Oscillation of second-order nonlinear neutral delay dynamic equations on time scales

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Received 22 June 2004; received in revised form 4 December 2004

Abstract

In this paper, some sufficient conditions for oscillation of the second-order nonlinear neutral delay dynamic equation

\[ (r(t)([y(t) + p(t)y(t - \tau)]^\gamma)^{\frac{1}{\gamma}} + f(t, y(t - \delta))) = 0, \]

on a time scale \( \mathbb{T} \) are established; here \( \gamma \geq 1 \) is an odd positive integer with \( r(t) \) and \( p(t) \) are rd-continuous functions defined on \( \mathbb{T} \). Our results as a special case when \( \mathbb{T} = \mathbb{R} \) and \( \mathbb{T} = \mathbb{N} \), involve and improve some well-known oscillation results for second-order neutral delay differential and difference equations. When \( \mathbb{T} = h\mathbb{N} \) and \( \mathbb{T} = q^\mathbb{N} = \{ t : t = q^k, k \in \mathbb{N}, q > 1 \} \), i.e., for generalized neutral delay differential and \( q \)-neutral delay difference equations our results are essentially new and also can be applied on different types of time scales, e.g., \( \mathbb{T} = \mathbb{N}^2 = \{ t^2 : t \in \mathbb{N} \} \) and \( \mathbb{T} = \mathbb{T}_n = \{ t_n : n \in \mathbb{N}_0 \} \) where \( \{ t_n \} \) is the set of harmonic numbers. Some examples illustrating our main results are given.

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MSC: 34K11; 39A10; 39A99 (34A99, 34C10, 39A11)

Keywords: Oscillation; Second-order neutral nonlinear dynamic equation; Time scales

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1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. Thesis in 1988 [18] in order to unify continuous and discrete analysis (see [18]). Not only can this theory of the so-called “dynamic equations” unify the theories of differential equations and difference equations but also it is able to extend these classical cases to cases “in between”, e.g., to the so-called $q$-difference equations. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [5]).

Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors has expounded on various aspects of this new theory, see the paper by Agarwal et al. [1] and the references cited therein. The books on the subject of time scale, i.e., measure chain, by Bohner and Peterson [5,6] summarize and organize much of time scale calculus, and in the next section, we recall some of the main tools used in the subsequent sections of this paper.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of different types of dynamic equations on time scales. We refer the reader to the papers [2–4,7–14,21–23] and the reference cited therein.

In this paper, we are concerned with oscillation of solutions of the second-order nonlinear neutral delay dynamic equation

$$
(r(t)[y(t) + p(t)y(t-\tau)]^d)' + f(t, y(t-\delta)) = 0,
$$

(1.1)
on a time scale $\mathbb{T}$.

Since we are interested in asymptotic behavior of solutions we will suppose that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)$.

Throughout this paper we assume that: $\gamma \geq 1$ is an odd positive integer, $\tau$, $\delta$ are positive constants such that the delay functions $\tau(t) := t - \tau < t$ and $\delta(t) := t - \delta < t$ satisfy $\tau(t): \mathbb{T} \to \mathbb{T}$ and $\delta(t): \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, $r(t)$ and $p(t)$ are real valued rd-continuous positive functions defined on $\mathbb{T}$,

\begin{align*}
\text{(h1)} & \quad \int_{t_0}^{\infty} \frac{1}{r(t)} \Delta t = \infty, \quad 0 \leq p(t) < 1, \\
\text{(h2)} & \quad f(t, u): \mathbb{T} \times \mathbb{R} \to \mathbb{R} \text{ is continuous function such that } uf(t, u) > 0 \text{ for all } u \neq 0 \text{ and there exists a nonnegative function } q(t) \text{ defined on } \mathbb{T} \text{ such that } |f(t, u)| \geq q(t)|u|^\gamma.
\end{align*}

By a solution of Eq. (1.1), we mean a nontrivial real-valued function $y(t)$ which has the properties $y(t) + p(t)y(t-\tau) \in C^1_{rd}[t_0, \infty)$ and $r(t)[y(t) + p(t)y(t-\tau)]^\gamma \in C^1_{rd}[t_0, \infty)$, $t_0 \geq t_0$ and satisfying (1.1) for all $t \geq t_0$. Our attention is restricted to those solutions of Eq. (1.1) which exist on some half line $[t_0, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1| > 0$ for any $t_1 \geq t_0$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

We note that if $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^{(d)}(t) = f'(t)$ and (1.1) becomes the second-order neutral delay differential equation

$$
(r(t)[y(t) + p(t)y(t-\tau)]')' + f(t, y(t-\delta)) = 0, \quad t \in \mathbb{T}.
$$

(1.2)
If $\mathbb{T} = \mathbb{N}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $y^A(t) = \Delta y(t) = y(t + 1) - y(t)$ and (1.1) becomes the second-order neutral delay difference equation

$$\Delta(r(t)(\Delta[y(t) + p(t)y(t - \tau)])^2) + f(t, y(t - \delta)) = 0, \quad t \in \mathbb{T}. \quad (1.3)$$

If $\mathbb{T} = h\mathbb{N}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$,

$$y^A(t) = \Delta_h y(t) = \frac{y(t + h) - y(t)}{h},$$

and (1.1) becomes the second-order neutral delay difference equation

$$\Delta_h(r(t)(\Delta_h[y(t) + p(t)y(t - \tau)])^2) + f(t, y(t - \delta)) = 0, \quad t \in \mathbb{T}. \quad (1.4)$$

If $\mathbb{T} = q\mathbb{N} = \{t : t = q^n, n \in \mathbb{N}, q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q - 1)t$,

$$y^A(t) = \Delta_q y(t) = \frac{y(qt) - y(t)}{(q - 1)t},$$

and (1.1) becomes the second-order $q$-neutral delay difference equation

$$\Delta_q(r(t)(\Delta_q[y(t) + p(t)y(t - \tau)])^2) + f(t, y(t - \delta)) = 0, \quad t \in \mathbb{T}. \quad (1.5)$$

If $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, we have $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$,

$$y^A(t) = \Delta_N y(t) = \frac{y((\sqrt{t} + 1)^2) - y(t)}{1 + 2\sqrt{t}},$$

and (1.1) becomes the second-order neutral delay difference equation

$$\Delta_N(r(t)(\Delta_N[y(t) + p(t)y(t - \tau)])^2) + f(t, y(t - \delta)) = 0, \quad t \in \mathbb{T}. \quad (1.6)$$

If $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where $\{t_n\}$ is the set of harmonic numbers defined by

$$t_0 = 0, \quad t_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}_0,$$

we have $\sigma(t_n) = t_{n+1}$, $\mu(t_n) = \frac{1}{n+1}$, $y^A(t) = \Delta_{t_n} y(t_n) = (n + 1)\Delta y(t_n)$ and (1.1) becomes the second-order neutral delay difference equation

$$\Delta_{t_n}(r(t_n)(\Delta_{t_n}[y(t_n) + p(t_n)y(t_n - \tau)])^2) + f(t_n, y(t_n - \delta)) = 0, \quad t_n \in \mathbb{T}. \quad (1.7)$$

Numerous oscillation criteria have been established for second-order neutral delay differential and difference equations (1.2) and (1.3). See for examples [15,16,19,20,24] and the references cited therein.

