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Journal of Complexity 20 (2004) 46–74
Journal of COMPLEXITY
http://www.elsevier.com/locate/jco

Finite-order weights imply tractability of multivariate integration

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Received 18 August 2003; accepted 13 November 2003

Abstract

Multivariate integration of high dimension s occurs in many applications. In many such applications, for example in finance, integrands can be well approximated by sums of functions of just a few variables. In this situation the superposition (or effective) dimension is small, and we can model the problem with *finite-order weights*, where the weights describe the relative importance of each distinct group of variables up to a given order (where the order is the number of variables in a group), and ignore all groups of variables of higher order.

In this paper we consider multivariate integration for the anchored and unanchored (non-periodic) Sobolev spaces equipped with finite-order weights. Our main interest is tractability and strong tractability of QMC algorithms in the worst-case setting. That is, we want to find how the minimal number of function values needed to reduce the initial error by a factor ε depends on s and ε^{-1}. If there is no dependence on s, and only polynomial dependence on ε^{-1}, we have strong tractability, whereas with polynomial dependence on both s and ε^{-1} we have tractability.

We show that for the anchored Sobolev space we have strong tractability for *arbitrary* finite-order weights, whereas for the unanchored Sobolev space we have tractability for all *bounded* finite-order weights. In both cases, the dependence on ε^{-1} is quadratic. We can improve the dependence on ε^{-1} at the expense of polynomial dependence on s. For finite-order weights, we may achieve almost linear dependence on ε^{-1} with a polynomial dependence on s whose degree is proportional to the order of the weights.

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doi:10.1016/j.jco.2003.11.003

We show that these tractability bounds can be achieved by shifted lattice rules with generators computed by the component-by-component (CBC) algorithm. The computed lattice rules depend on the weights. Similar bounds can also be achieved by well-known low discrepancy sequences such as Halton, Sobol and Niederreiter sequences which do not depend on the weights. We prove that these classical low discrepancy sequences lead to error bounds with almost linear dependence on n^{-1} and polynomial dependence on d. We present explicit worst-case error bounds for shifted lattice rules and for the Niederreiter sequence. Better tractability and error bounds are possible for finite-order weights, and even for general weights if they satisfy certain conditions. We present conditions on general weights that guarantee tractability and strong tractability of multivariate integration.

MSC: 65C05; 65D30

Keywords: Lattice rules; Multivariate integration; QMC algorithms; Tractability

1. Introduction

In mathematical modeling of many problems, it is observed that although the number of input variables can be very large, functions mainly depend on groups of just a few variables at a time. For instance, the functions arising in finance often depend on groups of two or three variables, see [1,23,24], in the sense that a function of $\mathbf{x} = (x_1, x_2, \ldots, x_s)$ with large or very large s can be well approximated by

$$f(\mathbf{x}) = \sum_{u \subseteq \{1,\ldots,s\},\ |u|=q^*} f_u(\mathbf{x}_u) \tag{1}$$

with a relatively small value q^*. Here $\mathbf{x}_u$ is the $|u|$-dimensional vector of components x_j of the vector $\mathbf{x}$ for $j \in u$, and f_u is a function of $|u|$ variables. This means that f can be represented as a sum of $\binom{s}{q^*}$ functions, each of which is a function of at most q^* variables. It is often said that such a function f has *superposition* (or effective) dimension q^*. If q^* is small, say 2 or 3, then f has small superposition dimension even though its nominal dimension s can be arbitrarily large.

Assume for convenience that all f_u in (1) have continuous mixed first partial derivatives. Then f has continuous mixed first derivatives but the only non-zero partial derivatives $(\prod_{j \in u} \frac{\partial}{\partial x_j}) f$ are those for which $|u| \leqslant q^*$. In this case, we should embed f into a space H_s of functions of superposition dimension q^*. In this paper we achieve this by choosing H_s as the anchored or unanchored Sobolev space equipped with *finite-order weights* whose order corresponds to the superposition dimension; see Section 2 for formal definitions. Here we only mention that for any $u \subseteq \{1, \ldots, s\}$, we assign a weight $\gamma_{s,u}$ that moderates the importance of the term f_u. As in [2], we say that the weights $\{\gamma_{s,u}\}$ are *finite-order* if there exists an integer q such that

$$\gamma_{s,u} = 0 \quad \text{for all } s \text{ and for all } u \text{ with } |u| > q, \tag{2}$$

and say that the *order* is q^* if q^* is the smallest integer q with this property. We believe that finite-order weights capture the essence of many high-dimensional problems of computational importance, and that in many cases q^* is small.

We study the problem of multivariate integration over the s-dimensional unit cube for functions from the space H_s:

$$I_s(f) := \int_{[0,1]^s} f(\mathbf{x})\, d\mathbf{x}, \quad \forall f \in H_s.$$

The dimension s can be very large. For instance, dimensions of hundreds or thousands are common in computational finance, see [12]. For large s, classical methods based on the Cartesian product of a one-dimensional integration rule such as trapezoidal rule, Simpson's rule, Gaussian rule, etc., are not efficient due to the curse of dimensionality. That is, to guarantee that the error is at most ε we must compute roughly $\varepsilon^{-s/r}$ function values, where r is the smoothness of integrands from H_s. In our case $r = 1$ and we have exponential dependence on s.

In this situation the Monte Carlo (MC) and quasi-Monte Carlo (QMC) algorithms become the only known viable numerical methods for high-dimensional integration. These algorithms take the average of function values over selected points as the approximation of the integral:

$$Q_{n,s}(f) := Q_{n,s}(f; P_n) := \frac{1}{n} \sum_{k=0}^{n-1} f(\mathbf{x}_k), \tag{3}$$

where $P_n := \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$ is a set of random points for MC, and a set of deterministic points for QMC. The advantage of MC is that for square-integrable functions their convergence order $O(n^{-1/2})$ is independent of the dimension s. However, this rate of convergence is slow, and the implied factor in the big O notation may depend exponentially on s, see [19].

Some QMC algorithms have proved to be very efficient for high-dimensional integration, as reported in many papers, see for instance [12]. Furthermore, they converge with the improved rate of convergence $O(n^{-1+\delta})$ for any positive δ and with the implied factor dependent on δ and independent of s. This holds for weighted Sobolev classes, and with even faster convergence possible for weighted Korobov spaces of periodic smooth functions, see e.g., [2,18].

An important example of QMC algorithms is given by lattice rules, see [9,13]. For example, a rank-1 lattice rule has the form

$$Q_{n,s}(f) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(\left\{\frac{k\mathbf{z}}{n}\right\}\right), \tag{4}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_s)$ is the generating vector with no factor in common with n, and the notation $\{\mathbf{x}\}$ means the vector whose jth component is the fractional part of

x_j. A shifted rank-1 lattice rule has the form

$$Q_{n,s}(f) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\left\{\frac{k\mathbf{z}}{n} + \mathbf{\Delta}\right\}\right), \tag{5}$$

where $\mathbf{\Delta} \in [0,1)^s$ is the shift.

Other important QMC algorithms are the rules based on digital nets, such as the Sobol [20] or Niederreiter point sets [9]. The convergence order of lattice rules or digital nets for functions from a Korobov or Sobolev class is of the form $O(n^{-a}(\log_2 n)^b)$ for some $a > \frac{1}{2}$ and b of order s. The $(\log_2 n)^b$ factor can be overwhelming, and the implied factor can also depend exponentially on s, thus when s is large the error bound is only useful for extremely large n.

It is important to know *when* and *why* QMC works or does not work for relatively small n, i.e., in the non-asymptotic regime, especially for high dimensions. It is also important to characterize function classes for which QMC is better than MC, and to construct QMC point sets that are efficient for functions of such classes. Such questions are studied in the field of information-based complexity, see [21] for a survey.

Here we study the worst-case error for f in the unit ball of H_s. Tractability of multivariate integration means that we need only polynomially many in s and ε^{-1} function evaluations to reduce the initial error by a factor ε, whereas strong tractability means that the number of function values has a bound independent of s and polynomially dependent on ε^{-1}. By the initial error we mean the worst-case error when no function evaluations are allowed, which is just the norm of the integration functional I_s in H_s. The tractability of multivariate integration has been extensively studied recently. For example, non-constructive results can be found in [16,17] and constructive results in [7,14,22,25].

Tractability usually does not hold in the Hilbert case for spaces such as classical Sobolev or Korobov spaces where all variables play the same role, see [11]. To obtain tractability we must have *weights* that moderate the importance of successive variables or of different groups of variables. This corresponds to *weighted* spaces, and in particular, we have weighted Sobolev or Korobov spaces. The first results were for *product* weights, where each variable x_j was moderated by the weight $\gamma_{s,j}$, sometimes with no dependence on s, i.e., $\gamma_{s,j} = \gamma_j$, and the groups of variables were moderated by $\gamma_{s,u} = \prod_{j\in u}\gamma_{s,j}$, see [16] and the survey paper [10]. Usually, we have tractability or strong tractability for spaces with product weights iff

$$\sup_s \frac{\sum_{j=1}^s \gamma_{s,j}}{\log_2(1+s)} < \infty \quad \text{or} \quad \sup_s \sum_{j=1}^s \gamma_{s,j} < \infty,$$

respectively. For product weights the underlying function spaces H_s are *tensor product* Hilbert spaces, i.e., $H_s = H_{s,1} \otimes H_{s,2} \otimes \cdots \otimes H_{s,s}$ with $H_{s,j}$ being a Hilbert space of univariate functions. We stress that product weights do not always capture well the spaces of functions with small superposition dimension.

To describe the most general situation it is desirable to allow general weights $\gamma_{s,u}$ that describe the relative importance of each distinct subset u of the variables, and that may depend on the dimension s. General weights have been used previously for Sobolev spaces in [7,8] and for Korobov spaces in [2]. Such an approach gives much more freedom to study problems of different character. In particular, special choices of the weights such as finite-order weights allow modeling of situations where only some specific combinations of variables are important.

The anchored and unanchored Sobolev spaces considered here are defined for general weights and are generalizations of spaces considered for product weights in [3,5,8]. For the anchored spaces, we show strong tractability for *arbitrary* finite-order weights, whereas for the unanchored Sobolev space, we show tractability for bounded but otherwise *arbitrary* finite-order weights. The reason for these different results can be explained by the difference between their initial errors. For the anchored case, the initial error grows with the weights, see (11), and the ratio of the worst-case error of an efficient QMC algorithm to the initial error is independent of finite-order weights, and is bounded by an exponential function of the order q^*. We prove that exponential dependence on the order is present for any QMC algorithm. However, as long as q^* is relatively small, we can tolerate an exponential dependence on q^*. For the unanchored case, the initial error is always 1, independently of the choice of weights, and the worst-case error of an efficient QMC algorithm is polynomial in s only if the finite-order weights are bounded.

