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Longest Chains in the Lattice of Integer Partitions ordered by Majorization

CURTIS GREENE AND DANIEL J. KLEITMAN

A method is given for finding a chain of maximum length between two partitions $\lambda \leqslant \mu$ in the lattice of integer partitions, ordered by majorization. The main result is that chains in which covers of a certain kind (called 'H-steps') precede covers of another kind (called 'V-steps') all have the same length, and this length is maximal.

1. INTRODUCTION

Let $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots\}$ and $\mu = \{\mu_1 \geqslant \mu_2 \geqslant \cdots\}$ be partitions of n, a positive integer. We say that λ *majorizes* μ if $\lambda_1 \geqslant \mu_1$, $\lambda_1 + \lambda_2 \geqslant \mu_1 + \mu_2, \ldots, \lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$, etc. The terminology is due to Hardy, Littlewood, and Polya [2], although the idea appears earlier in the works of Schur, Muirhead, and others. It has been shown to have many applications to problems in combinatorics, statistics, algebra, geometry, matrix theory, among other fields. A thorough account of the history and applications of the theory can be found in [3].

For n a fixed positive integer, define $(P_n, \leqslant)$ to be the partially ordered set consisting of all partitions of n, ordered by majorization. P_n enjoys many special combinatorial properties: for example, it is a lattice (with top element $\{n\}$ and bottom element $\{1^n\}$), and is self-dual (under the map which sends each partition λ to its conjugate λ^*). A discussion of these and other elementary properties of P_n can be found in [1] or [3].

On the negative side, P_n fails to have a rank function: indeed it is arguably the most important 'natural' family of combinatorial posets which is not ranked. (A poset P is *ranked* if there exists a *rank function* $r: P \to Z$ such that $x \leqslant y$ implies all maximal chains from x to y contain $r(y) - r(x) + 1$ elements.) In the absence of a rank function, the notion of *height* is often a useful substitute: if $\mu \leqslant \lambda$ define $h(\mu, \lambda)$ to be the length (=number of elements minus 1) of the longest chain from μ to λ. The purpose of this note is to characterize the height function $h(\mu, \lambda)$ of P_n.

It turns out that, while P_n has maximal chains of length as little as $2n-3$ (see [1]), the longest chain has length asymptotic to $cn^{3/2}$ (c a constant). In Theorem 12 we give a simple algorithmic description of the longest chain between arbitrary partitions, $\mu \leqslant \lambda$. Many of the lemmas used in the proof of the main theorem are of some interest in their own right. For example, we define and discuss properties of an interesting closure operator on P_n which maps partitions onto partitions with distinct parts. Another somewhat surprisng result is that, while intervals can contain maximal chains of different lengths, all chains of a certain natural type (called 'HV-chains') have the same length. Thus, in a weak sense, P_n is 'ranked' by chains of this kind.

EXAMPLE. Figure 1 illustrates the lattice P_7. The reader will note maximal chains in P_7 with both 11 and 12 elements.

2. RESULTS

In any poset P, a pair $x < y$ is called a *cover* if there exists no z such that $x < z < y$. Covers in P_n are characterized as follows:

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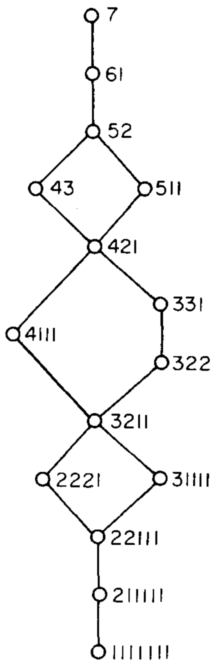


P_7

FIGURE 1.

LEMMA 1 (Muirhead [4]; see [1] or [3]). *Let $\mu < \lambda$ in P_n. Then λ covers μ if and only if there exist indices $i < j$ such that*
(a) $\mu_i = \lambda_i - 1$, $\mu_j = \lambda_j + 1$, *and* $\mu_k = \lambda_k$ *for* $k \neq i, j$, *and*
(b) *either* $j = i+1$ *or* $\lambda_i = \lambda_j + 2$.