As a special case of Eq. (1.2), Grammatikopoulos et al. [16] considered the second-order linear neutral delay equation

$$[y(t) + p(t)y(t - \tau)]'' + q(t)y(t - \delta) = 0, \quad t \geq t_0, \quad (1.8)$$
and proved that
If \( q(t) > 0 \), \( 0 \leq p(t) < 1 \) and
\[
\int_{t_0}^{\infty} q(s)[1 - p(s - \delta)] \, ds = \infty,
\]
(1.9)
then every solution of Eq. (1.8) oscillates.

Note that condition (1.9) cannot be applied to the second-order neutral delay equation
\[
[y(t) + p(t)y(t - \tau)]'' + \frac{\beta}{t^2} y(t - \delta) = 0, \quad t \geq t_0,
\]
(1.10)
where \( \beta > 0 \) and \( 0 \leq p(t) < 1 \).

Graef et al. [15] considered the second-order nonlinear neutral delay differential equation
\[
[y(t) + p(t)y(t - \tau)]'' + q(t)f(y(t - \delta)) = 0, \quad t \geq t_0,
\]
(1.11)
and extended condition (1.9) and proved that:
If \( q(t) > 0 \), \( 0 \leq p(t) < 1 \), \( f(u) \geq Ku \), and there exists a positive function \( \zeta(t) \) such that
\[
\lim_{t \to \infty} \sup \int_{t_0}^{t} \left(K\zeta(s)q(s)(1 - p(s - \delta)) - \frac{\zeta'(s)}{4\zeta(s)}\right) \, ds = \infty,
\]
(1.13)
then every solution of Eq. (1.11) oscillates.

Kubiaczyk and Saker [19], considered Eq. (1.11) and improved the results established by Grammatikopoulos et al. [16] and Graef et al. [15] and proved that:
If \( q(t) > 0 \), \( 0 \leq p(t) < 1 \), \( f(n, u) \geq K_u \), and there exists positive sequence \( \zeta(t) \) such that
\[
\lim_{t \to \infty} \sup \int_{t_0}^{t} \left(K\zeta(s)Q(s) - \frac{r(s - \delta)(\Delta\zeta(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1}\zeta'(s)}\right) \, ds = \infty,
\]
(1.14)
where
\[
Q(t) = q(t)(1 - p(t - \delta))^7.
\]
Then every solution of Eq. (1.3) oscillates.
Recently, Sun and Saker [24] considered the nonlinear neutral delay difference equation (1.3) and proved that:

If \( \gamma > 0 \) is a quotient of odd positive integers, \( 0 \leq p(t) < 1, |f(n, u)| \geq q(n)|u^\gamma| \),

\[
\sum_{t=t_0}^{\infty} \left( \frac{1}{r(t)} \right)^{1/\gamma} = \infty,
\]

and there exists positive sequences \( \phi(t) \) and \( \phi(t) \) such that

\[
\lim_{t \to \infty} \sup_{t-1} \sum_{s=t_0}^{t-1} \left[ x(s)\phi(s)Q(s) - \frac{r(s - \delta) (x^\sigma)^{\gamma + 1} A^{\gamma + 1}(s)}{(\gamma + 1)^{\gamma + 1} x^\sigma(s)\phi^\gamma(s)} \right] = \infty,
\] (1.15)

where

\[ Q(t) = q(t)(1 - p(t - \delta))^{\gamma}, \quad A(s) = \frac{\phi(s)(x^A(s))_+}{x^\sigma} + (\phi^A(s))_+, \]

\( (x^A(t))_+ = \max\{x^A(t), 0\} \) and \( (\phi^A(t))_+ = \max\{\phi^A(t), 0\} \). Then every solution of Eq. (1.3) oscillates.

Also in [20,24] the authors established some oscillation criteria of Kamenev and Philos types.

Our aim in this paper, is to establish some sufficient conditions for oscillation of Eq. (1.1) on time scales. Our results, as special case when \( T = \mathbb{R} \), involve the results established by Kubiaczyk and Saker [19] and also involve and improve the results established by Grammatikopoulos et al. [16] and Graef et al. [15] for second order nonlinear neutral delay differential equation (1.2). When \( T = \mathbb{N} \), our results involve the results of Saker [20] and Sun and Saker [24] when \( \gamma \geq 1 \) for second-order nonlinear delay difference equation (1.3). Our oscillation results for equations (1.4)–(1.7) are essentially new.

The paper is organized as follows: In the next section, we present some basic definitions concerning the calculus on time scales. In Section 3, we will use the Riccati transformation technique to prove our main oscillation results. In Section 4, we will apply our oscillation results for equations (1.2)–(1.7) to establish some sufficient conditions for oscillation. In Section 5, we present some examples to illustrate our main results.

### 2. Some preliminaries on time scales

A time scale \( T \) is an arbitrary nonempty closed subset of the real numbers \( \mathbb{R} \). On any time scale \( T \), we define the forward and backward jump operators by

\[
\sigma(t) := \inf\{s \in T : s > t\}, \quad \rho(t) := \sup\{s \in T, s < t\}.
\] (2.1)

A point \( t \in T \), \( t > \inf T \), is said to be left-dense if \( \rho(t) = t \), right-dense if \( \rho(t) < \sup T \) and \( \sigma(t) = t \), left-scattered if \( \rho(t) < t \) and right-scattered if \( \sigma(t) > t \). The graininess function \( \mu \) for a time scale \( T \) is defined by

\[ \mu(t) = \sigma(t) - t. \]

A function \( f : T \to \mathbb{R} \) is called \( rd \)-continuous function provided it is continuous at right-dense points in \( T \) and its left-sided limits exist (finite) at left-dense points in \( T \). The set of \( rd \)-continuous functions \( f : T \to \mathbb{R} \) is denoted by \( C_{rd} = C_{rd}(T) = C_{rd}(T, \mathbb{R}) \).
The set of functions \( f : \mathbb{T} \to \mathbb{R} \) that are differentiable and whose derivative is \( rd \)-continuous function is denoted by \( C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}) \). The function \( H(t, s) \) is \( rd \)-continuous function if \( H \) is \( rd \)-continuous function in \( t \) and \( s \).

A function \( p : \mathbb{T} \to \mathbb{R} \) is called positively regressive (we write \( p \in \mathbb{R}^+ \)) if it is \( rd \)-continuous function and satisfies \( 1 + \mu(t)p(t) > 0 \) for all \( t \in \mathbb{T} \).