The results we mentioned above hold with tractability bounds depending on ε^{-2}. We can improve the dependence on ε^{-1} at the expense of polynomial dependence on s. For finite-order weights, we may achieve almost linear dependence on ε^{-1} with a polynomial dependence on s having degree proportional to q^*.

We show also that the tractability bounds can be achieved by shifted lattice rules with generators computed by a component-by-component (CBC) algorithm, see [14,15] for product weights and [2] for general weights. We stress that the CBC algorithm depends on the weights.

For given finite-order weights, the cost of computing the generator by the CBC algorithm is polynomial in s and ε^{-1}. We stress that the shifted lattice rules from the CBC algorithm are not fully constructive since for the computed generator we only know that a good shift $\mathbf{\Delta}$ exists but do not know how to efficiently compute a good $\mathbf{\Delta}$.

Similar tractability bounds can also be achieved by well-known low discrepancy sequences, such as Halton, Sobol and Niederreiter sequences, which have the additional attraction that they do not depend on the weights. We prove that these classical low discrepancy sequences lead to tractability error bounds with almost linear dependence on ε^{-1} and polynomial dependence on s for arbitrary finite-order weights. We present explicit worst-case error bounds for the Niederreiter sequence as well as for shifted lattice rules from the CBC algorithm.

We also present conditions on general weights, which are sufficient to obtain tractability or strong tractability of multivariate integration in the anchored and unanchored Sobolev spaces.

The results on finite-order weights help to explain why QMC algorithms are so efficient for high-dimensional integration. We hope that finite-order weights will also lead to interesting tractability or strong tractability results for other multivariate problems.

This paper is organized as follows. In Section 2, we introduce the anchored and unanchored Sobolev spaces with general weights. In Section 3, sufficient conditions on tractability and strong tractability are established for general weights. In addition, the shifted lattice rules constructed by the CBC algorithm are used to establish improved tractability or strong tractability error bounds, with possibly the optimal convergence order $O(n^{-1+\delta})$ for arbitrary $\delta > 0$. In Section 4 we study finite-order weights. For such weights the shifted lattice rules constructed by the CBC algorithm achieve tractability or strong tractability error bounds under weak conditions, or even under no conditions on the weights. Lower bounds on the normalized error are also studied. In Section 5, it is shown that QMC algorithms based on some well-known low discrepancy point sets, such as the Niederreiter point set, achieve the optimal convergence $O(n^{-1+\delta})$ either independently of the dimension, or polynomially dependent on the dimension, for the case of finite-order weights.

2. Weighted Sobolev spaces with general weights

Let H_s be a Hilbert space of functions defined on $[0,1]^s$ with norm $||\cdot||_{H_s}$. We assume that I_s is a continuous linear functional in H_s. Since our Hilbert spaces H_s must allow point evaluation of functions $f \in H_s$, we restrict ourselves to reproducing kernel Hilbert spaces.

Define the worst-case error of the algorithm $Q_{n,s}(f;P_n)$, with $n \geqslant 1$, by its worst-case performance over the unit ball of H_s:

$$e(P_n;H_s) := \sup\{\, |I_s(f) - Q_{n,s}(f;P_n)| : f \in H_s, ||f||_{H_s} \leqslant 1 \,\}.$$

For $n = 0$, we do not sample the function, and define the initial error as

$$e(0;H_s) := \sup\{|I_s(f)| : f \in H_s, ||f||_{H_s} \leqslant 1\} = ||I_s||.$$

For $\varepsilon \in (0,1)$, let $n(\varepsilon, H_s)$ be the smallest n for which there exists an algorithm $Q_{n,s}(f;P_n)$ such that $e(P_n;H_s) \leqslant \varepsilon e(0;H_s)$. Multivariate integration in spaces H_s is said to be *QMC-tractable* if there are non-negative numbers C, p and q such that

$$n(\varepsilon, H_s) \leqslant C\varepsilon^{-p} s^q \quad \forall \varepsilon \in (0,1) \text{ and } \forall s \geqslant 1. \tag{6}$$

The numbers p and q are called ε- and *s-exponents of QMC-tractability*; we stress that they are not defined uniquely. Multivariate integration is said to be *QMC-strongly tractable* if $q = 0$ in (6). The infimum of the numbers p satisfying (6) with $q = 0$ is called the *ε-exponent of QMC-strong tractability*.

In this paper we restrict ourselves to QMC algorithms, defined in (3), the study of tractability for more general algorithms being left for future research. To simplify the description, we will be using tractability and strong tractability as shortened versions of QMC-tractability and QMC-strong tractability.

Specifically, we consider the reproducing kernel Hilbert spaces, denoted by $H(K_s)$, with the reproducing kernel

$$K_s(\mathbf{x}, \mathbf{y}) = \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} \eta_j(x_j, y_j), \tag{7}$$

where

$$\eta_j(x, y) = \tfrac{1}{2} B_2(\{x - y\}) + (x - \tfrac{1}{2})(y - \tfrac{1}{2}) + \mu_j(x) + \mu_j(y) + m_j, \tag{8}$$

and $\gamma_{s,u}$ are arbitrary non-negative numbers. We can also allow some $\gamma_{s,u}$ to be zero, by taking the limiting case of positive $\gamma_{s,u}$. In (7), we use the convention that the product for $u = \emptyset$ is taken as 1, and without loss of generality we assume that $\gamma_{s,\emptyset} = 1$.

In (8), $B_2(x) := x^2 - x + \frac{1}{6}$ is the Bernoulli polynomial of degree 2, μ_j is a function with bounded derivative in $[0, 1]$ such that $\int_0^1 \mu_j(x)\,dx = 0$, and the number m_j is given by

$$m_j := \int_0^1 (\mu_j'(x))^2\,dx.$$

We are interested in the following two choices for the function $\mu_j(x)$ in (8):

- (A): $\mu_j(x) = \max(x, a_j) - \frac{1}{2}x^2 - \frac{1}{2}a_j^2 - \frac{1}{3}$, with arbitrary $a_j \in [0, 1]$.
- (B): $\mu_j(x) = 0,\ j = 1, \dots, s$.

These two choices lead to two different kinds of Sobolev spaces:

- Choice (A) leads to an *anchored Sobolev kernel*, denoted by $K_{s,\mathrm{A}}$, and is given by (7) with

  $$\eta_j(x, y) = \begin{cases} \min(|x - a_j|, |y - a_j|) & \text{for } (x - a_j)(y - a_j) > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

  This reproducing kernel Hilbert space is called the *anchored Sobolev space*, and is denoted by $H(K_{s,\mathrm{A}})$. Note that $\eta_j(a_j, y) = \eta_j(x, a_j) = 0$. The point $\mathbf{a} = (a_1, \dots, a_s)$ is called the *anchor*. In this case, $m_j = a_j^2 - a_j + \frac{1}{3}$. Clearly,

  $$\tfrac{1}{12} \leqslant m_j \leqslant \tfrac{1}{3}.$$

- Choice (B) leads to an *unanchored Sobolev kernel*, denoted by $K_{s,\mathrm{B}}$, and is given by (7) with

  $$\eta_j(x, y) = \tfrac{1}{2} B_2(\{x - y\}) + (x - \tfrac{1}{2})(y - \tfrac{1}{2})$$

  since $m_j = 0$. Note that $\int_0^1 \eta_j(x, y)\,dy = 0$ for all $y \in [0, 1]$.

  This reproducing kernel Hilbert space is called the *unanchored Sobolev space*, and is denoted by $H(K_{s,\mathrm{B}})$.

The anchored and unanchored Sobolev spaces defined above are generalizations of the reproducing kernel Hilbert spaces considered in [3,5]. The difference is that we now allow the weights $\gamma_{s,u}$ to depend not only on the dimension s, but also on each group of variables u. When the weights $\gamma_{s,u}$ are *product* weights, i.e.,

$$\gamma_{s,u} = \prod_{j \in u} \gamma_{s,j} \quad \text{for some } \{\gamma_{s,j}\}, \tag{10}$$

then kernel (7) can be written as

$$K_s(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s} (1 + \gamma_{s,j}\, \eta_j(x_j, y_j)),$$

where here and elsewhere in the paper we use the obvious identity

$$\prod_{j \in v} (b_j + c_j) = \sum_{u \subseteq v} \prod_{k \in v \setminus u} b_k \prod_{j \in u} c_j \quad \forall\ b_j, c_j \in \mathbb{R}.$$

The space $H(K_s)$ is in this case the tensor product of spaces of univariate functions. Such function spaces have been studied in many papers.

For general weights $\{\gamma_{s,u}\}$, it can be checked, as in the case of product weights, that the inner product in the space $H(K_{s,\mathrm{A}})$ is

$$(f,g) = \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u}^{-1} \int_{[0,1]^{|u|}} \frac{\partial^{|u|} f(\mathbf{x}_u, \mathbf{a}_{-u})}{\partial \mathbf{x}_u} \frac{\partial^{|u|} g(\mathbf{x}_u, \mathbf{a}_{-u})}{\partial \mathbf{x}_u}\, d\mathbf{x}_u,$$

where $|u|$ denotes the cardinality of u, $\mathbf{x}_u$ denotes the $|u|$-dimensional vector of components x_j with $j \in u$, and $\mathbf{x}_{-u}$ denotes the vector $\mathbf{x}_{\{1,\dots,s\}\setminus u}$; moreover $(\mathbf{x}_u, \mathbf{a}_{-u})$ denotes an s-dimensional vector whose jth component is x_j if $j \in u$ and a_j if $j \notin u$. For $u = \emptyset$, we use the convention that $\int_{[0,1]^\emptyset} f(\mathbf{x}_\emptyset, \mathbf{a}_{-\emptyset})\, d\mathbf{x}_\emptyset = f(\mathbf{a})$.

For the space $H(K_{s,\mathrm{B}})$ with general weights, it can be checked, as in the case of product weights, that the inner product is

$$(f,g) = \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u}^{-1} \int_{[0,1]^{|u|}} \left(\int_{[0,1]^{s-|u|}} \frac{\partial^{|u|} f(\mathbf{x})}{\partial \mathbf{x}_u}\, d\mathbf{x}_{-u} \right) \left(\int_{[0,1]^{s-|u|}} \frac{\partial^{|u|} g(\mathbf{x})}{\partial \mathbf{x}_u}\, d\mathbf{x}_{-u} \right) d\mathbf{x}_u,$$

with the term corresponding to $u = \emptyset$ interpreted as $\int_{[0,1]^s} f(\mathbf{x})\, d\mathbf{x} \int_{[0,1]^s} g(\mathbf{x})\, d\mathbf{x}$.

The difference between the inner products is that for terms indexed by u, the components of $\mathbf{x}$ not in u are anchored at $\mathbf{a}$ for the space $H(K_{s,\mathrm{A}})$ while the same components are integrated over $[0,1]$ for the space $H(K_{s,\mathrm{B}})$.