In other words, λ covers μ if and only if μ is obtained from λ by 'lowering' a cell in the Ferrers diagram of λ to the next available position. Condition (b) says that the transfer must either be from some λ_i to λ_{i+1}, or from some λ_i^* to λ_{i-1}^*, where λ^* denotes the partition conjugate to λ. Let us denote the operation of transferring a single cell from λ_i to λ_j by $[i \to j]$, with the convention that this symbol will only be used when the resulting sequence of parts remains monotone. The result of applying $[i \to j]$ to λ will be denoted by $\lambda[i \to j]$. Thus $[i \to j]$ represents a cover iff there is no 'legal' $[i' \to j']$ with $[i', j']$ a proper subinterval of $[i, j]$. Figure 2 illustrates two examples of covers in P_{12}. Here we adopt the convention of representing Ferrers diagrams with vertical parts, i.e. with λ_1 cells in the first column, λ_2 in the second, etc.

When a cell is moved by $[i \to j]$, it obviously moves at least one unit in both the horizontal and vertical directions. In fact, $[i \to j]$ represents a cover precisely when one (or perhaps both) of these displacements is a single unit. Let us call $[i \to j]$ an *H-step* if the horizontal displacement is one unit (i.e. if $j = i+1$), and a *V-step* if the vertical displacement is one unit (i.e. if $\lambda_i = \lambda_j + 2$). Note that $[i \to j]$ can be both an H-step and a V-step (if $j = i+1$ and $\lambda_i = \lambda_j + 2$). An *H-chain* is a sequence of H-steps, and a *V-chain* is a sequence of V-steps. Define the *H-weight* $w_H(\lambda)$ of a partition λ to be the total horizontal displacement of all the cells from the left-most column, that is,

$$w_H(\lambda) = \sum \binom{\lambda_i^*}{2} = \sum (i-1)\lambda_i.$$

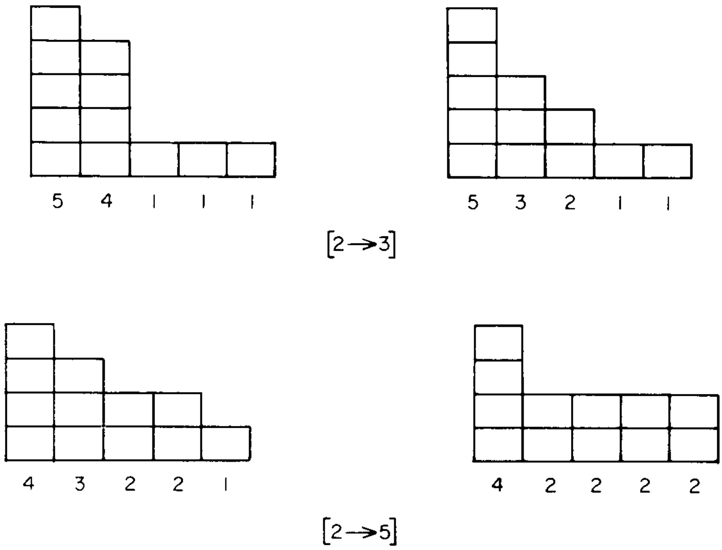


FIGURE 2.

Similarly, define the *V-weight* $w_V(\lambda)$ to be the total vertical displacement from the bottom row,

$$w_V = \sum \binom{\lambda_i}{2} = \sum (i-1)\lambda_i^*$$

For example, if $\lambda = \{4, 3, 1\}$, then $w_H(\lambda) = 5$, and $w_V(\lambda) = 9$, as shown in Figure 3:

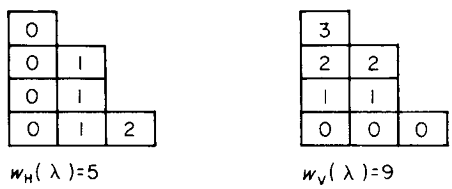


FIGURE 3.

