} x[g(t)] + a(t)|\tanh x[g(t)]|^{\lambda}\tanh x[g(t)] \\ &\quad - p(t)|x[g(t)]|^{\mu}\operatorname{sgn} x[g(t)], \end{aligned} \tag{N,δ}$$

where $\delta = \pm 1$, λ and μ are real constants, $\lambda > 0$ and $\mu > \lambda + 1$, the functions a, g, p, and q are as in equation (E,δ) and conditions (1) and (2) hold. As in the proof of Theorems 1 and 2, one can easily see that equations (N,1) and (N,-1) are reduced respectively to the following inequalities:

$$\left\{\frac{dx(t)}{dt} + Q(t)|x[g(t)]|^{\lambda}\right\}\operatorname{sgn} x[g(t)] \leq 0 \tag{11}$$

and

$$\left\{\frac{dx(t)}{dt} - Q(t)|x[g(t)]|^{\lambda}\right\}\operatorname{sgn} x[g(t)] \geq 0, \tag{12}$$

where $Q(t)$ is defined in (3).

Now, it is easy to see that (11) is oscillatory if one of the following conditions holds:

(I) $\lambda = 1$, $g(t) \leq t$ and $g'(t) \geq 0$ for $t \geq t_0$, and

$$\liminf_{t\to\infty} \int_{g(t)}^{t} Q(s)\,ds > \frac{1}{e},$$

(II) $0 < \lambda < 1$ and

$$\int_{R_g} Q(s)\,ds = \infty,$$

where $R_g = \{t \in [t_0, \infty) : t_0 \leq g(t) \leq t\}$.

Also, we see that (12) is oscillatory if one of the following conditions holds:

(III) $\lambda = 1$, $g(t) \geq t$, and $g'(t) \geq 0$ for $t \geq t_0$, and

$$\liminf_{t\to\infty} \int_t^{g(t)} Q(s)\,ds > 1,$$

(IV) $\lambda > 1$ and

$$\int_{A_g} Q(s)\,ds = \infty,$$

where $A_g = \{t \in [t_0, \infty) : g(t) \geq t\}$.

Thus, the oscillation of the equation (N,δ), $\delta = \pm 1$ follows from those for inequalities (11) and (12).

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