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ScienceDirect
Journal of Functional Analysis 244 (2007) 342–348
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup

Jose A. Conejero [a,1], V. Müller [b,2], A. Peris [c,*,1]

[a] *Departament de Matemàtica Aplicada & IMPA-UPV, Facultat d'Informàtica, Universitat Politècnica de València, E-46022 València, Spain*
[b] *Mathematical Institute, Czech Academy of Sciences, Zitná 25, 115 67 Prague 1, Czech Republic*
[c] *Departament de Matemàtica Aplicada & IMPA-UPV, ETS Arquitectura, Universitat Politècnica de València, E-46022 València, Spain*

Received 13 June 2006; accepted 12 December 2006

Available online 24 January 2007

Communicated by D. Voiculescu

Abstract

Let $\{T_t\}_{t\geqslant 0}$ be a hypercyclic strongly continuous semigroup of operators. Then each T_t $(t>0)$ is hypercyclic as a single operator, and it shares the set of hypercyclic vectors with the semigroup. This answers in the affirmative a natural question concerning hypercyclic C_0-semigroups. The analogous result for frequent hypercyclicity is also obtained.

Keywords: Hypercyclic vectors; C_0-semigroups; Frequently hypercyclic operators; Chaotic dynamics

1. Introduction

A continuous linear operator $T : X \to X$ on a topological vector space X is said to be *hypercyclic* if there is a vector $x \in X$ (called a hypercyclic vector) whose orbit under T, $Orb(T, x) := \{T^n x\colon n \in \mathbb{N}\}$, is dense in X.

[*] Corresponding author. Fax: +34 963877669.
E-mail addresses: aconejero@mat.upv.es (J.A. Conejero), muller@math.cas.cz (V. Müller), aperis@mat.upv.es (A. Peris).
[1] Supported in part by MEC and FEDER, Project MTM2004-02262 and Research Net MTM2006-26627-E.
[2] Partially supported by grant No. 201/06/0128 of GA ČR.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.12.008

In [1] Ansari proved that all the powers of a hypercyclic operator T are also hypercyclic. Moreover, they share the same hypercyclic vectors with T. Recall that Ansari [2] and Bernal-González [8] showed that every infinite-dimensional separable Banach space admits a hypercyclic operator. This result was also extended to the non-normable Fréchet case by Bonet and Peris [9]. For more details about hypercyclic operators see the surveys [10,21,22].

In the continuous case, a one-parameter family $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ of continuous linear operators in $L(X)$ is a *strongly continuous semigroup* (or C_0-semigroup) of operators in $L(X)$ if $T_0 = I$, $T_t T_s = T_{t+s}$ for all $t, s \geqslant 0$, and $\lim_{t \to s} T_t x = T_s x$ for all $s \geqslant 0$, $x \in X$. For further information about C_0-semigroups we refer the reader to the books [20,28].

A C_0-semigroup $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ is said to be hypercyclic if $Orb(\mathcal{T}, x) := \{T_t x\colon t \geqslant 0\}$ is dense in X for some $x \in X$. The investigation of hypercyclic semigroups was initiated by Desch, Schappacher and Webb in [18]. So far, several specific examples of hypercyclic strongly continuous semigroups have been studied, see for example [16,18,19,24,29]. In [7] Bermúdez, Bonilla and Martinón proved that every separable infinite-dimensional Banach space admits a hypercyclic semigroup. This result was extended to Fréchet spaces ($\neq \omega$) in [13].

Given $T \in L(X)$, let us denote by $HC(T)$ the set of all hypercyclic vectors of T, and analogously, denote by $HC(\mathcal{T})$ the set of hypercyclic vectors of a C_0-semigroup $\mathcal{T}$. It is easy to see that if $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ is a C_0-semigroup and some operator T_t in the semigroup is hypercyclic, then the semigroup $\mathcal{T}$ itself is hypercyclic.

When one analyzes the converse situation (from the continuous to the discrete case), as a consequence of an old result of Oxtoby and Ulam [27] it is possible to establish that, if $x \in HC(\mathcal{T})$, then there exists a residual set $G \subset \mathbb{R}_+$, such that $x \in HC(T_t)$ for all $t \in G$ (see, e.g., [12]). The point here is whether $G = \mathbb{R}_+$. That is, if $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ is a hypercyclic C_0-semigroup, is every operator T_t, $t > 0$, hypercyclic? This problem was explicitly stated in [7].