Obviously, I_s is well defined for $H(K_s)$, where here and later K_s represents either the anchored Sobolev kernel $K_{s,\mathrm{A}}$ or the unanchored Sobolev kernel $K_{s,\mathrm{B}}$. Due to the linearity of $I_s - Q_{n,s}$, we have the error bound

$$|I_s(f) - Q_{n,s}(f)| \leqslant e(P_n; H(K_s)) \|f\|_{H(K_s)} \quad \forall f \in H(K_s).$$

As is well known, the square worst-case error can be written in terms of the reproducing kernel

$$e^2(P_n; H(K_s)) = \int_{[0,1]^{2s}} K_s(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} - \frac{2}{n} \sum_{k=0}^{n-1} \int_{[0,1]^s} K_s(\mathbf{x}_k, \mathbf{x})\, d\mathbf{x} + \frac{1}{n^2} \sum_{k,l=0}^{n-1} K_s(\mathbf{x}_k, \mathbf{x}_l).$$

The square of the initial error $e^2(0; H(K_s))$ is given by

$$e^2(0; H(K_s)) = \int_{[0,1]^{2s}} K_s(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} = \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} m_j. \tag{11}$$

In particular, for the unanchored Sobolev space $H(K_{s,\mathrm{B}})$ the initial error is always $\gamma_{s,\emptyset} = 1$ independently of the weights.

3. Tractability for general weights

In this section we study the existence of efficient QMC algorithms that achieve tractability or strong tractability error bounds for weighted Sobolev spaces with general weights. We also show that a suitable adaptation of the component-by-component (CBC) algorithm for non-product weights proposed in [2] achieves tractability error bounds.

3.1. Existence of efficient QMC algorithms

We first prove the existence of efficient QMC algorithms by an averaging argument. Let K_s be the anchored Sobolev kernel $K_{s,\mathrm{A}}$ or the unanchored Sobolev kernel $K_{s,\mathrm{B}}$. It is known that the square of the *average worst-case error* for the Hilbert space $H(K_s)$ given by

$$(e_{n,s}^{\mathrm{avg}})^2 := \int_{[0,1]^{ns}} e^2(\{\mathbf{x}_i\}; H(K_s))\, d\mathbf{x}_0 \cdots d\mathbf{x}_{n-1},$$

has the explicit expression

$$(e_{n,s}^{\mathrm{avg}})^2 = \frac{1}{n} \left(\int_{[0,1]^s} K_s(\mathbf{x}, \mathbf{x})\, d\mathbf{x} - \int_{[0,1]^{2s}} K_s(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \right).$$

In our case with $K = K_s$ this expression becomes

$$(e_{n,s}^{\mathrm{avg}})^2 = \frac{1}{n} \left(\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} \left(m_j + \frac{1}{6} \right) - \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} m_j \right). \tag{12}$$

Using the mean value theorem, we easily deduce the following theorem. (Recall that in the definition of tractability and strong tractability we compare the worst-case error with the initial error, which is given by (11).)

Theorem 1. (A) *Consider the anchored Sobolev space* $H(K_{s,\mathrm{A}})$ *with an arbitrary anchor* $\mathbf{a}$ *and arbitrary weights* $\{\gamma_{s,u}\}$. *Then there exists a point set* P_n *for which*

$$e(P_n; H(K_{s,\mathrm{A}})) \leqslant \frac{1}{\sqrt{n}} \left(\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} \left(m_j + \frac{1}{6} \right) - \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} m_j \right)^{1/2}.$$

Therefore, if

$$\sup_{s=1,2,\dots} \left(\frac{\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} (m_j + 1/6)}{\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} m_j} \right) < \infty,$$

then the multivariate integration problem in spaces $H(K_{s,\mathrm{A}})$ *is strongly tractable with* ε-*exponent at most* 2.

(B) *Consider the unanchored Sobolev space* $H(K_{s,\mathrm{B}})$ *with arbitrary weights* $\{\gamma_{s,u}\}$. *Then there exists a point set* P_n *for which*

$$e(P_n; H(K_{s,\mathrm{B}})) \leqslant \frac{1}{\sqrt{n}} \left(\sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} 6^{-|u|} \right)^{1/2}.$$

Therefore, if

$$\sup_{s=1,2,\dots} \left(\sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} 6^{-|u|} \right) < \infty,$$

then multivariate integration problem in spaces $H(K_{s,\mathrm{B}})$ *is strongly tractable with* ε-*exponent at most* 2.

Note that if the anchor is $\mathbf{a} = (1, \dots, 1)$, then Theorem 1(A) reduces to a result in [8]. If the weights $\{\gamma_{s,u}\}$ are product, see (10), then Theorem 1 reduces to the results in [3]; furthermore, if the product weights $\{\gamma_{s,j}\}$ are independent of the dimension s, i.e., $\gamma_{s,j} = \gamma_j$, then Theorem 1 reduces to the results in [16] for the anchored space with anchor $\mathbf{a} = (1, \dots, 1)$, and reduces to results in [18] for the unanchored space.

3.2. Results for weighted Korobov spaces

Theorem 1 indicates the existence of a QMC algorithm whose convergence order is $n^{-1/2}$, which is known to be not optimal. To improve the convergence order we will present a shifted lattice rule which allows us to obtain even an optimal order of convergence. To do this, we need to recall some facts and results for *weighted Korobov* spaces of periodic functions defined on $[0,1]^s$. The weighted Korobov space is a Hilbert space with the reproducing kernel

$$K_{s,\beta,\alpha}(\mathbf{x},\mathbf{y}) = 1 + \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \beta_{s,u} \prod_{j\in u} \left({\sum_{h=-\infty}^{\infty}}' \frac{e^{2\pi i h(x_j-y_j)}}{|h|^\alpha} \right). \tag{13}$$

Here $\alpha > 1$ is a smoothness parameter, and $\beta := \{\beta_{s,u}\}$ is a weight sequence with non-negative weights $\beta_{s,u}$. The prime on the sum indicates that the $h=0$ term is omitted. The inner product of the weighted Korobov space is

$$(f,g) = \sum_{\mathbf{h}\in\mathbf{Z}^s} r_\alpha(\beta,\mathbf{h}) \hat{f}(\mathbf{h})\, \overline{\hat{g}(\mathbf{h})},$$

where $\mathbf{Z}^s$ stands for s-dimensional integer vectors, $\hat{f}(\mathbf{h})$ denotes the Fourier coefficient,

$$\hat{f}(\mathbf{h}) = \int_{[0,1]^s} f(\mathbf{x}) \exp(-2\pi\, i\, \mathbf{h}\cdot\mathbf{x})\, d\mathbf{x}$$

and

$$r_\alpha(\beta,\mathbf{h}) = \begin{cases} 1 & \text{if } \mathbf{h} = \mathbf{0}, \\ \beta_{s,u_\mathbf{h}}^{-1} \prod_{j\in u_\mathbf{h}} |h_j|^\alpha & \text{if } \mathbf{h} \neq \mathbf{0}, \end{cases}$$

with $u_\mathbf{h} := \{j : h_j \neq 0\}$. The weighted Korobov space is denoted by $H(K_{s,\beta,\alpha})$.

Multivariate integration for the space $H(K_{s,\beta,\alpha})$ can be solved by a component-by-component (CBC) algorithm, see [15] for product weights and [2] for general weights. This algorithm constructs a generator of the lattice rule as follows:

Component-by-component (CBC) algorithm

Suppose n is a prime number and suppose the weights $\{\beta_{s,u}\}$ *are given. The generator* $\bar{\mathbf{z}} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_s)$ *is found as follows*:

1. *Set the first component* $\bar{z}_1$ *to* 1.
2. *For* $t = 2, 3, \dots, s$ *and known* $\bar{z}_1, \dots, \bar{z}_{t-1}$, *find* $\bar{z}_t \in \{1, \dots, n-1\}$ *such that the square of the worst-case error*,

$$e^2(P(1,\bar{z}_2,\dots,\bar{z}_{t-1},\bar{z}_t); H(K_{t,\beta,\alpha})) = \frac{1}{n}\sum_{k=0}^{n-1} \sum_{\emptyset\neq u\subseteq\{1,\dots,t\}} \beta_{s,u} \prod_{j\in u} \left(\sum_{h=-\infty}^{\infty} \frac{e^{2\pi i h k \bar{z}_j/n}}{|h|^\alpha} \right) \tag{14}$$

is minimized, where $P(1,\bar{z}_2,\dots,\bar{z}_t)$ *is the rank*-1 *lattice point set with the generator* $(1,\bar{z}_2,\dots,\bar{z}_t)$.

The cost of the CBC algorithm is exponential in s for arbitrary weights. Indeed, we have to sum $2^t - 1$ terms as part of Step 2 of the algorithm. The total cost would require $O(s2^s n)$ operations, making the algorithm impossible to use for large s and n. But the problem is much easier for some special weights, such as order-dependent weights (where the weights depend on u only through the cardinality of u), or finite-order weights, see (2). In these cases, the cost of the CBC algorithm is polynomial in s and n.

It is shown in [2] that for any $\tau \in [1, \alpha)$ the lattice rule constructed by the CBC algorithm satisfies

$$e(P(\bar{\mathbf{z}}); H(K_{s,\beta,\alpha})) \leqslant C(s,\tau)(n-1)^{-\tau/2}, \tag{15}$$

where $C(s,\tau)$ is given by

$$C(s,\tau) = \left(\sum_{\emptyset \neq u \subseteq \{1,\ldots,s\}} \beta_{s,u}^{1/\tau} \, (2\zeta(\alpha/\tau))^{|u|} \right)^{\tau/2}, \tag{16}$$

and $\zeta(x) = \sum_{j=1}^{\infty} j^{-x}$ for $x > 1$ is the Riemann zeta function.

3.3. *Shift-invariant kernels of weighted Sobolev spaces*

To use the CBC algorithm for weighted Sobolev spaces, we need to recall the relationship between weighted Sobolev and weighted Korobov spaces. This is done by using the concept of a shift-invariant kernel, see [8], defined as follows.

For an arbitrary reproducing kernel K, the associated shift-invariant kernel K^{sh} is

$$K^{\mathrm{sh}}(\mathbf{x},\mathbf{y}) := \int_{[0,1]^s} K(\{\mathbf{x}+\boldsymbol{\Delta}\}, \{\mathbf{y}+\boldsymbol{\Delta}\}) \, d\boldsymbol{\Delta}.$$

The kernel K^{sh} is *shift-invariant*, i.e., for arbitrary $\boldsymbol{\Delta} \in [0,1)^s$

$$K^{\mathrm{sh}}(\mathbf{x},\mathbf{y}) = K^{\mathrm{sh}}(\{\mathbf{x}+\boldsymbol{\Delta}\}, \{\mathbf{y}+\boldsymbol{\Delta}\}) \quad \forall \mathbf{x},\mathbf{y} \in [0,1]^s.$$

It is shown in [8] that for a point set $P_n \subseteq [0,1]^s$, we have

$$\int_{[0,1]^s} e^2(P_n+\boldsymbol{\Delta}; H(K)) \, d\boldsymbol{\Delta} = e^2(P_n; H(K^{\mathrm{sh}})), \tag{17}$$

where $P_n + \boldsymbol{\Delta} := \{\{\mathbf{x}_k + \boldsymbol{\Delta}\},\ k = 0, \ldots, n-1\}$. By the mean value theorem, this implies that there exists a shift $\boldsymbol{\Delta} \in [0,1)^s$ such that

$$e(P_n+\boldsymbol{\Delta}; H(K)) \leqslant e(P_n; H(K^{\mathrm{sh}})).$$

Thus $e(P_n; H(K^{\mathrm{sh}}))$ is an upper bound on the value of $e(P_n+\boldsymbol{\Delta}; H(K))$ with a properly chosen shift $\boldsymbol{\Delta}$.