We say that \( f \) is increasing, decreasing, nondecreasing, and nonincreasing on \([a, b]\) if \( t_1, t_2 \in [a, b] \) and \( t_2 > t_1 \) imply \( f(t_2) > f(t_1), \ f(t_2) < f(t_1), \ f(t_2) \geq f(t_1), \) and \( f(t_2) \leq f(t_1) \), respectively.

Let \( f \) be a differentiable function on \([a, b]\). Then \( f \) is increasing, decreasing, nondecreasing, and non-increasing on \([a, b]\) if \( f^A(t) > 0, \ f^A(t) < 0, \ f^A(t) \geq 0 \), and \( f^A(t) \leq 0 \) for all \( t \in [a, b] \), respectively.

For a function \( f : \mathbb{T} \to \mathbb{R} \) (the range \( \mathbb{R} \) of \( f \) may be actually replaced by any Banach space) the (delta) derivative is defined by

\[
f^A(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},
\]
if \( f \) is continuous at \( t \) and \( t \) is right-scattered. If \( t \) is not right-scattered then the derivative is defined by

\[
f^A(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},
\]
provided this limit exists.

A function \( f : [a, b] \to \mathbb{R} \) is said to be differentiable if its derivative exists, and a useful formula is

\[
f^\sigma = f(\sigma(t)) = f(t) + \mu(t)f^A(t).
\]

We will make use of the following product and quotient rules for the derivative of the product \( fg \) and the quotient \( f/g \) (where \( gg^\sigma \neq 0 \)) of two differentiable functions \( f \) and \( g \)

\[
(fg)^A = f^A g + f^\sigma g^A = fg^A + f^A g^\sigma,
\]

\[
\left(\frac{f}{g}\right)^A = \frac{f^A g - fg^A}{gg^\sigma}.
\]

For \( t_0, b \in \mathbb{T} \), and a differentiable function \( f \), the Cauchy integral of \( f^A \) is defined by

\[
\int_{t_0}^{b} f^A(t) \Delta t = f(b) - f(a).
\]

An integration by parts formula reads

\[
\int_{t_0}^{b} f(t) g^A(t) \Delta t = [f(t) g(t)]_{t_0}^{b} - \int_{t_0}^{b} f^A(t) g^\sigma \Delta t,
\]
and infinite integral is defined as

\[
\int_{t_0}^{\infty} f(t) \Delta t = \lim_{b \to \infty} \int_{t_0}^{b} f(t) \Delta t.
\]
3. Main results

In this section, by employing the Riccati transformation technique we establish the oscillation criteria for Eq. (1.1). To prove our main results, we will use the formula

\[ (x^\gamma(t))^A = \gamma \int_0^1 [hx^\gamma + (1 - h)x]^{\gamma-1} dh x^A(t), \]

which is a simple consequence of Keller’s chain rule [5, Theorem 1.90]. Also, we need the following lemma.

**Lemma 3.1.** Let \( f(u) = Bu - Au^{\frac{\gamma+1}{\gamma}} \), where \( A > 0 \) and \( B \) are constants, \( \gamma \) is a positive integer. Then \( f \) attains its maximum value on \( \mathbb{R} \) at \( u^* = \left( \frac{B}{A(\gamma+1)} \right)^{\gamma} \), and

\[ \max_{u \in \mathbb{R}} f = f(u^*) = \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}. \]

**Theorem 3.1.** Assume that \((h_1)\) and \((h_2)\) hold. Furthermore assume that there exist positive rd-continuous \( \Delta \)-differentiable functions \( x(t) \) and \( \phi(t) \) such that

\[ \lim_{t \to \infty} \sup \int_{t_0}^t \left[ x(s)\phi(s)Q(s) - \frac{r(s - \delta)(x^\gamma)^{\gamma+1}C^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}x^\gamma(s)\phi^\gamma(s)} \right] \Delta s = \infty, \] (3.1)

where

\[ Q(s) := q(s)(1 - p(s - \delta))^{\gamma}, \quad C(s) := \frac{\phi(s)x^A(s)}{x^\gamma} + (\phi^A(s))_+, \]

\( (x^A(t))_+ := \max\{x^A(t), 0\} \) and \( (\phi^A(t))_+ := \max\{\phi^A(t), 0\} \). Then every solution of Eq. (1.1) oscillates on \([t_0, \infty)\).

**Proof.** Suppose to the contrary that \( y(t) \) is a nonoscillatory solution of (1.1) and let \( t_0 \geq t_1 \) be such that \( y(t) \neq 0 \) for all \( t \geq t_1 \). Without loss of generality, we may assume that \( y \) is an eventually positive solution of (1.1) with \( y(t - N) > 0 \) where \( N = \max\{\tau, \delta\} \) for all \( t \geq t_1 \) sufficiently large. Set

\[ x(t) := y(t) + p(t)y(t - \tau). \] (3.2)

In view of (1.1) and \((h_2)\), we have

\[ (r(t)(x^A(t))^\gamma)^A + q(t)y^\gamma(t - \delta) \leq 0, \quad \text{for all } t \geq t_1, \] (3.3)

and this implies that \( r(t)(x^A(t))^\gamma \) is an eventually decreasing function, since \( q(t) > 0 \). We first show that \( r(t)(x^A(t))^\gamma \) is eventually nonnegative. Indeed, since \( q(t) \) is a positive function, the decreasing function \( r(t)(x^A(t))^\gamma \) is either eventually positive or eventually negative. Suppose there exists an integer \( t_2 \geq t_1 \) such that \( r(t_1)(x^A(t_1))^\gamma = c < 0 \), then from (3.3) we have \( r(t)(x^A(t))^\gamma < r(t_1)(x^A(t_1))^\gamma = c \) for \( t \geq t_2 \), hence

\[ x^A(t) \leq c^{\frac{1}{\gamma}} \left( \frac{1}{r(t)} \right)^{\frac{1}{\gamma}}, \] (3.4)
which implies by \((h_1)\) that
\[
x(t) \leq x(t_2) + c \int_{t_2}^t \left( \frac{1}{r(s)} \right)^{\frac{1}{r}} \Delta s \to -\infty \text{ as } t \to \infty.
\] (3.5)

This contradicts the fact that \(x(t) > 0\) for all \(t \geq t_1\). Hence \(r(t)(x^A(t))^{\gamma} A\) is eventually nonnegative. Therefore, we see that there is some \(t_1\) such that
\[
x(t) > 0, \quad x^A(t) \geq 0, \quad (r(t)(x^A(t))^{\gamma}) A < 0, \quad t \geq t_1.
\] (3.6)

This implies that
\[
y(t) = x(t) - p(t)y(t - \tau) = x(t) - p(t)[x(t - \tau) - p(t - \tau)y(t - 2\tau)] \\
\geq x(t) - p(t)x(t - \tau) \geq (1 - p(t))x(t).
\]

Then for \(t \geq t_2 = t_1 + \delta\), we have
\[
y(t - \delta) \geq (1 - p(t - \delta))x(t - \delta).
\]