Our main result is the solution to this problem in the affirmative. To do this we will adapt an argument due to León-Saavedra and Müller [26] on rotations of hypercyclic operators. This approach is not new: several authors have tried to use similar arguments to the ones in [26] for the C_0-semigroups context without success (e.g., [14,17,25]). The key point in the proof, proceeding by contradiction, is to construct a pair of continuous maps $f : HC(\mathcal{T}) \to \mathbb{T}$ and $g : \mathbb{D} \to HC(\mathcal{T})$ such that $f \circ g|_{\mathbb{T}}$ is homotopically nontrivial. Such a point has resisted previous attempts (notice that the homotopy in [17] does not yield any contradiction, which results in a serious gap), and it is finally solved here.

A new trend in hypercyclicity was recently opened by the work of Bayart and Grivaux. Motivated by Birkhoff's ergodic theorem, they introduced the notion of frequent hypercyclicity [5,6], by quantifying the frequency with which an orbit meets open sets. To be precise, let us define the *lower density* of a set $A \subset \mathbb{N}$ by $\underline{dens}(A) := \liminf_{N \to \infty} \#\{n \leqslant N\colon n \in A\}/N$. An operator $T \in L(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that, for every non-empty open subset $U \subset X$, the set $\{n \in \mathbb{N}\colon T^n x \in U\}$ has positive lower density. Each such a vector x is called a *frequently hypercyclic vector* for T, and the set of all frequently hypercyclic vectors is denoted by $FHC(T)$.

Analogously, if we define the lower density of a measurable set $M \subset \mathbb{R}_+$ by $\underline{Dens}(M) := \liminf_{N \to \infty} \mu(M \cap [0, N])/N$, where μ is the Lebesgue measure on $\mathbb{R}_+$, then a C_0-semigroup $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ in $L(X)$ is said to be frequently hypercyclic if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the set $\{t \in \mathbb{R}_+\colon T_t x \in U\}$ has positive lower density. As before, we denote by $FHC(\mathcal{T})$ the set of all hypercyclic vectors of $\mathcal{T}$. In both cases, frequent hypercyclicity is stronger than hypercyclicity. See also [4,11,23] for further details concerning frequently hypercyclic operators and C_0-semigroups.

We prove that, if a C_0-semigroup $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ is frequently hypercyclic, then every single operator $T_t \neq I$ is frequently hypercyclic.

From now on, X stands for an F-space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of either real or complex numbers; by an F-space we mean a metrizable and complete topological vector space. Let $\mathcal{U}_0(X)$ be a base of open balanced neighbourhoods of the origin in X. Within this context, any C_0-semigroup $\mathcal{T} = \{T_t : X \to X\}_{t \geqslant 0}$ is *locally equicontinuous*, i.e., for any $t_0 > 0$, the family of operators $\{T_t\colon\ t \in [0, t_0]\}$ is equicontinuous. This fact will be used repeatedly throughout the paper. We would like to point out that there is no simplification in the proofs if we assume that X is a Banach space, and that our results remain valid for general topological vector spaces X if we assume that $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ is locally equicontinuous.

2. Hypercyclic operators and semigroups

We begin this section with some technical results. The first one is an adaptation to F-spaces of a result of Costakis and Peris [15], using ideas of Wengenroth [30]. See also [13].

Lemma 2.1. *Let $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ be a hypercyclic semigroup in $L(X)$. Then $T_t - \lambda I$ has dense range for all $t > 0$ and $\lambda \in \mathbb{K}$.*

Proof. Fix arbitrarily $\lambda \in \mathbb{K}$ and $t_0 > 0$. We assume $L := \overline{(T_{t_0} - \lambda I)(X)} \neq X$, and consider the quotient map $q : X \to X/L$, which satisfies $q \circ (T_{t_0} - \lambda I) = 0$. Inductively, this yields $q \circ T_{t_0}^n = \lambda^n q$ for all $n \in \mathbb{N}$. Consider $x \in HC(\mathcal{T})$, and define $M := q(Orb(\mathcal{T}, x)) = \{\lambda^n q(T_s x)\colon n \in \mathbb{N}_0, s \in [0, t_0]\}$, which is dense by the definition of q. Now we distinguish two cases.

The case $|\lambda| \leqslant 1$. Since $\{T_s x\colon\ s \in [0, t_0]\}$ is bounded in X, M must be bounded, so that it cannot be dense. A contradiction.