Consider first the unanchored Sobolev space $H(K_{s,\mathrm{B}})$. Its associated shift-invariant kernel can be easily found:

$$\begin{aligned} K_{s,\mathrm{B}}^{\mathrm{sh}}(\mathbf{x},\mathbf{y}) &:= \int_{[0,1]^s} K_{s,\mathrm{B}}(\{\mathbf{x}+\boldsymbol{\Delta}\}, \{\mathbf{y}+\boldsymbol{\Delta}\}) \, d\boldsymbol{\Delta} \\ &= 1 + \sum_{\emptyset \neq u \subseteq \{1,\ldots,s\}} \gamma_{s,u} \prod_{j \in u} B_2(\{x_j - y_j\}) \\ &= 1 + \sum_{\emptyset \neq u \subseteq \{1,\ldots,s\}} \gamma_{s,u}^{\mathrm{B}} \prod_{j \in u} \sum_{h=-\infty}^{\infty} \frac{e^{2\pi i h (x_j - y_j)}}{h^2}, \end{aligned}$$

where

$$\gamma^{\mathrm{B}}_{s,u} = \frac{\gamma_{s,u}}{(2\pi^2)^{|u|}}. \tag{18}$$

Here we used the fact that the Bernoulli polynomial of degree 2 can be expressed as

$$B_2(x) = \frac{1}{2\pi^2}\sum_{h=-\infty}^{\infty} \frac{e^{2\pi i h x}}{h^2}, \quad x \in [0,1].$$

This means that the shift-invariant kernel $K^{\mathrm{sh}}_{s,\mathrm{B}}$ of the unanchored Sobolev space is just the reproducing kernel of the weighted Korobov space with the weights $\beta^{\mathrm{B}} = \{\gamma^{\mathrm{B}}_{s,u}\}$ and $\alpha = 2$, see (13), i.e.,

$$K^{\mathrm{sh}}_{s,\mathrm{B}} \;=\; K_{s,\beta^{\mathrm{B}},2}. \tag{19}$$

For the anchored Sobolev kernel $K_{s,\mathrm{A}}(\mathbf{x},\mathbf{y})$, its associated shift-invariant kernel can also be found after some computation

$$\begin{aligned} K^{\mathrm{sh}}_{s,\mathrm{A}}(\mathbf{x},\mathbf{y}) &= 1 + \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j\in u} \left[B_2(\{x_j - y_j\}) + m_j\right] \\ &= e^2(0; H(K_{s,\mathrm{A}}))\left(1 + \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma^{\mathrm{A}}_{s,u} \prod_{j\in u} \left[2\pi^2 B_2(\{x_j - y_j\})\right]\right), \end{aligned} \tag{20}$$

where

$$\gamma^{\mathrm{A}}_{s,u} = \frac{1}{(2\pi^2)^{|u|}\, e^2(0; H(K_{s,\mathrm{A}}))} \sum_{u \subseteq v \subseteq \{1,\dots,s\}} \gamma_{s,v} \prod_{j \in v\setminus u} m_j. \tag{21}$$

Thus apart from the factor $e^2(0; H(K_{s,\mathrm{A}}))$, the shift-invariant kernel $K^{\mathrm{sh}}_{s,\mathrm{A}}(\mathbf{x},\mathbf{y})$ is just the Korobov reproducing kernel (13) for the weights $\beta^{\mathrm{A}} = \{\gamma^{\mathrm{A}}_{s,u}\}$ and the parameter $\alpha = 2$, i.e.,

$$K^{\mathrm{sh}}_{s,\mathrm{A}} = e^2(0; H(K_{s,\mathrm{A}}))\, K_{s,\beta^{\mathrm{A}},2}. \tag{22}$$

The worst-case errors of any QMC algorithm in the spaces $H(K^{\mathrm{sh}}_{s,\mathrm{A}})$ and $H(K_{s,\beta^{\mathrm{A}},2})$ are therefore related by

$$\frac{e(P_n; H(K^{\mathrm{sh}}_{s,\mathrm{A}}))}{e(0; H(K_{s,\mathrm{A}}))} = e(P_n; H(K_{s,\beta^{\mathrm{A}},2})). \tag{23}$$

The initial error $e(0; H(K_{s,\mathrm{A}}))$ is given by (11).

We summarize the analysis of this subsection in the following lemma.

Lemma 2. *The shift-invariant kernel of the anchored Sobolev kernel* $K_{s,\mathrm{A}}$ *or of the unanchored Sobolev kernel* $K_{s,\mathrm{B}}$ *is related to the weighted Korobov kernel* $K_{s,\beta,2}$ *by* (22) *or* (19), *respectively, with the weights* $\{\beta_{s,u}\}$ *given by*

$$\beta_{s,u} := \begin{cases} \gamma_{s,u}^{\mathrm{A}} & \text{if } K_s = K_{s,\mathrm{A}}, \\ \gamma_{s,u}^{\mathrm{B}} & \text{if } K_s = K_{s,\mathrm{B}}. \end{cases}$$

3.4. Shifted lattice rules with high convergence order

We can now easily combine the constructive results for weighted Korobov spaces from [3], see (15) and (16), with Lemma 2 and relation (17), to obtain the following theorem.

Theorem 3. *Let n be a prime number.*

(A) *Let* P_n^{A} *be the lattice point set with the generator constructed by the CBC algorithm with the weights* $\gamma_{s,u}^{\mathrm{A}}$ *and parameter* $\alpha = 2$. *Then there exists a shift* $\boldsymbol{\Delta} \in [0,1)^s$ *such that for any* $\tau \in [1,2)$

$$\frac{e(P_n^{\mathrm{A}} + \boldsymbol{\Delta}; H(K_{s,\mathrm{A}}))}{e(0; H(K_{s,\mathrm{A}}))} \leqslant \left(\sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} (\gamma_{s,u}^{\mathrm{A}})^{1/\tau} (2\zeta(2/\tau))^{|u|} \right)^{\tau/2} (n-1)^{-\tau/2}.$$

(B) *Let* P_n^{B} *be the lattice point set with the generator constructed by the CBC algorithm with the weights* $\gamma_{s,u}^{\mathrm{B}}$ *and parameter* $\alpha = 2$. *Then there exists a shift* $\boldsymbol{\Delta} \in [0,1)^s$ *such that for any* $\tau \in [1,2)$

$$e(P_n^{\mathrm{B}} + \boldsymbol{\Delta}; H(K_{s,\mathrm{B}})) \leqslant \left(\sum_{\emptyset \neq u \subseteq \{1\dots,s\}} (\gamma_{s,u}^{\mathrm{B}})^{1/\tau} (2\zeta(2/\tau))^{|u|} \right)^{\tau/2} (n-1)^{-\tau/2}.$$

From the last theorem there follows the following tractability theorem.

Theorem 4. (A) *Consider the anchored Sobolev space* $H(K_{s,\mathrm{A}})$ *with an arbitrary anchor* $\mathbf{a}$. *Assume that for some* $\tau \in [1,2)$ *and some* $q \geqslant 0$, *we have*

$$\sup_{s=1,2,\dots} \frac{\left(s^{-q} \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u}^{1/\tau} \prod_{j \in u} (2\zeta(2/\tau)(2\pi^2)^{-1/\tau} + m_j^{1/\tau})\right)^{\tau}}{\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} m_j} < \infty.$$

Then if $q = 0$ *we have strong tractability and the* ε*-exponent is at most* $2/\tau$, *and if* $q > 0$ *we have tractability with* ε*-exponent* $2/\tau$ *and s-exponent* q.

(B) *Consider the unanchored Sobolev space* $H(K_{s,\mathrm{B}})$. *Assume that for some* $\tau \in [1,2)$ *and some* $q \geqslant 0$, *we have*

$$\sup_{s=1,2,\dots} \left(s^{-q} \sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u}^{1/\tau} (2\pi^2)^{-|u|/\tau} (2\zeta(2/\tau))^{|u|} \right) < \infty.$$

Then if $q = 0$ *we have strong tractability and the* ε*-exponent is at most* $2/\tau$, *and if* $q > 0$ *we have tractability with* ε*-exponent* $2/\tau$ *and s-exponent* q.

Proof. Part (B) follows immediately from Theorem 3. To prove Part (A), we know from Theorem 3 there exists a shift $\boldsymbol{\Delta} \in [0,1)^s$ such that

$$\frac{e(P_n^{\mathrm{A}} + \boldsymbol{\Delta}; H(K_{s,\mathrm{A}}))}{e(0; H(K_{s,\mathrm{A}}))} \leqslant C_{\mathrm{A}}(s,\tau)\,(n-1)^{-\tau/2}$$

with

$$C_{\mathrm{A}}(s,\tau) := \left(\sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} (\gamma_{s,u}^{\mathrm{A}})^{1/\tau} (2\zeta(2/\tau))^{|u|} \right)^{\tau/2}.$$

Let $W_\tau = 2\zeta(2/\tau)\,(2\pi^2)^{-1/\tau}$. Inserting expression (21) for $\gamma_{s,u}^{\mathrm{A}}$ into the expression for $C_A(s,\tau)$ and then using Jensen's inequality, we have

$$\begin{aligned} C_{\mathrm{A}}^2(s,\tau) &\leqslant \frac{1}{e^2(0; H(K_{s,\mathrm{A}}))} \left(\sum_{u \subseteq \{1,\dots,s\}} W_\tau^{|u|} \left(\sum_{v:\, u \subseteq v} \gamma_{s,v} \prod_{j \in v \setminus u} m_j \right)^{1/\tau} \right)^{\tau} \\ &\leqslant \frac{1}{e^2(0; H(K_{s,\mathrm{A}}))} \left(\sum_u W_\tau^{|u|} \left(\sum_{v:\, u \subseteq v} \gamma_{s,v}^{1/\tau} \prod_{j \in v \setminus u} m_j^{1/\tau} \right) \right)^{\tau} \\ &= \frac{\left(\sum_u \gamma_{s,u}^{1/\tau} \prod_{j \in u} (2\zeta(2/\tau)\,(2\pi^2)^{-1/\tau} + m_j^{1/\tau})\right)^\tau}{\sum_u \gamma_{s,u} \prod_{j \in u} m_j}. \end{aligned}$$

From this inequality the rest follows immediately. □

Theorems 3 and 4 state that for arbitrarily large s, shifted lattice rules with the generator constructed by the CBC algorithm and a suitable shift achieve a convergence order $n^{-\tau/2}$. If τ can be arbitrarily close to 2, we may achieve almost the same convergence as for the univariate case, which is n^{-1}, and the difficulty of the s-dimensional integration is roughly the same as for the univariate one.