From (3.3) and the last inequality, we obtain
\[
(r(t)(x^A(t))^{\gamma}) A + q(t)(1 - p(t - \delta))^{\gamma} x^A(t - \delta) \leq 0, \quad \text{for } t \geq t_2.
\] (3.7)

Define the function \(w(t)\) by the Riccati substitution
\[
w(t) := x(t) \left( \frac{r(t)(x^A(t))^{\gamma}}{x^A(t - \delta)} \right), \quad \text{for } t \geq t_2.
\] (3.8)

Then \(w(t) > 0\), and by using (2.5) and (2.6), we have
\[
w^A(t) = (r(x^A)^{\gamma}) \left[ \frac{x(t)}{x^A(t - \delta)} \right]^A + \frac{x(t)}{x^A(t - \delta)} (r(t)(x^A(t))^{\gamma}) A \\
= \frac{x(t)}{x^A(t - \delta)} (r(t)(x^A(t))^{\gamma}) A \\
+ (r(x^A)^{\gamma}) \left[ \frac{x^A(t - \delta)x^A(t) - x(t)(x^A(t - \delta))^A}{x^A(t - \delta)x^A(\sigma(t) - \delta)} \right].
\] (3.9)

In view of (3.7) and (3.9), we have
\[
w^A(t) \leq -x(t)Q(t) + (x^A(t))_+^{\sigma} w^{\sigma} \quad \text{for } t \geq t_2.
\] (3.10)

Using (3.6) and the Keller’s chain rule, we obtain
\[
(x^A(t))^A = \gamma \int_0^1 [hx^\sigma + (1 - h)x]^\gamma - 1 \, dh x^A(t) \\
\geq \gamma \int_0^1 [hx + (1 - h)x]^\gamma - 1 \, dh x^A(t) = \gamma (x(t))^\gamma - 1 x^A(t).
\] (3.11)
Then for \( t \geq t_2 \) sufficiently large, we have
\[
(x^\gamma(t - \delta))^A \geq \gamma x^{\gamma-1}(t - \delta)(x^A(t - \delta)). \tag{3.12}
\]
Also from (3.6), we have for \( t \geq t_2 \)
\[
r(t - \delta)(x^A(t - \delta))^\gamma \geq r(\sigma(t) - \delta))(x^A(\sigma(t) - \delta))^\gamma \geq (r(x^A)^\gamma)^\sigma. \tag{3.13}
\]
It follows from (3.10), (3.12) and (3.13) that
\[
w^A(t) \leq - \alpha(t) Q(t) + \frac{(x^A(t))_+}{x_\sigma} w^\sigma - \frac{\gamma \zeta(t)(r^{\frac{\gamma+1}{\gamma}}(x^A)^{\gamma+1})^\sigma}{r^{\frac{1}{\gamma}}(t - \delta)x(t - \delta)x^\gamma(\sigma(t) - \delta)}.
\]
Since \( x^A(t) \geq 0 \), we have \( x(\sigma(t) - \delta) \geq x(t - \delta) \), and this implies that
\[
w^A(t) \leq - \alpha(t) Q(t) + \frac{(x^A(t))_+}{x_\sigma} w^\sigma - \frac{\gamma \zeta(t)(r^{\frac{\gamma+1}{\gamma}}(x^A)^{\gamma+1})^\sigma}{r^{\frac{1}{\gamma}}(t - \delta)x^{\gamma+1}(\sigma(t) - \delta)}. \tag{3.14}
\]
Then from (3.8) and (3.14), we obtain
\[
w^A(t) \leq - \alpha(t) Q(t) + \frac{(x^A(t))_+}{x_\sigma} w^\sigma - \frac{\gamma \zeta(t)(r^{\frac{\gamma+1}{\gamma}}(x^A)^{\gamma+1})^\sigma}{r^{\frac{1}{\gamma}}(t - \delta)(x_\sigma)^{\gamma+1}}(w^\sigma)^{\frac{\gamma+1}{\gamma}}. \tag{3.15}
\]
Multiplying (3.15) by \( \phi(s) \) and integrating from \( t_2 \) to \( t \) \((t \geq t_2)\), we have
\[
\int_{t_2}^t \phi(s) \alpha(s) Q(s) \Delta s \leq - \int_{t_2}^t \phi(s) w^A(s) \Delta s + \int_{t_2}^t \phi(s) \frac{(x^A(s))_+}{x_\sigma} w^\sigma \Delta s
\]
\[
- \int_{t_2}^t \frac{\gamma \phi(s) \alpha(s)}{r^{\frac{1}{\gamma}}(t - \delta)(x_\sigma)^{\gamma+1}}(w^\sigma)^{\frac{\gamma+1}{\gamma}} \Delta s. \tag{3.16}
\]
Using the integration by parts, we obtain
\[
- \int_{t_2}^t \phi(s) w^A(s) \Delta s = -\phi(s) w(s) \big|_{t_2}^t + \int_{t_2}^t (\phi(s))^A w^\sigma \Delta s. \tag{3.17}
\]
From (3.16) and (3.17), we have
\[
\int_{t_2}^t \phi(s) \alpha(s) Q(s) \Delta s \leq w(t_2)\phi(t_2) + \int_{t_2}^t \left[ \frac{\phi(s)(x^A(s))_+}{x_\sigma} + (\phi^A(s))_+ \right] w^\sigma(s) \Delta s
\]
\[
- \int_{t_1}^t \frac{\gamma \phi(s) \alpha(s)}{r^{\frac{1}{\gamma}}(t - \delta)(x_\sigma)^{\gamma+1}}(w^\sigma)^{\frac{\gamma+1}{\gamma}} \Delta s. \tag{3.18}
\]
Setting
\[
B = \left[ \frac{\phi(s)(x^A(s))_+}{x_\sigma} + (\phi^A(s))_+ \right] \text{ and } A = \frac{\gamma \phi(s) \alpha(s)}{r^{\frac{1}{\gamma}}(t - \delta)(x_\sigma)^{\gamma+1}} \text{ and } u = w^\sigma,
\]
and applying Lemma 3.1 in (3.18), we find that

\[
\int_{t_2}^{t'} \phi(s)z(s)Q(s)\Delta s \leq w(t_2)\phi(t_2) + \int_{t_1}^{t'} \frac{r(s - \delta)(x^\gamma(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} (x(s))^{\gamma}\phi^\gamma(s)} \Delta s. \tag{3.20}
\]

Hence

\[
\int_{t_2}^{t'} \left[ \phi(s)z(s)Q(s) - \frac{r(s - \delta)(x^\gamma(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} (x(s))^{\gamma}\phi^\gamma(s)} \right] \Delta s < \phi(t_2)w(t_2), \tag{3.21}
\]

which contradicts condition (3.1). Then every solution of (1.1) oscillates. The proof is complete. \( \square \)

**Remark 3.1.** From Theorem 3.1, we can establish different sufficient conditions for oscillation of (1.1) by different choices of \( \phi(t) \) and \( \phi(t) \). For instance, if \( \phi(t) = 1 \) for \( t \geq t_0 \), we have the following result.