The case $|\lambda| > 1$. Fix an arbitrary $y \in L$ with $q(y) \neq 0$. There exists an $r > 0$ such that $q(T_r x) \neq 0$. We pick $U \in \mathcal{U}_0(X/L)$ satisfying $q(T_r x) \notin U$. The equicontinuity of $\{T_s\colon\ s \in [0, t_0]\}$ yields the existence of $V \in \mathcal{U}_0(X)$ such that $q(T_t(V)) \subset U$, $t \in [0, t_0]$. Fix $t' > r$ with $T_{t'} x \in V$. We write $t' = mt_0 - t + r$, for some $m \in \mathbb{N}$ and $t \in [0, t_0]$. Since $|\lambda| > 1$, we have $\lambda^m q(T_r x) \notin U$. On the other hand,

$$\lambda^m q(T_r x) = q(T_{mt_0 + r} x) = q\big(T_t(T_{t'} x)\big) \in q\big(T_t(V)\big) \subset U,$$

which is a contradiction. □

An easy consequence of the previous lemma is the following.

Corollary 2.2. *Let $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ be a hypercyclic semigroup in $L(X)$. If $t > 0$, $(\lambda_1, \lambda_2) \neq (0, 0)$ and $x \in HC(\mathcal{T})$, then $\lambda_1 x + \lambda_2 T_t x \in HC(\mathcal{T})$.*

Theorem 2.3. *Let $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ be a hypercyclic semigroup in $L(X)$, and let $x \in HC(\mathcal{T})$. Then $x \in HC(T_{t_0})$ for every $t_0 > 0$.*

Proof. Without loss of generality, we may assume that $t_0 = 1$. Indeed, we can consider the semigroup $\widetilde{\mathcal{T}} = \{\widetilde{T}_t\}_{t \geqslant 0}$ in $L(X)$, with $\widetilde{T}_t := T_{tt_0}$ for every $t \geqslant 0$. Clearly, $x \in HC(\widetilde{\mathcal{T}})$ and $\widetilde{T}_1 = T_{t_0}$.

Let $\mathbb{T} := \{z \in \mathbb{C}\colon\ |z| = 1\}$ denote the unit circle, $\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| \leqslant 1\}$ the closed unit disc, and let $\mathbb{R}_+ := \{t \in \mathbb{R}\colon\ t \geqslant 0\}$.

We define the map $\rho : \mathbb{R}_+ \to \mathbb{T}$ by $\rho(t) := e^{2\pi i t}$. For every pair $u, v \in X$ let

$$F_{u,v} := \Big\{\lambda \in \mathbb{T}\colon\ \exists (t_n)_n \subset \mathbb{R} \text{ with } \lim_n t_n = \infty,\ \lim_n T_{t_n} u = v, \text{ and } \lim_n \rho(t_n) = \lambda\Big\}.$$

Our proof is divided into several steps.

Step 1. *If $u \in HC(\mathcal{T})$, then $F_{u,v} \neq \emptyset$ for all $v \in X$.* Since u is hypercyclic for $\mathcal{T}$, we can find an unbounded increasing sequence $\{t_k\}_k$ in $\mathbb{R}_+$, such that $T_{t_k} u$ converges to v. By passing to a subsequence, if necessary, we may assume that $(\rho(t_k))_k$ is convergent. Its limit is an element of $F_{u,v}$.

Step 2. *If $\lim_k v_k = v$, $\lambda_k \in F_{u,v_k}$, and $\lim_k \lambda_k = \lambda$, then $\lambda \in F_{u,v}$.* (In particular, $F_{u,v}$ is a closed set for each $u, v \in X$.) Indeed, for each k we select $t_k > k$ such that $\lim_k (T_{t_k} u - v_k) = 0$ and $\lim_k |\rho(t_k) - \lambda_k| = 0$. It is easy to see that $\lim_k T_{t_k} u = v$ and that $\lim_k \rho(t_k) = \lambda$.