We stress that the CBC algorithm described above is *not* fully constructive, since we only know that there exists a shift for which the generator computed by the CBC algorithm leads to desired error bounds. The simultaneous construction of both a lattice vector and a shift with a polynomial cost is given in [14] for the anchored Sobolev space with $\mathbf{a} = \mathbf{1}$ and for product weights. However the proven convergence rate for this construction is only $n^{-1/2}$. The construction of a shift vector preserving better rates of convergence is open, and left for future research.

4. Tractability for finite-order weights

The theorems of the previous section are for general weights. In particular, we may apply them to the finite-order weights defined in (2). As we shall see, the tractability conditions greatly simplify for finite-order weights, and there are some (positive) surprises.

4.1. Existence of efficient QMC algorithms

We now show that for the anchored Sobolev spaces, strong tractability holds for *arbitrary* finite-order weights. For the unanchored Sobolev space we get tractability, not strong tractability, and only under the additional (reasonable) assumption that the finite-order weights are uniformly bounded.

Theorem 5. (A) *Consider the anchored Sobolev space* $H(K_{s,\mathrm{A}})$. *For arbitrary finite-order weights of order* q^*, *there exists a point set* P_n *such that*

$$\frac{e(P_n;H(K_{s,\mathrm{A}}))}{e(0;H(K_{s,\mathrm{A}}))}\leqslant\frac{1}{\sqrt{n}}\left[\max_{u:\ |u|\leqslant q^*}\prod_{j\in u}\left(1+\frac{1}{6m_j}\right)-1\right]^{1/2}\leqslant\frac{1}{\sqrt{n}}(3^{q^*}-1)^{1/2}.$$

Hence, the minimal number $n(\varepsilon,H(K_{s,\mathrm{A}}))$ *of function evaluation needed to reduce the initial error by a factor* ε *with a QMC algorithm is bounded by*

$$n(\varepsilon,H(K_{s,\mathrm{A}}))\leqslant\lceil\varepsilon^{-2}(3^{q^*}-1)\rceil.$$

Thus for arbitrary finite-order weights we have strong tractability with ε*-exponent at most* 2.

(B) *Consider the unanchored Sobolev space* $H(K_{s,\mathrm{B}})$. *If the finite-order weights* $\{\gamma_{s,u}\}$ *of order* q^* *satisfy* $\gamma_{s,u}\leqslant\Gamma^*$ *for all s and for all* $u\subseteq\{1,\dots,s\}$, *then there exists a point set* P_n *such that*

$$e(P_n;H(K_{s,\mathrm{B}}))\leqslant G(s)n^{-1/2},$$

where

$$G(s)=\left(\Gamma^*\sum_{\ell=1}^{q^*}\binom{s}{\ell}6^{-\ell}\right)^{1/2}=(\Gamma^*)^{1/2}\frac{s^{q^*/2}}{((q^*)!6^{q^*})^{1/2}}(1+O(s^{-1})).$$

Hence,

$$n(\varepsilon,H(K_{s,\mathrm{B}}))\leqslant\lceil\varepsilon^{-2}G^2(s)\rceil.$$

Thus we have tractability with ε*-exponent* 2 *and s-exponent* q^*.

Proof. For the anchored Sobolev space $H(K_{s,\mathrm{A}})$, we have from (11) and Theorem 1(A) that there exists a point set P_n for which

$$e^2(P_n;H(K_{s,\mathrm{A}}))\leqslant\frac{\rho_s-1}{n}e^2(0;H(K_{s,\mathrm{A}})),$$

where

$$\rho_s = \frac{\sum_{u:\ |u| \leqslant q^*} \gamma_{s,u} \prod_{j\in u} (m_j + 1/6)}{\sum_{u:\ |u| \leqslant q^*} \gamma_{s,u} \prod_{j\in u} m_j}$$
$$= \frac{\sum_{u:\ |u| \leqslant q^*} \gamma_{s,u} \Big(\prod_{j\in u} m_j\Big)\Big(\prod_{j\in u}\Big(1+\frac{1}{6m_j}\Big)\Big)}{\sum_{u:\ |u| \leqslant q^*} \gamma_{s,u} \prod_{j\in u} m_j}$$
$$\leqslant \max_{u:\ |u|\leqslant q^*} \prod_{j\in u}\left(1+\frac{1}{6m_j}\right).$$

Since $1/12 \leqslant m_j \leqslant \frac{1}{3}$, we have $\rho_s \leqslant 3^{q^*}$ independently of s and independently of the weights $\gamma_{s,u}$. Thus there exists a point set P_n such that

$$\frac{e^2(P_n; H(K_{s,\mathrm{A}}))}{e^2(0; H(K_{s,\mathrm{A}}))} \leqslant \frac{1}{n}\left[\max_{u:\ |u|\leqslant q^*} \prod_{j\in u}\left(1+\frac{1}{6m_j}\right) - 1\right] \leqslant \frac{1}{n}(3^{q^*}-1).$$

Therefore, we have strong tractability with ε-exponent at most 2.

We now prove the second part. For the unanchored Sobolev space $H(K_{s,\mathrm{B}})$ we have from Theorem 1(B) that there exists a point set P_n for which

$$e^2(P_n; H(K_{s,\mathrm{B}})) \leqslant \frac{1}{n} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} 6^{-|u|} \leqslant \frac{1}{n} \sum_{0<|u|\leqslant q^*} \Gamma^* 6^{-|u|}$$
$$= \frac{\Gamma^*}{n} \sum_{\ell=1}^{q^*} \binom{s}{\ell} 6^{-\ell} = G^2(s) n^{-1},$$

where

$$G^2(s) = \Gamma^* \sum_{\ell=1}^{q^*} \binom{s}{\ell} 6^{-\ell} = \Gamma^* \binom{s}{q^*} 6^{-q^*}(1+O(s^{-1})) = \Gamma^* \frac{s^{q^*}}{(q^*)!6^{q^*}}(1+O(s^{-1})).$$

Thus there exists a QMC algorithm such that

$$e(P_n; H(K_{s,\mathrm{B}})) \leqslant G(s) n^{-1/2}.$$

Noting that the initial error for the unanchored space is 1, we have tractability with ε-exponent 2 and s-exponent q^*. This completes the proof. □

4.1.1. Lower bounds

For the anchored Sobolev space we have strong tractability for arbitrary finite-order weights. However, the error bounds as well as the minimal number $n(\varepsilon, H(K_{s,\mathrm{A}}))$ of function values depend exponentially on q^*. Hence, if the order q^* is large, the corresponding minimal number may be huge. We now show that the exponential growth is indeed present for some finite-order weights of order q^*, and this holds for *any* QMC algorithm, or equivalently for any point set P_n.

We provide a lower bound on the worst case error of any QMC algorithm in the space $H(K_{s,\mathrm{A}})$, and conclude that the minimal number $n(\varepsilon, H(K_{s,\mathrm{A}}))$ of function values must depend exponentially on q^*. The proof technique used in the next theorem is based on the reproducing kernel being pointwise non-negative, as in [16]. This assumption is obviously true for the anchored Sobolev space since $K_s(\mathbf{x},\mathbf{y})\geqslant 1$ for all $\mathbf{x},\mathbf{y}\in[0,1]^s$. For the unanchored Sobolev space, the kernel takes also negative values, and therefore we are unable to provide a corresponding lower bound in this case.

Theorem 6. *Consider the anchored Sobolev spaces $H(K_{s,\mathrm{A}})$ with an arbitrary anchor $\mathbf{a}=(a_1,\dots,a_s)$. There are finite-order weights $\{\gamma_{s,u}\}$ of arbitrary order $q^*\leqslant s$ such that for any point set P_n we have*

$$\frac{e^2(P_n;H(K_{s,\mathrm{A}}))}{e^2(0;H(K_{s,\mathrm{A}}))}\geqslant 1-2c_{q^*}n\geqslant 1-2\left(\frac{8}{9}\right)^{q^*}n,$$

where

$$c_{q^*}=\min_{u\subseteq\{1,\dots,s\},\,|u|=q^*}\prod_{j\in u}\frac{8\max^3(a_j,1-a_j)}{27(a_j^2-a_j)+9}\in\left[\left(\frac{4}{9}\right)^{q^*},\left(\frac{8}{9}\right)^{q^*}\right].$$

Hence,

$$n(\varepsilon,H(K_{s,\mathrm{A}})\geqslant\frac{1-\varepsilon^2}{2c_{q^*}}\geqslant\frac{1-\varepsilon^2}{2}\left(\frac{9}{8}\right)^{q^*},$$

which depends exponentially on q^*.

Proof. Since the kernel $K_{s,\mathrm{A}}(\mathbf{x},\mathbf{y})$ is always positive, see (9), we may use Lemma 4 of [16]. This lemma states that

$$\frac{e^2(P_n;H(K_{s,\mathrm{A}}))}{e^2(0;H(K_{s,\mathrm{A}}))}\geqslant 1-n\Lambda_s^2, \tag{24}$$

where

$$\Lambda_s^2=\max_{\mathbf{x}\in[0,1]^s}\frac{h_s^2(\mathbf{x})}{e^2\,(0;H(K_{s,\mathrm{A}}))K_{s,\mathrm{A}}(\mathbf{x},\mathbf{x})} \tag{25}$$

with

$$h_s(\mathbf{x})=\int_{[0,1]^s}K_{s,\mathrm{A}}(\mathbf{x},\mathbf{y})\,d\mathbf{y}.$$

Clearly,

$$K_{s,\mathrm{A}}(\mathbf{x},\mathbf{x})=\sum_{u\subseteq\{1,\dots,s\}}\gamma_{s,u}\prod_{j\in u}|x_j-a_j|.$$

By direct computation we have

$$h_s(\mathbf{x})=\sum_{u\subseteq\{1,\dots,s\}}\gamma_{s,u}\prod_{j\in u}[|x_j-a_j|w_j(x_j)], \tag{26}$$

where

$$w_j(x_j) := \begin{cases} 1 - x_j/2 - a_j/2 & \text{if } x_j > a_j, \\ x_j/2 + a_j/2 & \text{if } x_j \leqslant a_j. \end{cases}$$

For $u \subseteq \{1, \dots, s\}$, define

$$a_{s,u} = \gamma_{s,u}^{1/2} \prod_{j \in u} |x_j - a_j|^{1/2} \quad \text{and} \quad b_{s,u} = \gamma_{s,u}^{1/2} \prod_{j \in u} [|x_j - a_j|^{1/2} w_j(x_j)].$$