**Corollary 3.1.** Assume that \((h_1)\) and \((h_2)\) hold. Let \( z(t) \) be as defined in Theorem 3.1 and

\[
\limsup_{t \to \infty} \int_{t_0}^{t} \left[ z(s)Q(s) - \frac{r(s - \delta)(x^\gamma(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} (x(s))^{\gamma}\phi^\gamma(s)} \right] \Delta s = \infty. \tag{3.22}
\]

Then every solution of Eq. (1.1) oscillates on \([t_0, \infty)\).

**Remark 3.2.** From Corollary 3.1, we can obtain different conditions for oscillation of (1.1) by different choices of \( z(t) \).

For instance, if \( z(t) = t \) and \( z(t) = 1 \) for \( t \geq t_0 \), we have the following results respectively.

**Corollary 3.2.** Assume that \((h_1)\) and \((h_2)\) hold. Furthermore, assume that

\[
\limsup_{t \to \infty} \int_{t_0}^{t} \left[ sq(s)(1 - p(s - \delta))^\gamma - \frac{r(s - \delta)(x^\gamma(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} x^\gamma(s)} \right] \Delta s = \infty. \tag{3.23}
\]

Then, every solution of Eq. (1.1) oscillates on \([t_0, \infty)\).

**Corollary 3.3.** Assume that \((h_1)\) and \((h_2)\) hold. If

\[
\int_{t_0}^{\infty} q(s)(1 - p(s - \delta))^\gamma \Delta s = \infty. \tag{3.24}
\]

Then every solution of Eq. (1.1) oscillates on \([t_0, \infty)\).

The following theorem gives Philos-type oscillation criteria for Eq. (1.1). First, let us introduce now the class of functions \( \Phi \) which will be extensively used in the sequel.
Theorem 3.2. Assume that \( h_1 \) and \( h_2 \) hold. Let \( x(t) \) be as defined in Theorem 3.1 and let \( h, H : D \to \mathbb{R} \) be rd-continuous functions such that \( H \) belongs to the class \( \mathcal{R} \) and

\[
\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^t \left[ H(t, s)x(s)Q(s) - \frac{(x^\sigma)^{\gamma + 1}r(s - \delta)C(t,s)Q(s)}{x^\sigma(s)(\gamma + 1)^{\gamma + 1}H(t,s)} \right] \Delta s = \infty,
\]

where

\[
C(t, s) := \frac{H(t, s)(x^A(s))}{x^\sigma} + H^A(t, s).
\]

Then every solution of Eq. (1.1) oscillates on \( [t_0, \infty) \).

**Proof.** Suppose to the contrary that \( y(t) \) is a nonoscillatory solution of (1.1) and let \( t_1 > t_0 \) be such that \( y(t) \neq 0 \) for all \( t > t_1 \), so without loss of generality, we may assume that \( y \) is an eventually positive solution of (1.1) with \( y(t - N) > 0 \) where \( N = \max\{\tau, \delta\} \) for all \( t > t_1 \) sufficiently large. We proceed as in the proof of Theorem 3.1 to prove that there exists \( t_2 > t_1 \) such that (3.15) holds for \( t > t_2 \). From (3.15), it follows that

\[
\int_{t_2}^t H(t, s)x(s)Q(s)\Delta s \leq - \int_{t_2}^t H(t, s)x^\sigma(s)\Delta s + \int_{t_2}^t \frac{H(t, s)(x^A(s))}{x^\sigma} w^\sigma\Delta s
\]

\[- \int_{t_2}^t \frac{\gamma x(t)H(t, s)}{(x^\sigma)^{\gamma + 1}r(\frac{1}{r} + (s - \delta))} (w^\sigma)^{\gamma + 1} \Delta s.
\]

Using integration by parts formula (2.7), we have

\[
\int_{t_2}^t H(t, s)w^\sigma(s)\Delta s = H(t, s)w(s)|_{t_2}^t - \int_{t_2}^t H^A(t, s)w^\sigma\Delta s
\]

\[= - H(t, t_2)w(t_2) - \int_{t_2}^t H^A(t, s)w^\sigma\Delta s.
\]

where \( H(t, t) = 0 \). Substituting from (3.28) in (3.27), we get

\[
\int_{t_2}^t H(t, s)x(s)Q(s)\Delta s \leq H(t, t_2)w(t_2) + \int_{t_2}^t H^A(t, s)w^\sigma\Delta s + \int_{t_2}^t \frac{H(t, s)(x^A(s))}{x^\sigma} w^\sigma\Delta s
\]

\[- \int_{t_2}^t \frac{\gamma x(t)H(t, s)}{(x^\sigma)^{\gamma + 1}r(\frac{1}{r} + (s - \delta))} (w^\sigma)^{\gamma + 1} \Delta s.
\]
Hence,
\[
\int_{t_2}^{t} H(t, s) \varphi(s) Q(s) \Delta s \leq H(t, t_2) w(t_2) + \int_{t_2}^{t} \left[ \frac{H(t, s) (\varphi^\sigma(s))}{\varphi^\sigma} + H^A_s (t, s) \right] w^\sigma \Delta s
\]
\[
- \int_{t_2}^{t} \frac{\gamma \varphi(t) H(t, s)}{(\varphi^\sigma)^{1+1/\gamma}} \left( w^\sigma \right)^{1+1/\gamma} \Delta s.
\]
(3.29)

Therefore by using Lemma 3.1 in (3.29), with
\[
A = \frac{\gamma \varphi(t) H(t, s)}{(\varphi^\sigma)^{1+1/\gamma}} \left( w^\sigma \right)^{1+1/\gamma}
\]
and \(B = \frac{H(t, s) (\varphi^\sigma(s))}{\varphi^\sigma} + H^A_s (t, s)\) and \(u = w^\sigma\), we have
\[
\int_{t_2}^{t} H(t, s) \varphi(s) Q(s) \Delta s \leq H(t, t_2) w(t_2) + \int_{t_2}^{t} \left[ \frac{(\varphi^\sigma)^{1+1/\gamma} r(s - \delta) [C(t, s)]^\gamma + 1}{\varphi^\gamma(s) (\gamma + 1)^{1+1/\gamma}} H((t, s))^\gamma \right] \Delta s.
\]
(3.30)

Then for all \(t \geq t_2\), we have
\[
\int_{t_2}^{t} \left[ H(t, s) \varphi(s) Q(s) - \frac{(\varphi^\sigma)^{1+1/\gamma} r(s - \delta) [C(t, s)]^\gamma + 1}{\varphi^\gamma(s) (\gamma + 1)^{1+1/\gamma}} H((t, s))^\gamma \right] \Delta s < H(t, t_2) w(t_2),
\]
(3.31)

and this implies that
\[
\frac{1}{H(t, t_2)} \int_{t_2}^{t} \left[ H(t, s) \varphi(s) Q(s) - \frac{(\varphi^\sigma)^{1+1/\gamma} r(s - \delta) [C(t, s)]^\gamma + 1}{\varphi^\gamma(s) (\gamma + 1)^{1+1/\gamma}} H((t, s))^\gamma \right] \Delta s < w(t_2),
\]
for all large \(t\), which contradicts (3.25). Then every solution of (1.1) oscillates. The proof is complete. \(\square\)

As an immediate consequence of Theorem 3.2 we have the following.