Step 3. *If $u, v, w \in X$, $\lambda \in F_{u,v}$, and $\mu \in F_{v,w}$, then $\lambda\mu \in F_{u,w}$.* Given $U \in \mathcal{U}_0(X)$ and $\varepsilon > 0$, take $U' \in \mathcal{U}_0(X)$ such that $U' + U' \subset U$. Find t_1 such that $T_{t_1} v - w \in U'$ and $|\rho(t_1) - \mu| < \varepsilon$. Pick $V \in \mathcal{U}_0(X)$ and $t_2 > 0$ satisfying $T_{t_1}(V) \subset U'$, $T_{t_2} u - v \in V$, and $|\rho(t_2) - \lambda| < \varepsilon$. Then

$$T_{t_1+t_2} u - w = T_{t_1}(T_{t_2} u - v) + (T_{t_1} v - w) \in T_{t_1}(V) + U' \subset U, \quad \text{and}$$
$$\big|\rho(t_1 + t_2) - \lambda\mu\big| = \big|\rho(t_1)\rho(t_2) - \lambda\mu\big| \leqslant \big|\rho(t_1) - \mu\big| \cdot \big|\rho(t_2)\big| + |\mu| \cdot \big|\rho(t_2) - \lambda\big| < 2\varepsilon.$$

Hence $\lambda\mu \in F_{u,w}$.

Fix now $x \in HC(\mathcal{T})$. By Steps 1, 2 and 3, $F_{x,x}$ is a nonempty closed subsemigroup of $\mathbb{T}$. Firstly, suppose that $F_{x,x} = \mathbb{T}$. Then, given $y \in X$, by Steps 1 and 3 we get $F_{x,y} = \mathbb{T}$. In particular $1 \in F_{x,y}$, which yields the existence of a sequence $(t_n)_n \subset \mathbb{R}_+$ tending to infinity such that $\lim_n T_{t_n} x = y$ and $\lim_n \rho(t_n) = 1$. Write t_n as $t_n = k_n + \varepsilon_n$ with $k_n \in \mathbb{N}$ and $\varepsilon_n \in [-1/2, 1/2]$. Then $\lim_n \varepsilon_n = 0$. Let $U \in \mathcal{U}_0(X)$. We fix $U', V \in \mathcal{U}_0(X)$ with $U' + U' \subset U$ and $T_s(V) \subset U'$, $0 \leqslant s \leqslant 2$. Let $n \in \mathbb{N}$ be large enough such that $T_{t_n} x - y \in V$ and $T_{1-\varepsilon_n} y - T_1 y \in U'$. Then we have

$$\begin{aligned} T_{k_n+1} x - T_1 y &= T_{1-\varepsilon_n}(T_{t_n} x - y) + (T_{1-\varepsilon_n} - T_1) y \\ &\in T_{1-\varepsilon_n}(V) + U' \subset U' + U' \subset U. \end{aligned}$$

Hence $T_1 y \in \overline{Orb(T_1, x)}$. Since T_1 has dense range and $y \in X$ is arbitrary, then x is hypercyclic for T_1.

For the rest of the proof we assume that $F_{x,x} \neq \mathbb{T}$, and we will show that it leads to a contradiction.

Step 4. *There exists some $k \in \mathbb{N}$ such that, for each $y \in HC(\mathcal{T})$, there is $\lambda \in \mathbb{T}$ satisfying $F_{x,y} = \{\lambda z\colon\ z^k = 1\}$.* It turns out that there is $k \in \mathbb{N}$ such that $F_{x,x} = \{z \in \mathbb{T}\colon\ z^k = 1\}$. Indeed, given $z \in F_{x,x}$, the set $\{z^n\colon\ n \in \mathbb{N}\}$ is either dense in $\mathbb{T}$ or finite. Since it is contained in the closed semigroup $F_{x,x} \neq \mathbb{T}$, it should be finite. Now, given $y \in HC(\mathcal{T})$, $\lambda \in F_{x,y}$, and $\mu \in F_{y,x}$, by Step 3, $\lambda F_{x,x} \subset F_{x,y}$, and $\mu F_{x,y} \subset F_{x,x}$, then $\#(F_{x,y}) = \#(F_{x,x})$. This implies that $F_{x,y} = \lambda F_{x,x}$.

Step 5. *There is a continuous function $h:\mathbb{D}\to\mathbb{T}$, whose restriction to the unit circle is homotopically nontrivial. A contradiction.*

Let us recall that two maps $f,g:X\to Y$ are *homotopic* if there is a continuous map $H:X\times[0,1]\to Y$ such that $H(x,0)=f(x)$ and $H(x,1)=g(x)$, $x\in X$. f is *homotopically trivial* if it is homotopic to a constant map. If f is homotopically trivial, then so are all its restrictions. Any continuous map $f:\mathbb{D}\to Y$ is homotopically trivial. We say that a continuous map $g:\mathbb{T}\to\mathbb{T}$ has *index n* ($n\in\mathbb{Z}$), if it is homotopic to the map $z\mapsto z^n$. Any continuous map $g:\mathbb{T}\to\mathbb{T}$ has some index, and it is homotopically trivial if and only if it has index 0. We refer the reader to, e.g., [3].