From (26), using Cauchy's inequality we have

$$\begin{aligned} h_s^2(\mathbf{x}) &= \left(\sum_{u \subseteq \{1,\dots,s\}} a_{s,u} b_{s,u} \right)^2 \\ &\leqslant \sum_{u \subseteq \{1,\dots,s\}} a_{s,u}^2 \sum_{u \subseteq \{1,\dots,s\}} b_{s,u}^2 \\ &= K_{s,\mathrm{A}}(\mathbf{x}, \mathbf{x}) \sum_{u \subseteq \{1,\dots,s\}} b_{s,u}^2. \end{aligned}$$

Based on this and (25) we find

$$\begin{aligned} \Lambda_s^2 &\leqslant \max_{\mathbf{x} \in [0,1]^s} \frac{\sum_{u \subseteq \{1,\dots,s\}} b_{s,u}^2}{e^2(0; H(K_{s,\mathrm{A}}))} \\ &= \max_{\mathbf{x} \in [0,1]^s} \frac{\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} [|x_j - a_j|\, w_j^2(x_j)]}{\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} m_j} \\ &\leqslant \frac{\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} W_j}{\sum_{u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} m_j}, \end{aligned} \tag{27}$$

where

$$W_j := \max_{x_j \in [0,1]} [|x_j - a_j| w_j^2(x_j)] = \max\left(\frac{8a_j^3}{27}, \frac{8(1 - a_j)^3}{27} \right).$$

Recall that for the anchored space $H(K_{s,\mathrm{A}})$ we have $m_j = a_j^2 - a_j + \frac{1}{3}$. As functions of a_j, both W_j and m_j are symmetric with respect to $a_j = \frac{1}{2}$; moreover, for $a_j \in [\frac{1}{2}, 1]$,

$$\frac{W_j}{m_j} = \frac{8a_j^3}{27(a_j^2 - a_j + \frac{1}{3})} \in \left[\frac{4}{9}, \frac{8}{9} \right], \quad j = 1, \dots, s.$$

The minimal value of this ratio is obtained for $a_j = \frac{1}{2}$ and the maximal one for $a_j = 1$.

We are ready to define the finite-order weights for which Theorem 5 holds. As always $\gamma_{s,\emptyset} = 1$. Let u^* be a subset for which c_{q^*} is attained, i.e., $c_{q^*} = \prod_{j \in u^*} 8 \max^3(a_j, 1 - a_j)/(27(a_j^2 - a_j) + 9)$. Then we take the weights $\gamma_{s,u^*} = \beta$ for some $\beta > 0$ and $\gamma_{s,u} = 0$ for all other u. From (27) we have

$$\Lambda_s^2 \leqslant \frac{1 + \beta \prod_{j \in u^*} W_j}{1 + \beta \prod_{j \in u^*} m_j}.$$

Take β so large[1] that

$$\frac{1+\beta\prod_{j\in u^*} W_j}{1+\beta\prod_{j\in u^*} m_j}\leqslant 2\prod_{j\in u^*}\frac{W_j}{m_j}=2c_{q^*}. \tag{28}$$

Then

$$\Lambda_s^2\leqslant 2c_{q^*}\in[2(\tfrac{4}{9})^{q^*},2(\tfrac{8}{9})^{q^*}].$$

From (24) we have that

$$\frac{e^2(P_n;H(K_{s,\mathrm{A}}))}{e^2(0;H(K_{s,\mathrm{A}}))}\geqslant 1-n\Lambda_s^2\geqslant 1-2nc_{q^*}\geqslant 1-2\left(\frac{8}{9}\right)^{q^*}n,$$

as claimed. □

Combining this theorem with Theorem 5, we see that for the anchored Sobolev spaces and some finite-order weights of order $q^*\leqslant s$, the minimal number $n(\varepsilon,H(K_{s,\mathrm{A}}))$ of function values is bounded for any anchor $\mathbf{a}$ by

$$\frac{1-\varepsilon^2}{2}\left(\frac{9}{8}\right)^{q^*}\leqslant n(\varepsilon,H(K_{s,\mathrm{A}}))\leqslant\frac{3^{q^*}-1}{\varepsilon^2}.$$

Moreover, for any anchor $\mathbf{a}$ in which the first q^* components have the value $\frac{1}{2}$, the minimal number is bounded by

$$\frac{1-\varepsilon^2}{2}\left(\frac{9}{4}\right)^{q^*}\leqslant n(\varepsilon,H(K_{s,\mathrm{A}}))\leqslant\frac{3^{q^*}-1}{\varepsilon^2}.$$

These bounds depend exponentially on q^*. Theoretically, if q^* is large the minimal number of function values is huge. For example, for $s\geqslant q^*=300$, $\varepsilon=\frac{1}{2}$ and the anchor $(\frac{1}{2},\dots,\frac{1}{2})$, we have

$$n(\tfrac{1}{2},H(K_{s,\mathrm{A}}))\geqslant\tfrac{3}{8}(\tfrac{9}{4})^{q^*}>1.5\times 10^{105}.$$

However, for many practical problems q^* is small, say, $q^*\leqslant 3$ or 5. In such cases, we may tolerate exponential dependence on q^*.

4.2. Shifted lattice rules with higher convergence order

The next theorem, which is a corollary of Theorem 3, shows that lattice rules constructed by the CBC algorithm for finite-order weights achieve tractability or strong tractability error bounds with high order of convergence under the same conditions on the weights as in Theorem 5.

Theorem 7. *Let n be a prime number.*

(A) *Consider the anchored Sobolev space $H(K_{s,\mathrm{A}})$ with arbitrary finite-order weights $\{\gamma_{s,u}\}$ of order q^*. Let P_n^{A} be the lattice point set with the generator found by the CBC*

[1] Clearly, the number 2 in (28) can be replaced by any number greater than 1.

algorithm using the weights $\beta_{s,u} := \gamma_{s,u}^{\mathrm{A}}$, *see* (21), *and parameter* $\alpha = 2$. *Then there exists a shift* $\mathbf{\Delta} \in [0,1)^s$ *such that for any* $\tau \in [1,2)$,

$$\frac{e(P_n^{\mathrm{A}} + \mathbf{\Delta}; H(K_{s,\mathrm{A}}))}{e(0; H(K_{s,\mathrm{A}}))} \leqslant \left(1 + 2\zeta\left(\frac{2}{\tau}\right)\left(\frac{6}{\pi^2}\right)^{1/\tau}\right)^{q^*\tau/2} \times \left(\frac{s^{q^*}}{(q^*)!}\right)^{(\tau-1)/2} (1 + O(s^{-1}))(n-1)^{-\tau/2},$$

with the implied constant independent of s. When $\tau = 1$ *the* $1 + O(s^{-1})$ *factor is absent, and the bound is independent of the dimension s.*

Hence, for arbitrary finite-order weights of order q^*, *we have strong tractability with* ε*-exponent at most* 2, *and tractability with* ε*-exponent* $2/\tau$ *and s-exponent* $q^*(1 - 1/\tau)$.

(B) *Consider the unanchored Sobolev space* $H(K_{s,\mathrm{B}})$. *Assume that finite-order weights* $\{\gamma_{s,u}\}$ *of order* q^* *are uniformly bounded by* Γ^*. *Let* P_n^{B} *be the lattice point set with the generator found by the CBC algorithm using the weights* $\beta_{s,u} := \gamma_{s,u}^{\mathrm{B}}$, *see* (18), *and parameter* $\alpha = 2$. *Then there exists a shift* $\mathbf{\Delta} \in [0,1)^s$ *such that for any* $\tau \in [1,2)$,

$$e(P_n^{\mathrm{B}} + \mathbf{\Delta}; H(K_{s,\mathrm{B}})) \leqslant \left(\frac{\Gamma^*}{(2\pi^2)^{q^*}}\right)^{1/2} \frac{(2\zeta(2/\tau)s)^{q^*\tau/2}}{((q^*)!)^{\tau/2}} (1 + O(s^{-1}))(n-1)^{-\tau/2},$$

with the implied constant independent of s.

Hence, for arbitrary bounded finite-order weights of order q^*, *we have tractability with* ε*-exponent* $2/\tau$, *and s-exponent* q^*.

Proof. For fixed $\tau \in [1,2)$, define

$$c_{s,u} := \gamma_{s,u} \prod_{j \in u} m_j,$$

$$W_\tau := 2\zeta(2/\tau)(2\pi^2)^{-1/\tau},$$

$$\mathscr{M}_1 := \max_{u:\ 1 \leqslant |u| \leqslant q^*} \prod_{j \in u} \left(1 + \frac{W_\tau}{m_j^{1/\tau}}\right) \leqslant (1 + W_\tau 12^{1/\tau})^{q^*}.$$

From Theorem 3 there exists a shift $\mathbf{\Delta} \in [0,1)^s$ such that

$$\frac{e(P_n^{\mathrm{A}} + \mathbf{\Delta}; H(K_{s,\mathrm{A}}))}{e(0; H(K_{s,\mathrm{A}}))} \leqslant C_{\mathrm{A}}(s,\tau)(n-1)^{-\tau/2}$$

with

$$C_{\mathrm{A}}(s,\tau) := \left(\sum_{u:\ 1 \leqslant |u| \leqslant q^*} (\gamma_{s,u}^{\mathrm{A}})^{1/\tau} \left(2\zeta\left(\frac{2}{\tau}\right)\right)^{|u|}\right)^{\tau/2}.$$

From the proof of Theorem 4 we know that

$$C_{\rm A}^2(s,\tau) \leqslant \frac{1}{e^2(0;H(K_{s,\rm A}))}\left(\sum_{u:\ |u|\leqslant q^*} \gamma_{s,u}^{1/\tau} \prod_{j\in u}(W_\tau + m_j^{1/\tau})\right)^\tau.$$

Hence,

$$C_{\rm A}^2(s,\tau) \leqslant \frac{1}{e^2(0;H(K_{s,\rm A}))}\left(\sum_{u:\ |u|\leqslant q^*} c_{s,u}^{1/\tau} \prod_{j\in u}\left(1+\frac{W_\tau}{m_j^{1/\tau}}\right)\right)^\tau \leqslant \mathscr{M}_1^\tau H(s,\tau),$$

where, using $e^2(0;H(K_{s,\rm A})) = \sum_{u:\ |u|\leqslant q^*} c_{s,u}$, we have

$$H(s,\tau) := \frac{\left(\sum_{u:\ |u|\leqslant q^*} c_{s,u}^{1/\tau}\right)^\tau}{\sum_{u:\ |u|\leqslant q^*} c_{s,u}}.$$

Using Hölder's inequality with τ we obtain

$$\begin{aligned} H(s,\tau) &\leqslant \frac{\left(\sum_{u:\ |u|\leqslant q^*} c_{s,u}\right)^{\tau/\tau}\left(\sum_{u:\ |u|\leqslant q^*} 1\right)^{\tau(1-1/\tau)}}{\sum_{u:\ |u|\leqslant q^*} c_{s,u}} \\ &= \left(\sum_{\ell=1}^{q^*}\binom{s}{\ell}\right)^{\tau-1} = \left(\frac{s^{q^*}}{(q^*)!}\right)^{\tau-1}(1+O(s^{-1})). \end{aligned}$$

Using the bound on $\mathscr{M}_1$, we conclude the first part of the proof.