**Corollary 3.4.** Let assumption (3.25) in Theorem 3.2 be replaced by
\[
\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \varphi(s) Q(s) \Delta s = \infty,
\]
\[
\lim_{t \to \infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ \frac{(\varphi^\sigma)^{1+1/\gamma} r(s - \delta) [C(t, s)]^\gamma + 1}{\varphi^\gamma(s) (\gamma + 1)^{1+1/\gamma}} H((t, s))^\gamma \right] \Delta s < \infty.
\]

Then every solution of Eq. (1.1) oscillates on \([t_0, \infty)\).

**Remark 3.3.** With an appropriate choice of the functions \(H\) and \(h\) one can establish a number of oscillation criteria for Eq. (1.1) on different types of time scales. For example if \(H(t, s) = (t - s)^m, (t, s) \in D\) with \(m > 1\). It is clear that \(H\) belongs to the class \(\mathfrak{H}\). Now, we claim that
\[
((t - s)^m)^A_s \leq -m(t - \sigma(s))^{m-1}.
\]
(3.32)
We consider the following two cases:

**Case 1:** If $\mu(t) = 0$, then
\[
((t-s)^m)^{A_t} = -m(t-s)^{m-1}.
\]

**Case 2:** If $\mu(t) \neq 0$, then we have
\[
((t-s)^m)^{A_t} = \frac{1}{\mu(s)}[((t-\sigma(s))^m) - ((t-s)^m)]
= -\frac{1}{\sigma(s) - s}[((t-s)^m) - ((t-\sigma(s))^m)].
\]

Using Hardy et al. inequality (cf. [17])
\[
x^m - y^m \geq y^{m-1}(x-y) \quad \text{for all } x \geq y > 0 \text{ and } m \geq 1,
\]
we have
\[
[((t-s)^m) - ((t-\sigma(s))^m)] \geq m((t-s)^{m-1}(\sigma(s) - s)).
\]
Then, from (3.33) and (3.34), we have
\[
((t-s)^m)^{A_t} \leq -m((t-\sigma(s))^{m-1},
\]
and this proves (3.32).

From the above claim and Theorem 3.2, we have the following Kamenev-type oscillation criteria for Eq. (1.1).

**Corollary 3.5.** Assume that (h1) and (h2) hold. Let $\alpha(t)$ be as defined in Theorem 3.1. If for $m > 1$
\[
\limsup_{t \to \infty} \frac{1}{tm} \int_{t_0}^{t} \left[ (t-s)^m \alpha(s) Q(s) - \frac{r(s-\delta)(\alpha^\sigma)^{\gamma+1} K^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}(\alpha(s))^{\gamma}(t-s)^{m\gamma}} \right] \Delta s = \infty,
\]
where
\[
K(t,s) := (t-s)^m \frac{(\alpha^\sigma(s))^{\gamma+1}}{\alpha^\sigma} - m(t-\sigma(s))^{m-1}, \quad t \geq s \geq t_0.
\]
Then every solution of Eq. (1.1) oscillates on $[t_0, \infty)$.

Also, one can use the factorial function $H(t,s) = (t-s)^{(k)}$ where $t^{(k)} = t(t-1) \ldots (t-k+1)$, $t^{(0)} = 1$ and establish new oscillation criteria for Eq. (1.1). In this case
\[
H^{A_t}(t-s)^{(k)} = \frac{(t-\sigma(s))^{(k)} - (t-s)^{(k)}}{\mu(s)}
= -\frac{(t-s)^{(k)} - (t-\sigma(s))^{(k)}}{\mu(s)} \geq -k(t-s)^{(k-1)}.
\]
4. Applications

In this section, from Theorem 3.1 we give some sufficient conditions for oscillation of Eqs. (1.2)–(1.7). The oscillation criteria of Kamenev and Philos types from Theorem 3.2 and Corollary 3.4 for Eqs. (1.2) and (1.3) are similar to the results that established in [19,20,24] and for Eqs. (1.4)–(1.7) are essentially new and due to the limited space the details are left to the interested reader.

In the case when \( T = \mathbb{R} \), we have the following oscillation criteria for Eq. (1.2).

**Corollary 4.1.** Consider (1.2) with the assumptions:

(i) \( r, p : I \to \mathbb{R}^+ = [0, \infty), I = [t_0, \infty) \subset \mathbb{R}^+ \) are continuous functions with \( 0 \leq p(t) < 1 \),
\[
\int_{t_0}^\infty \left( \frac{1}{r(t)} \right)^\gamma \, dt = \infty,
\]

(ii) \( f(t, u) : I \times \mathbb{R} \to \mathbb{R} \) is continuous function such that \( uf(t, u) > 0 \) for all \( u \neq 0 \) and there exists a nonnegative function \( q(t) \) defined on \( I \) such that \( |f(t, u)| \geq q(t)|u| \). Furthermore, assume that there exists a positive function \( \gamma(t) \) such that
\[
\lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t'} \left[ x(s)q(s)(1 - p(s - \delta))^{\gamma - 1} - \frac{r(s - \delta)(x'(s))^{\gamma + 1}}{(\gamma + 1)^{\gamma + 1} \tilde{x}(s)} \right] \, ds = \infty.
\]

Then every solution of Eq. (1.2) oscillates.

As a special case of Corollary 4.1, we see that:

If \( \gamma = 1, f(u) = u \) and \( x = 1 \), we obtain the oscillation condition (1.9) of Grammatikopoulos et al. [16].

If \( x = 1 \) and \( f(u) = q(t)|u| \), we obtain the oscillation condition (1.12) of Graef et al. [15].

If \( \gamma = 1, r = 1 \) and \( f(u) \geq Ku \), we obtain the oscillation condition (1.13) of Kubiaczyk and Saker [19].

So our results involve the results established by Kubiaczyk and Saker [19] and involve and improve the results established by Grammatikopoulos et al. [16] and Graef et al. [15] for oscillation of second-order neutral delay differential equations.

In the case when \( T = \mathbb{N} \) and \( \gamma \geq 1 \), we have the following oscillation criteria of Sun and Saker [24].