Consider the function $f:HC(\mathcal{T})\to\mathbb{T}$ as $f(y):=\lambda^k$, where $\lambda\in F_{x,y}$. Clearly, by Steps 2 and 4, f is well defined and continuous. Besides, $f(x)=1$ and, since $x\in HC(\mathcal{T})$, then $T_tx\in HC(\mathcal{T})$ for every $t\geqslant 0$ by Corollary 2.2. Therefore it easily follows that $e^{2\pi it}\in F_{x,T_tx}$ and $f(T_tx)=e^{2\pi itk}$ for every $t\geqslant 0$.

We will find $g:\mathbb{D}\to HC(\mathcal{T})$ such that $h:=f\circ g$ is the desired function which will give the contradiction. We first define $g:\mathbb{T}\to HC(\mathcal{T})$, and then extend it to $\mathbb{D}$. To do this, since f is continuous at x, we find $U\in\mathcal{U}_0(X)$ such that $|f(y)-1|<1$ if $y\in HC(\mathcal{T})$ and $y-x\in U$. We now fix $t_0>1$ satisfying $T_{t_0}x-x\in U$. Let us define $g:\mathbb{T}\to HC(\mathcal{T})$ by

$$g\big(e^{2\pi it}\big):=\begin{cases}T_{2tt_0}x & \text{if } 0\leqslant t<1/2,\\ (2t-1)x+(2-2t)T_{t_0}x & \text{if } 1/2\leqslant t<1.\end{cases}$$

Clearly, g is well defined and continuous. By Corollary 2.2, we have $g(\mathbb{T})\subset HC(\mathcal{T})$. Since U is balanced, $g(e^{2\pi it})-x\in U$, for $1/2\leqslant t<1$. This implies $|f(g(e^{2\pi it}))-1|<1$, $1/2\leqslant t<1$. Moreover $f(g(e^{2\pi it}))=e^{4\pi itt_0k}$, $0\leqslant t<1/2$, which yields that the index of $f\circ g$ at 0 is between $[t_0]k$ and $([t_0]+1)k$ (depending on the difference $t_0-[t_0]$).

We extend the function g to $\mathbb{D}$ by defining $g(z):=(1-|z|)x+|z|g(z/|z|)$ for each $z\neq 0$, and $g(0)=x$. Clearly, this extension is also continuous on $\mathbb{D}$, and $g(z)\in HC(\mathcal{T})$ for every $z\in\mathbb{D}$ since $g(z)$ is a non-zero linear combination of x and T_tx, for some $0<t\leqslant t_0$ (Corollary 2.2).

To sum up, we have a continuous function $h:=f\circ g:\mathbb{D}\to\mathbb{T}$, such that its restriction to the unit circle is homotopically nontrivial, a contradiction. □

3. Frequently hypercyclic operators and semigroups

In this section we prove the analogous result for the stronger concept of frequent hypercyclicity. We first need a technical lemma concerning the frequently hypercyclic vectors of a C_0-semigroup $\mathcal{T}$.

Lemma 3.1. *Let $\mathcal{T}=\{T_t\}_{t\geqslant 0}$ be a frequently hypercyclic semigroup in $L(X)$, and let $x\in FHC(\mathcal{T})$. For every $k\in\mathbb{N}$, $y\in X$, and $U\in\mathcal{U}_0(X)$*

$$\underline{Dens}\bigg(\bigg\{t\in\bigcup_{n\in\mathbb{N}}[n-1/k,n):\ T_tx-y\in U\bigg\}\bigg)>0.$$

Proof. Clearly, $T_{j/k}x\in HC(\mathcal{T})$ for every $j=0,\dots,k-1$, and even more, $T_{j/k}x\in HC(T_1)$ by Theorem 2.3. Fix $U,U'\in\mathcal{U}_0(X)$ such that $U'+U'\subset U$, and $y\in X$. Then there are some $n_j\in\mathbb{N}$ such that $T_{n_j+j/k}x-y\in U'$, $j=0,\dots,k-1$. Besides, there is some $V\in\mathcal{U}_0(X)$ such that $T_s(V)\subset U'$ if $s\leqslant N_0:=\max\{n_j:\ j=0,\dots,k-1\}+1$.