We now prove the second part. From Theorem 4 there exists a shift $\mathbf{\Delta}\in[0,1)^s$ such that for any $\tau\in[1,2)$,

$$e(P_{\rm B}+\mathbf{\Delta};H(K_{s,\rm B})) \leqslant C_{\rm B}(s,\tau)(n-1)^{-\tau/2},$$

where

$$\begin{aligned} C_{\rm B}(s,\tau) &:= \left(\sum_{\phi\neq u\subseteq\{1,\dots,s\}} (\gamma_{s,u}^{\rm B})^{1/\tau}\left(2\zeta\left(\frac{2}{\tau}\right)\right)^{|u|}\right)^{\tau/2} \\ &= \left(\sum_{\phi\neq u\subseteq\{1,\dots,s\}} (\gamma_{s,u})^{1/\tau} W_\tau^{|u|}\right)^{\tau/2} \\ &\leqslant (\Gamma^*)^{1/2}\left(\sum_{\ell=1}^{q^*}\binom{s}{\ell} W_\tau^\ell\right)^{\tau/2} \\ &\leqslant \left(\frac{\Gamma^*}{(2\pi^2)^{q^*}}\right)^{1/2}\frac{(2\zeta(2/\tau)s)^{q^*\tau/2}}{((q^*)!)^{\tau/2}}(1+O(s^{-1})), \end{aligned}$$

as claimed. □

The factors $C_A(s,\tau)$ and $C_B(s,\tau)$ in Theorem 7 for $\tau = 1$ are very close to or even the same as those in Theorem 5.

For $\tau = 1$, the convergence order of shifted lattice rules is $n^{-1/2}$ and the error bounds are independent of the dimension s for the anchored Sobolev spaces, and polynomially dependent for the unanchored Sobolev space. For $\tau > 1$, the rate of convergence of shifted lattice rules is improved to $n^{-\tau/2}$ but the error bounds depend polynomially on the dimension s for both the anchored and unanchored Sobolev spaces.

We stress that Theorem 7 holds for *arbitrary* finite-order weights in the case of the anchored Sobolev spaces, and for *arbitrary but bounded* finite-order weights in the case of the unanchored Sobolev spaces. Better results than those presented in Theorem 7 are possible to obtain if we assume stronger conditions on the finite-order weights as in Theorem 4.

5. Tractability using low discrepancy sequences

Lattice rules constructed by the CBC algorithm have good theoretical properties. However, these lattice rules depend on n as well as on the weights, since we minimize the worst-case error which depends on both n and the weights, see also [14]. In general, when the weights change, the lattice rules also change. These properties may make those rules inconvenient for applications, since for different problems even for fixed n we may need different weights and therefore different lattice rules. It may be a challenging problem to construct a "universal" lattice rule which is "good" for all, or at least for many, choices of weights.

An alternative approach is to fix a sequence of point sets $\{P_n\}$ for $n = 1, 2, \ldots,$ and then to investigate the worst case error bounds for Sobolev spaces with different weights.

It is natural to take the point sets to be the leading n terms of one of the well-known low discrepancy sequences such as Halton, Sobol or Niederreiter. This approach has been already proposed in [7,22]. We use this approach for both general and finite-order weights, choosing to study explicitly the Niederreiter sequence. We make use of a lemma proved in [22], involving the L_∞-star discrepancy of projections of $P_n = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1}\}$. We recall that the L_∞-star discrepancy of P_n is defined by

$$D^*(P_n) = \sup_{\mathbf{x}\in[0,1)^s} |\mathrm{disc}(\mathbf{x}; P_n)|,$$

where $\mathrm{disc}(\mathbf{x}; P_n)$ is the local discrepancy given by

$$\mathrm{disc}(\mathbf{x}; P_n) := \frac{1}{n}|\{n\colon\ \mathbf{x}_n \in [\mathbf{0}, \mathbf{x})\}| - \prod_{j=1}^{s} x_j, \quad \mathbf{x}\in[0,1]^s.$$

Lemma 8. *Let b be a prime and let P_n be the first n points of the s-dimensional Niederreiter sequence in base b which is based on the first irreducible polynomial over the finite field F_b. Let P_n^u be the projection of P_n on the lower-dimensional space* $[0,1]^{|u|}$.

Then for any non-empty subset $u \subseteq \{1, \dots, s\}$, *the* L_∞*-star discrepancy of* P_n^u *satisfies*

$$D^*(P_n^u) \leqslant \frac{1}{n} \prod_{j \in u} [C_0 j \log_2(j+b) \log_2(bn)],$$

where C_0 *is a constant independent of* n, u *and* s.

Using this lemma, we will prove the following theorem.

Theorem 9. *Let* $H(K_s)$ *be the anchored Sobolev space* $H(K_{s,\mathrm{A}})$ *with an arbitrary anchor* $\mathbf{a}$, *or the unanchored Sobolev space* $H(K_{s,\mathrm{B}})$. *Let* P_n *be the point set consisting of the first n points of the s-dimensional Niederreiter sequence in base b. Then*

$$e^2(P_n; H(K_s)) \leqslant \frac{1}{n^2} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} [C_1 j \log_2(j+b) \log_2(bn)]^2, \tag{29}$$

where $C_1 = 2C_0$ *is a constant independent of n and s.*

Proof. For simplicity, we first consider the anchored Sobolev space with the anchor $\mathbf{a} = (1, \dots, 1)$. The corresponding kernel is, see (7) and (9),

$$K_{s,\mathrm{A}}(\mathbf{x}, \mathbf{y}) = 1 + \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} \min(1 - x_j, 1 - y_j).$$

The square of the worst-case error is in this case equal to the square of the weighted L_2-discrepancy, see [16], and is equal to

$$e^2(P_n; H(K_{s,\mathrm{A}})) = \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} \int_{[0,1]^{|u|}} \mathrm{disc}^2((\mathbf{x}_u, \mathbf{1}); P_n)\, d\mathbf{x}_u. \tag{30}$$

Obviously,

$$\int_{[0,1]^{|u|}} \mathrm{disc}^2((\mathbf{x}_u, \mathbf{1}); P_n)\, d\mathbf{x}_u \leqslant [D^*(P_n^u)]^2,$$

where P_n^u is the projection of P_n on $[0,1]^{|u|}$. From Lemma 8 we have

$$\int_{[0,1]^{|u|}} \mathrm{disc}^2((\mathbf{x}_u, \mathbf{1}); P_n)\, d\mathbf{x}_u \leqslant \frac{1}{n^2} \prod_{j \in u} [C_0 j \log_2(j+b) \log_2(bn)]^2.$$

Thus from (30) it follows that

$$e^2(P_n; H(K_{s,\mathrm{A}})) \leqslant \frac{1}{n^2} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} [C_0 j \log_2(j+b) \log_2(bn)]^2,$$

which proves the result for the case $\mathbf{a} = (1, \dots, 1)$.

For an arbitrary anchor $\mathbf{a} = (a_1, \dots, a_s)$, the proof is similar. It is useful to introduce some notation from [6]. The unit cube $[0,1]^s$ is partitioned into 2^s quadrants (some of them possibly degenerate) by the planes $x_j = a_j$ for $j = 1, \dots, s$. Given $\mathbf{x}$ in the interior of one of these quadrants, let $B(\mathbf{x}; \mathbf{a})$ denote the box with one corner at $\mathbf{x}$ and the opposite corner given by the unique vertex of $[0,1]^s$ which lies in the same quadrant as $\mathbf{x}$. Let $B_u(\mathbf{x}_u;\mathbf{a}_u)$ be the projection of $B(\mathbf{x};\mathbf{a})$ on $[0,1]^{|u|}$. Instead of (30), we now have, see [5,6],

$$e^2(P_n;H(K_{s,\mathrm{A}})) = \sum_{\emptyset\neq u\subseteq\{1,\dots,s\}} \gamma_{s,u}\int_{[0,1]^{|u|}} R_u^2(\mathbf{x}_u;\mathbf{a}_u)\,d\mathbf{x}_u, \tag{31}$$

with

$$R_u(\mathbf{x}_u;\mathbf{a}_u) = \frac{1}{n}|P_n^u\cap B_u(\mathbf{x}_u;\mathbf{a}_u)| - \mathrm{Vol}(B_u(\mathbf{x}_u;\mathbf{a}_u)) = (Q_{n,s}-I)\chi_{B_u(\mathbf{x}_u;\mathbf{a}_u)},$$

where χ_S denotes the indicator function for the set S. Clearly,

$$\begin{aligned}\int_{[0,1]^{|u|}} R_u^2(\mathbf{x}_u;\mathbf{a}_u)\,d\mathbf{x}_u &\leqslant \sup_{\mathbf{x}_u\in[0,1)^u} R_u^2(\mathbf{x}_u;\mathbf{a}_u)\\ &\leqslant \sup_{\mathbf{x}_u<\mathbf{y}_u}\left\{\frac{1}{n}|P_n^u\cap[\mathbf{x}_u,\mathbf{y}_u)| - \mathrm{Vol}([\mathbf{x}_u,\mathbf{y}_u))\right\}^2\\ &\leqslant 4^{|u|}(D^*(P_n^u))^2.\end{aligned}$$

The last step follows from the relation of the extreme discrepancy to the L_∞-star discrepancy, see [9]. It then follows from (31) and Lemma 8 that

$$e^2(P_n;H(K_{s,\mathrm{A}}))\leqslant\frac{1}{n^2}\sum_{\emptyset\neq u\subseteq\{1,\dots,s\}}\gamma_{s,u}\prod_{j\in u}[2C_0\,j\log_2(j+b)\log_2(b\,n)]^2.$$

We now consider the unanchored Sobolev space $H(K_{s,\mathrm{B}})$. It is known, see e.g., [5,16], that the worst-case error $e(P_n;H(K_{s,\mathrm{B}}))$ is the norm of the worst-case integrand $\xi(\mathbf{x}) := I_s(K_{s,\mathrm{B}}(\mathbf{x},\cdot)) - Q_{n,s}(K_{s,\mathrm{B}}(\mathbf{x},\cdot))$. By computing its norm, we find that (31) is now replaced by

$$e^2(P_n;H(K_{s,\mathrm{B}})) = \sum_{\emptyset\neq u\subseteq\{1,\dots,s\}}\gamma_{s,u}\int_{[0,1]^{|u|}}\left(\int_{[0,1]^{|u|}}\tilde{R}_u(\mathbf{x}_u;\mathbf{a}_u)\,d\mathbf{a}_u\right)^2 d\mathbf{x}_u, \tag{32}$$

where

$$\tilde{R}_u(\mathbf{x}_u;\mathbf{a}_u) := \left(\prod_{j\in u}\tau_j(x_j,a_j)\right)R_u(\mathbf{x}_u;\mathbf{a}_u)$$

and

$$\tau_j(x_j,a_j) := \begin{cases} 1 & \text{if } x_j<a_j,\\ 0 & \text{if } x_j=a_j,\\ -1 & \text{if } x_j>a_j.\end{cases}$$

In verifying (32) it may help to observe that for fixed $\mathbf{x}\in[0,1)^s$ the quantity $\tilde{R}_u(\mathbf{x}_u;\mathbf{a}_u)$ is a piecewise-constant function of $\mathbf{a}_u$.