**Corollary 4.2 (Sun and Saker [24]).** Consider (1.3) with the assumptions:

(i) \( r(t), p(t) \) are positive sequences, \( t \in I = [t_0, \infty) \subset \mathbb{N} \) with \( 0 \leq p(t) < 1 \),
\[
\sum_{t=t_0}^\infty \left( \frac{1}{r(t)} \right)^\gamma = \infty,
\]

(ii) \( f(t, u) : [t_0, \infty) \times \mathbb{R} \to \mathbb{R} \) is continuous function such that \( uf(t, u) > 0 \) for all \( u \neq 0 \) and there exists a nonnegative sequence \( q(t) \) such that \( |f(t, u)| \geq q(t)|u| \). Furthermore, assume that there exists positive sequences \( x(t) \) and \( \phi(t) \) such that
\[
\lim_{t \to \infty} \sup_{t_0} \sum_{s=t_0}^{t-1} \left[ x(s)\phi(s)Q(s) - \frac{r(s - \delta)(x'(s))^{\gamma + 1}}{(\gamma + 1)^{\gamma + 1} \tilde{x}(s)\phi'(s)} \right] = \infty,
\]
where
\[ Q(t) = q(t)(1 - p(t - \delta))^\gamma, \quad A(s) = \frac{\phi(s)(\Delta z(s))_+}{\sigma^\gamma} + (\Delta \phi(s))_+. \]

\((\Delta z(t))_+ = \max\{\Delta z(t), 0\} \text{ and } (\phi^A(t))_+ = \max\{\phi^A(t), 0\}.\) Then every solution of Eq. (1.3) oscillates.

As a special case of Corollary 4.2 when \(p = 1\), we have the following oscillation results of Saker [20].

**Corollary 4.3 (Saker [20]).** Consider (1.3) with the assumptions:

(i) \(r(t), p(t)\) are positive sequence, \(t \in I = [t_0, \infty) \subset \mathbb{N}\) with \(0 \leq p(t) < 1),

\[ \sum_{t=t_0}^{\infty} \left( \frac{1}{r(t)} \right)^{1/\gamma} = \infty, \]

(ii) \(f(t, u) : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}\) is continuous function such that \(uf(t, u) > 0\) for all \(u \neq 0\) and there exists a nonnegative sequence \(q(t)\) such that \(|f(t, u)| \geq q(t)|u|^\gamma\). Furthermore, assume that there exists a positive sequence \(z(t)\) such that

\[ \lim \sup_{t \to \infty} \sum_{s=t_0}^{t-1} \left[ x(s)s(q(s)(1 - p(s - \delta))^\gamma - \frac{r(s - \delta)(\Delta z(s))_+)^{\gamma+1}}{\gamma + 1} \right]^{\gamma+1} = \infty. \]

Then every solution of Eq. (1.3) oscillates.

The following oscillation results are essentially new for Eqs. (1.4)–(1.7).

**Corollary 4.4.** Consider (1.4) with the assumptions:

(i) \(r(t), p(t)\) are positive sequence, \(t \in I = [t_0, \infty) \subset h\mathbb{N}, h > 0\) with \(0 \leq p(t) < 1),

\[ \sum_{i=t_0 \pi}^{\infty} \left( \frac{1}{r(i\pi)} \right)^{1/\gamma} = \infty, \]

(ii) \(f(t, u) : I \times \mathbb{R} \to \mathbb{R}\) is continuous function such that \(uf(t, u) > 0\) for all \(u \neq 0\) and there exists a nonnegative sequence \(q(t)\) such that \(|f(t, u)| \geq q(t)|u|^\gamma\). Furthermore, assume that there exists a positive sequence \(z(t)\) such that

\[ \lim \sup_{t \to \infty} \sum_{i=t_0}^{t-1} \left[ z(i\pi)(i\pi(q(i\pi)(1 - p(i\pi - \delta))^\gamma - \frac{r(i\pi - \delta)(\Delta z(i\pi))_+)^{\gamma+1}}{\gamma + 1} \right]^{\gamma+1} = \infty. \]

Then every solution of Eq. (1.4) oscillates.
Corollary 4.5. Consider (1.5) with the assumptions:

(i) \( r(t), p(t) \) are positive sequences, \( t \in \mathbb{I} = [t_0, \infty) \subset q \mathbb{N} \) with \( 0 \leq p(t) < 1 \),
\[
\sum_{i=0}^{\infty} \left( \frac{1}{r(q^i)} \right)^{\frac{1}{\gamma}} = \infty,
\]

(ii) \( f(t, u) : \mathbb{I} \times \mathbb{R} \to \mathbb{R} \) is continuous function such that \( uf(t, u) > 0 \) for all \( u \neq 0 \) and there exists a nonnegative sequence \( q(t) \) such that \( |f(t, u)| \geq q(t)|u|^\gamma \). Furthermore, assume that there exists a positive sequence \( \gamma(t) \) such that
\[
\sum_{i=0}^{\infty} i \mu(q^i) \left[ \gamma(q^i)q(q^i)(1 - p(q^i - \delta))^\gamma - \frac{r(q^i - \delta)((\Delta q \gamma(q^i)))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} \gamma(q^i)} \right] = \infty.
\]

Then every solution of Eq. (1.5) oscillates.

Corollary 4.6. Consider (1.6) with the assumptions:

(i) \( r(t), p(t) \) are positive sequences, \( t \in \mathbb{I} = [t_0, \infty) \subset \mathbb{N}^2 \) with \( 0 \leq p(t) < 1 \),
\[
\sum_{t=t_0}^{\infty} \left( \frac{1}{r(t)} \right)^{\frac{1}{\gamma}} = \infty.
\]

(ii) \( f(t, u) : \mathbb{I} \times \mathbb{R} \to \mathbb{R} \) is continuous function such that \( uf(t, u) > 0 \) for all \( u \neq 0 \) and there exists a nonnegative sequence \( q(t) \) such that \( |f(t, u)| \geq q(t)|u|^\gamma \). Furthermore, assume that there exists a positive sequence \( \gamma(t) \) such that
\[
\sum_{t=t_0}^{\infty} (1 + 2\sqrt{t}) \left[ \gamma(t)q(t)(1 - p(t - \delta))^\gamma - \frac{r(t - \delta)((\Delta q \gamma(t)))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} \gamma(t)} \right] = \infty.
\]

Then every solution of Eq. (1.6) oscillates.

Corollary 4.7. Consider (1.7) with the assumptions:

(i) \( r(t), p(t) \) are positive sequences, \( t \in \mathbb{I} = [t_0, \infty) \subset \mathbb{T}_{t_n} \) with \( 0 \leq p(t) < 1 \),
\[
\sum_{n=0}^{\infty} \left( \frac{1}{r(t_n)} \right)^{\frac{1}{\gamma}} = \infty,
\]

(ii) \( f(t, u) : \mathbb{I} \times \mathbb{R} \to \mathbb{R} \) is continuous function such that \( uf(t, u) > 0 \) for all \( u \neq 0 \) and there exists a nonnegative sequence \( q(t) \) such that \( |f(t, u)| \geq q(t)|u|^\gamma \). Furthermore, assume that there exists a positive sequence \( \gamma(t) \) such that \( \Delta_t \gamma(t) \geq 0 \) and
\[
\sum_{n=0}^{\infty} \frac{1}{t_n + 1} \left[ \gamma(t_n)q(t_n)(1 - p(t_n - \delta))^\gamma - \frac{r(t_n - \delta)((\Delta_t \gamma(t_n)))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} \gamma(t_n)} \right] = \infty.
\]

Then every solution of Eq. (1.7) oscillates.
5. Examples

In this section, we give some examples to illustrate our main results. To obtain the conditions for oscillation, we will use the following facts:

\[ \int_{0}^{\infty} \frac{\Delta s}{s^\gamma} = \infty, \quad \text{if } 0 \leq \gamma \leq 1, \]

and

\[ \int_{0}^{\infty} \frac{\Delta s}{s^\gamma} < \infty, \quad \text{if } \gamma > 1. \]

For more details we refer the reader to Theorem 5.68 and Corollary 5.71 in [6].