Since $x \in FHC(\mathcal{T})$, we have $\underline{Dens}(\{t \in \mathbb{R}^+\colon\ T_t x - x \in V\}) > 0$. So there are $C > 0$ and $N_1 \in \mathbb{N}$ such that $\mu(\{t \leqslant N\colon\ T_t x - x \in V\}) \geqslant CN$ for every $N \geqslant N_1$.

For every $N \in \mathbb{N}$, let us define $L := \{t \leqslant N\colon\ T_t x - x \in V\}$. In addition, for every $j = 0, \ldots, k-1$, we define the sets $I_j := \bigcup_n [n + j/k, n + (j+1)/k)$, $L_j := L \cap I_j$, and the mapping $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ as $f_j(t) := t + n_{k-j-1} + (k-j-1)/k$. These mappings satisfy that $f_j(t) \in I_{k-1}$ for every $t \in L_j$, and

$$\begin{aligned} T_{f_j(t)} x - y &= T_{n_{k-j-1}+(k-j-1)/k}(T_t x - x) + (T_{n_{k-j-1}+(k-j-1)/k} x - y) \\ &\in T_{n_{k-j-1}+(k-j-1)/k}(V) + U' \subset U. \end{aligned}$$

Finally, for $N \geqslant N_0 + N_1$ we have

$$\begin{aligned} &\mu\big(\{t \leqslant 2N\colon\ T_t x - y \in U \text{ and } t \in I_{k-1}\}\big) \\ &\quad \geqslant \mu\left(\bigcup_{j=0}^{k-1} f_j(L_j)\right) \geqslant \sum_{j=0}^{k-1} \mu\big(f_j(L_j)\big)/k \\ &\quad = \sum_{j=0}^{k-1} \mu(L_j)/k = \mu(L)/k \geqslant CN/k. \end{aligned}$$

Hence $\underline{Dens}(\{t \in I_{k-1}\colon\ T_t x - y \in U\}) > 0$, and we are done. □

Theorem 3.2. *Let $\mathcal{T} = \{T_t\}_{t \geqslant 0}$ be a frequently hypercyclic semigroup in $L(X)$, and let $x \in FHC(\mathcal{T})$. Then $x \in FHC(T_{t_0})$ for every $t_0 > 0$.*

Proof. Without loss of generality, we may again assume that $t_0 = 1$ as in the proof of Theorem 2.3. Fix $y \in X$, $U \in \mathcal{U}_0(X)$, and select $k \in \mathbb{N}$, $U' \in \mathcal{U}_0(X)$, such that $U' + U' \subset U$ and $T_t y - y \in U'$ for every $0 \leqslant t \leqslant 1/k$. Since $\mathcal{T}$ is strongly continuous there is some $V \in \mathcal{U}_0(X)$ such that $T_t(V) \subset U'$ for every $0 \leqslant t \leqslant 1/k$. By the previous lemma, we know that $\underline{Dens}(\{t \in \bigcup_{n \in \mathbb{N}} [n - 1/k, n)\colon\ T_t x - y \in V\}) > 0$.

If $t \in [n - 1/k, n)$ for some $n \in \mathbb{N}$ and $T_t x - y \in V$, then we define $\eta_t := [t] + 1 - t$. Each η_t satisfies $0 < \eta_t \leqslant 1/k$, and $t + \eta_t \in \mathbb{N}$. So

$$T_{t+\eta_t} x - y = T_{\eta_t}(T_t x - y) + (T_{\eta_t} y - y) \in T_{\eta_t}(V) + U' \subset U.$$

Hence

$$\underline{dens}\big(\{n \in \mathbb{N}\colon\ T_n x - y \in U\}\big) \geqslant \underline{Dens}\left(\left\{t \in \bigcup_{n \in \mathbb{N}} [n - 1/k, n)\colon\ T_t x - y \in V\right\}\right) > 0. \qquad \square$$

Acknowledgment

The paper was written during the second author's stay at the Universitat Politècnica de València. He would like to express his gratitude for warm hospitality and perfect working conditions there.