Similarly to the above argument, we now use

$$\int_{[0,1]^{|u|}} \left(\int_{[0,1]^{|u|}} \tilde{R}_u(\mathbf{x}_u; \mathbf{a}_u)\, d\mathbf{a}_u \right)^2 d\mathbf{x}_u \leqslant \sup_{\mathbf{x}_u \in [0,1)^{|u|}} \ \sup_{\mathbf{a}_u \in [0,1)^{|u|}} R_u^2(\mathbf{x}_u; \mathbf{a}_u)$$
$$\leqslant \sup_{\mathbf{x}_u < \mathbf{y}_u} \left\{ \frac{1}{n} |P_n^u \cap [\mathbf{x}_u, \mathbf{y}_u)| - \mathrm{Vol}([\mathbf{x}_u, \mathbf{y}_u)) \right\}^2$$
$$\leqslant 4^{|u|} (D^*(P_n^u))^2.$$

It follows from (32) and Lemma 8 that

$$e^2(P_n; H(K_{s,\mathrm{B}})) \leqslant \frac{1}{n^2} \sum_{\emptyset \neq u \subseteq \{1,\dots,s\}} \gamma_{s,u} \prod_{j \in u} [2C_0\, j \log_2(j+b) \log_2(bn)]^2.$$

This completes the proof. □

We are ready to prove that the QMC algorithm using the Niederreiter sequence achieves a tractability or strong tractability error bound for finite-order weights.

Theorem 10. *Let P_n be the point set of the first n points of the s-dimensional Niederreiter sequence in base b.*

(A) *Consider the anchored Sobolev space $H(K_{s,\mathrm{A}})$ with an arbitrary anchor $\mathbf{a}$.*

- *For arbitrary finite-order weights $\{\gamma_{s,u}\}$ of order q^*, we have*

$$\frac{e(P; H(K_{s,\mathrm{A}}))}{e(0; H(K_{s,\mathrm{A}}))} \leqslant \frac{C_2 s^{q^*} \log_2^{q^*}(s+b) \log_2^{q^*}(bn)}{n},$$

  where C_2 is a constant independent of s and n.

  Hence, we have optimal convergence order, and tractability with ε-exponent arbitrarily close to 1, *and s-exponent arbitrarily close to q^*.*
- *If the finite-order weights $\{\gamma_{s,u}\}$ of order q^* satisfy*

$$\mathscr{M} := \sup_{s=1,2,\dots} \left(\frac{\sum_{u \subseteq \{1,\dots,s\},\, |u| \leqslant q^*} \gamma_{s,u} \prod_{j \in u} [j \log_2(j+b)]^2}{\sum_{u \subseteq \{1,\dots,s\},\, |u| \leqslant q^*} \gamma_{s,u} \prod_{j \in u} m_j} \right) < \infty, \qquad (33)$$

  then for arbitrary $\delta > 0$ there exists a constant C_δ independent of s and n such that

$$\frac{e(P_n; H(K_{s,\mathrm{A}}))}{e(0; H(K_{s,\mathrm{A}}))} \leqslant C_\delta n^{-1+\delta}.$$

  Hence, we have strong tractability with ε-exponent of strong tractability 1.

(B) *Consider the unanchored Sobolev space $H(K_{s,\mathrm{B}})$.*

- *For arbitrary bounded finite-order weights $\{\gamma_{s,u}\}$ of order q^* we have*

$$e(P; H(K_{s,\mathrm{B}})) \leqslant C_3 s^{q^*} \log_2^{q^*}(s+b) \log_2^{q^*}(bn) n^{-1},$$

  where C_3 is a constant independent of s and n.

Hence, we have optimal convergence order, and tractability with ε-exponent arbitrarily close to 1, *and s-exponent arbitrarily close to* q^*.

- *If the finite-order weights* $\{\gamma_{s,u}\}$ *of order* q^* *satisfy*

$$\sup_{s=1,2,\dots}\left(\sum_{u\subseteq\{1,\dots,s\},\ |u|\leqslant q^*}\gamma_{s,u}\prod_{j\in u}[j\log_2(j+b)]^2\right)<\infty,$$

then for arbitrary $\delta>0$ *there exists a constant* C'_δ *independent of s and n such that*

$$e(P_n;H(K_{s,\mathrm{B}}))\leqslant C'_\delta n^{-1+\delta}.$$

Hence, we have strong tractability with ε-exponent of strong tractability 1.

Proof. Consider the anchored Sobolev space $H(K_{s,\mathrm{A}})$. The square of the initial error is

$$e^2(0;H(K_{s,\mathrm{A}}))=\sum_{u\subseteq\{1,\dots,s\}}\gamma_{s,u}\prod_{j\in u}m_j. \tag{34}$$

For arbitrary finite-order weights $\{\gamma_{s,u}\}$ of order q^*, from Theorem 9 we have

$$\begin{aligned}\frac{e^2(P_n;H(K_{s,\mathrm{A}}))}{e^2(0;H(K_{s,\mathrm{A}}))}&\leqslant\frac{1}{n^2}\frac{\sum_{0<|u|\leqslant q^*}\gamma_{s,u}\prod_{j\in u}[C_1 j\log_2(j+b)\log_2(bn)]^2}{\sum_{0\leqslant|u|\leqslant q^*}\gamma_{s,u}\prod_{j\in u}m_j}\\&\leqslant\frac{12^{q^*}}{n^2}\frac{\sum_{0<|u|\leqslant q^*}\gamma_{s,u}\prod_{j\in u}[C_1 j\log_2(j+b)\log_2(bn)]^2}{\sum_{0\leqslant|u|\leqslant q^*}\gamma_{s,u}}\\&\leqslant\frac{12^{q^*}}{n^2}\max_{u:\ |u|\leqslant q^*}\prod_{j\in u}[C_1 j\log_2(j+b)\log_2(bn)]^2\\&\leqslant\frac{12^{q^*}C_1^{2q^*}}{n^2}(s\log_2(s+b))^{2q^*}(\log_2(bn))^{2q^*}.\end{aligned}$$

Therefore,

$$\frac{e(P_n;H(K_{s,\mathrm{A}}))}{e(0;H(K_{s,\mathrm{A}}))}\leqslant 2^{q^*}\sqrt{3^{q^*}}C_1^{q^*}s^{q^*}\log_2^{q^*}(s+b)\log_2^{q^*}(bn)n^{-1},$$

as claimed.

Now consider finite-order weights of order q^* satisfying (33). Clearly, bound (29) in Theorem 9 can be rewritten as

$$\begin{aligned}e^2(P_n;H(K_{s,\mathrm{A}}))&\leqslant\frac{1}{n^2}\sum_{\emptyset\neq u\subseteq\{1,\dots,s\}}\gamma_{s,u}\prod_{j\in u}[C_1 j\log_2(j+b)\log_2(bn)]^2\\&=\frac{1}{n^2}\sum_{\ell=1}^{q^*}\left([C_1\log_2(bn)]^{2\ell}\sum_{|u|=\ell}\left\{\gamma_{s,u}\prod_{j\in u}[j\log_2(j+b)]^2\right\}\right).\end{aligned} \tag{35}$$

For an arbitrary $\delta > 0$ define

$$B_\delta = \max_{\ell=1,2,\dots,q^*} \left[\left(\frac{C_1 \log_2 e}{2\delta} \right)^{2\ell} (2\ell)! \right].$$

It now follows from (34), (35) and (33) that for the anchored case we have

$$\begin{aligned}
\frac{e^2(P_n; H(K_{s,\mathrm{A}}))}{e^2(0; H(K_{s,\mathrm{A}}))} &\leqslant \frac{1}{n^2} \sum_{\ell=1}^{q^*} \left([C_1 \log_2(bn)]^{2\ell} \frac{\sum_{|u|=\ell} \left\{ \gamma_{s,u} \prod_{j\in u} [j \log_2(j+b)]^2 \right\}}{\sum_{0\leqslant |u| \leqslant q^*} \gamma_{s,u} \prod_{j\in u} m_j} \right) \\
&\leqslant \frac{\mathscr{M}}{n^2} \sum_{\ell=1}^{q^*} [C_1 \log_2(bn)]^{2\ell} \leqslant \frac{B_\delta \mathscr{M}}{n^2} \sum_{\ell=1}^{q^*} \frac{[2\delta \log_e(bn)]^{2\ell}}{(2\ell)!} \\
&\leqslant \frac{B_\delta \mathscr{M}}{n^2} \exp[2\delta \log_e(bn)] = C_\delta^2 n^{-2+2\delta},
\end{aligned}$$

where $C_\delta = \sqrt{B_\delta \mathscr{M}} b^\delta$. The case of the unanchored Sobolev space can be proven similarly. □

Similar results to those in Theorem 10 can be established for the Halton and Sobol sequences. Indeed, let P_n be the first n points of the s-dimensional Halton sequence based on the first s prime numbers, see [4]. Then it is proved in [7] that

$$D^*(P_n^u) \leqslant \frac{1}{n} \prod_{j\in u} [C_\mathrm{H}\, j \log_2(j+1) \log_2(en)],$$

for any non-empty subset $u \subseteq \{1, \dots, s\}$, with C_H being independent of u and s. For the Sobol sequence based on the first primitive polynomial, see [20], a similar bound is proved in [22], namely

$$D^*(P_n^u) \leqslant \frac{1}{n} \prod_{j\in u} [C_{\mathrm{Sob}}\, j \log_2(j+1) \log_2 \log_2(j+3) \log_2(2n)]$$

with C_{Sob} independent of u and s. These bounds are similar to the bound for the Niederreiter sequence. Therefore similar tractability and strong tractability results to those in Theorem 10 hold for the Halton and Sobol sequences.

Acknowledgments

We are grateful for comments from J.F. Traub, G.W. Wasilkowski and A.G. Werschulz. The support of the Australian Research Council under its Centre of Excellence Program and of the National Science Foundation of China is gratefully acknowledged. The last author is pleased to acknowledge the support of the National Science Foundation.

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