**Example 5.1.** Consider the following second-order neutral delay dynamic equation

\[ \left[ y(t) + \frac{1}{t + \delta} y(t - \tau) \right]^{\Delta A} + \frac{\lambda}{t^2} y(t - \delta) = 0, \quad t \in \mathbb{T}, \quad (5.1) \]

where \( \mathbb{T} \) is a time scale, and \( \lambda, \tau \) and \( \delta \) are nonnegative constants such that \( t - \tau \) and \( t - \delta \in \mathbb{T} \) and \( \lambda > 0 \) is a constant. In Eq. (5.1) \( \gamma = 1 \), \( r(t) = 1 \), \( f(t, u) = q(t)u \), \( q(t) = \frac{t}{t + \delta} \), \( p(t) = \frac{1}{t + \delta} \) and \( p(t - \delta) = \frac{1}{t} \). It is easy to see that the assumptions \((h_1)\) and \((h_2)\) hold. To apply Theorem 3.1, it remains to satisfy condition (3.1). By choosing \( \phi = 1 \) and \( \varphi(s) = s \), we have

\[
\lim_{t \to \infty} \sup \int_{0}^{t} \left[ \varphi(s) \varphi(s) Q(s) - \frac{r(s - \delta)}{(\gamma + 1)^{\gamma + 1}} \frac{C^{\gamma + 1}(s)}{\varphi(s) \varphi(s)} \right] \Delta s = \infty,
\]

\[
= \lim_{t \to \infty} \sup \int_{0}^{t} \left[ \frac{\lambda}{s} \left( 1 - \frac{1}{s} \right) - \frac{1}{4s} \right] \Delta s = \infty, \quad \text{if } \lambda > \frac{1}{4}.
\]

Hence, by Theorem 3.1 every solution of Eq. (5.1) oscillates if \( \lambda > \frac{1}{4} \).

**Example 5.2.** Consider the following second-order neutral nonlinear delay dynamic equation

\[ \left( \left( y(t) + \frac{t + \delta - 1}{t + \delta} y(t - \tau) \right)^{\Delta A} \right)^{\Delta A} + t^2 y(t - \delta) = 0, \quad t \in \mathbb{T}, \quad (5.2) \]

where \( \mathbb{T} \) is a time scale, with \( \gamma > 1 \) is an odd positive integer and \( \tau \) and \( \delta \) are nonnegative constants such that \( t - \tau \) and \( t - \delta \in \mathbb{T} \). In Eq. (5.2) \( r(t) \equiv 1 \), \( p(t) \equiv \frac{t + \delta - 1}{t + \delta} \), and \( q(t) = t^2 \). It is easy to see that the assumptions \((h_1)\) and \((h_2)\) hold. To apply Theorem 3.1, it remains to satisfy condition (3.1). By choosing
\( \phi = 1 \) and \( \varphi(s) = s \), we have
\[
\lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \left[ \varphi(s) \phi(s) Q(s) - \frac{r(s - \delta)(\varphi^\sigma)^{\gamma + 1}C^{\gamma + 1}(s)}{(\gamma + 1)^{\gamma + 1} \varphi^\sigma(s) \phi^\sigma(s)} \right] \Delta s
\]
\[
= \lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \left[ s^{1+\gamma - \gamma} - \frac{1}{(\gamma + 1)^{\gamma + 1} s^{\gamma}} \right] \Delta s
\]
\[
\geq \lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \left[ s^{1+\gamma - \gamma} - \frac{1}{(\gamma + 1)^{\gamma + 1} s^{\gamma}} \right] \Delta s = \infty,
\]
if \( \alpha - \gamma \geq -2 \). Therefore, Eq. (5.1) with \( \gamma > 1 \) is oscillatory when \( \alpha - \gamma \geq -2 \).

**Example 5.3.** Consider the following second-order nonlinear neutral delay dynamic equation
\[
\left( t + \delta \right)^{\gamma - 1} \left( y(t) + \frac{t + \delta - 1}{t + \delta} y(t - \tau) \right)^{A} + \beta t^{\gamma - 2} y^{\gamma}(t - \delta) = 0, \quad t \in \mathbb{T},
\]
where \( \mathbb{T} \) is a time scale, \( \beta > 0 \) and \( \gamma \geq 1 \) and is an odd positive integer and \( \tau \) and \( \delta \) are nonnegative constants such that \( \tau - \tau \) and \( t - \tau \) \( \in \mathbb{T} \). In Eq. (5.3) \( r(t) = (t + \delta)^{\gamma - 1} \), \( p(t) = \frac{t + \delta - 1}{t + \delta} \), and \( q(t) = \beta t^{\gamma - 2} \). It is easy to see that the assumptions \((h_1)\) and \((h_2)\) hold. To apply Theorem 3.1, it remains to satisfy condition (3.1). By choosing \( \phi = 1 \) and \( \varphi(s) = s \), we have
\[
\lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \left[ \varphi(s) \phi(s) Q(s) - \frac{r(s - \delta)(\varphi^\sigma)^{\gamma + 1}C^{\gamma + 1}(s)}{(\gamma + 1)^{\gamma + 1} \varphi^\sigma(s) \phi^\sigma(s)} \right] \Delta s
\]
\[
= \lim_{t \to \infty} \sup_{t_0} \int_{t_0}^{t} \left[ \beta \frac{t}{s} - \frac{1}{(\gamma + 1)^{\gamma + 1} s^{\gamma}} \right] \Delta s = \infty,
\]
if \( \beta > \frac{1}{(\gamma + 1)^{\gamma + 1}} \). Therefore, Eq. (5.3) with \( \gamma \geq 1 \) is oscillatory if \( \beta > \frac{1}{(\gamma + 1)^{\gamma + 1}} \).

Note that, none of the oscillation results that has been established by Grammatikopoulos et al. [16], Graef et al. [15] and Kubiaczyk and Saker [19] can be applied for Eqs. (5.2) and (5.3). So our results in the case when \( \mathbb{T} = \mathbb{R} \) are also new.

**Acknowledgements**

The author thanks the referees for their helpful suggestions.

**References**