References

[1] S.I. Ansari, Hypercyclic and cyclic vectors, J. Funct. Anal. 128 (2) (1995) 374–383.
[2] S.I. Ansari, Existence of hypercyclic operators on topological vector spaces, J. Funct. Anal. 148 (2) (1997) 384–390.
[3] M. Aubry, Homotopy Theory and Models, Birkhäuser-Verlag, Basel, 1995.
[4] C. Badea, S. Grivaux, Unimodular eigenvalues, uniformly distributed sequences and linear dynamics, Adv. Math., in press.
[5] F. Bayart, S. Grivaux, Hypercyclicité: le rôle du spectre ponctuelunimodulaire, C. R. Math. Acad. Sci. Paris 338 (2004) 703–708.
[6] F. Bayart, S. Grivaux, Frequently hypercyclic operators, Trans. Amer. Math. Soc. 358 (2006) 5083–5117.
[7] T. Bermúdez, A. Bonilla, A. Martinón, On the existence of chaotic and hypercyclic semigroups in Banach spaces, Proc. Amer. Math. Soc. 131 (8) (2003) 2435–2441.
[8] L. Bernal-González, On hypercyclic operators on Banach spaces, Proc. Amer. Math. Soc. 127 (4) (1999) 1003–1010.
[9] J. Bonet, A. Peris, Hypercyclic operators on non-normable Fréchet spaces, J. Funct. Anal. 159 (2) (1998) 587–595.
[10] J. Bonet, F. Martínez-Giménez, A. Peris, Linear chaos on Fréchet spaces, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (7) (2003) 1649–1655.
[11] A. Bonilla, K.G. Grosse-Erdmann, Frequently hypercyclic operators, Ergodic Theory Dynam. Systems, in press.
[12] J.A. Conejero, Operadores y semigrupos de operadores en espacios de Fréchet y espacios localmente convexos, PhD thesis, Universidad Politécnica de Valencia, 2004.
[13] J.A. Conejero, On the existence of transitive and topologically mixing semigroups, Bull. Belg. Math. Soc. Simon Stevin, in press.
[14] J.A. Conejero, A. Peris, Every operator in a hypercyclic semigroup is hypercyclic, manuscript, 2003.
[15] G. Costakis, A. Peris, Hypercyclic semigroups and somewhere dense orbits, C. R. Math. Acad. Sci. Paris 335 (2002) 895–898.
[16] R. deLaubenfels, H. Emamirad, V. Protopopescu, Linear chaos and approximation, J. Approx. Theory 105 (1) (2000) 176–187.
[17] W. Desch, W. Schappacher, Discrete subsemigroups of hypercyclic C_0-semigroups are hypercyclic, Ergodic Theory Dynam. Systems 26 (2006) 87–92.
[18] W. Desch, W. Schappacher, G.F. Webb, Hypercyclic and chaotic semigroups of linear operators, Ergodic Theory Dynam. Systems 17 (1997) 1–27.
[19] H. Emamirad, Hypercyclicity in the scattering theory for linear transport equation, Trans. Amer. Math. Soc. 350 (1998) 3707–3716.
[20] K.J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Grad. Texts in Math., vol. 194, Springer-Verlag, New York, 2000.
[21] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. 36 (3) (1999) 345–381.
[22] K.G. Grosse-Erdmann, Recent developments in hypercyclicity, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 97 (2) (2003) 273–286.
[23] K.G. Grosse-Erdmann, A. Peris, Frequently dense orbits, C. R. Math. Acad. Sci. Paris 341 (2005) 123–128.
[24] T. Kalmes, On chaotic C_0-semigroups and infinitely regular hypercyclic vectors, Proc. Amer. Math. Soc. 134 (2006) 2997–3002.
[25] T. Kalmes, J. Wengenroth, personal communication.
[26] F. León-Saavedra, V. Müller, Rotations of hypercyclic and supercyclic operators, Integral Equations Operator Theory 50 (2004) 385–391.
[27] J.C. Oxtoby, S.M. Ulam, Measure-preserving homeomorphisms and metrical transitivity, Ann. of Math. 42 (4) (1941) 874–920.
[28] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1992.
[29] V. Protopopescu, Y. Azmy, Topological chaos for a class of linear models, Math. Models Methods Appl. Sci. 2 (1992) 79–90.
[30] J. Wengenroth, Hypercyclic operators on non-locally convex spaces, Proc. Amer. Math. Soc. 131 (6) (2002) 1759–1761.