LEMMA 2. *H-chains and V-chains are maximum-length chains between their endpoints. In other words, if* $\lambda = \lambda^{(0)} > \lambda^{(1)} > \cdots > \lambda^{(L)} = \mu$ *is an H-chain (or a V-chain) then* $h(\mu, \lambda) = L$.

PROOF. Suppose the chain is an H-chain. Then $w_H(\mu) - w_H(\lambda) = L$, since each H-step increases $w_H(\lambda)$ by 1. On the other hand, if $\lambda = \lambda^{(0)} > \tau^{(1)} > \cdots > \tau^{(m)} = \mu$ is *any* chain, then $w_H(\tau^{(i+1)}) - w_H(\tau^{(i)}) \geq 1$ for each i, so that $m \leq w_H(\mu) - w_H(\lambda) = L$. Thus L is maximal.

Let us call a chain $\lambda = \lambda^{(0)} > \lambda^{(1)} > \cdots > \lambda^{(L)} = \mu$ an *HV-chain* if there exists an index i, $0 \leq i \leq L$, such that $\lambda^{(0)} > \lambda^{(1)} > \cdots > \lambda^{(i)}$ is an H-chain and $\lambda^{(i)} > \cdots > \lambda^{(L)}$ is a V-chain. By Lemma 2, such a chain has maximum length among all chains from λ to μ which pass through $\lambda^{(i)}$. The next lemma is crucial to the arguments which follow:

LEMMA 3. *Let $\lambda = \lambda^{(0)} > \lambda^{(1)} > \cdots > \lambda^{(L)} = \mu$ be any chain from λ to μ. Then there exists an HV-chain from λ to μ which has length at least L.*

PROOF. We can assume that each $\lambda^{(i)} > \lambda^{(i+1)}$ is a cover, that is, either an H-step or a V-step. Furthermore, it suffices to prove the lemma for chains $\lambda^{(k)} > \lambda^{(k+1)} > \lambda^{(k+2)}$ of length 2. By repeated application of this special case, one can eventually transform any chain of length L into an HV chain of length at least L. Assume that $\lambda^{(k)} > \lambda^{(k+1)}$ is a V-step which is not an H-step, and $\lambda^{(k+1)} > \lambda^{(k+2)}$ is an H-step which is not a V-step. Denote the corresponding transfer operators by $[p \to q]$ and $[r \to r+1]$, respectively. If p, q, r and $r+1$ are distinct (i.e. if the operators act on different parts) then $[p \to q]$ and $[r \to r+1]$ can be applied in either order, and the lemma follows immediately. Otherwise, there are three possibilities:
Case 1: $q = r$. In this case one can check that

$$[p \to q][q \to q+1] = [q \to q+1][q-1 \to q][p \to q-1]$$

which is an HV chain of length three.
Case 2: $p = r+1$. In this case,

$$[p \to q][p-1 \to p] = [p-1 \to p][p \to p+1][p+1 \to q]$$

which is again an HV chain of length three.
Case 3: $p = r$ or $q = r+1$. It is easy to check that these cases cannot occur.
This completes the proof of the lemma.

One last definition: if $\mu \leqslant \tau \leqslant \lambda$, let us say that τ is *H-reachable from* λ if there exists an H-chain from λ to τ. Similarly, τ is *V-reachable from* μ if there is a V-chain from μ to τ.

LEMMA 4. *Every interval $\mu \leqslant \lambda$ contains a unique smallest partition which is H-reachable from λ and a unique largest partition which is V-reachable from μ.*

PROOF: Let θ be a partition in the interval $\mu \leqslant \lambda$ which is H-reachable from λ, but such that no smaller partition in the interval has this property. Let us abbreviate the symbol for the operator $[i \to i+1]$ as simply $[i]$. Thus $\theta = \lambda[i_1][i_2]\ldots$ for some sequence of indices $i_1, i_2, \ldots$. Note that any 'legal' permutation of the operators $[i_1], [i_2], \ldots$ yields the same θ, i.e. θ is determined by the *multiset* $\{i_1, i_2, \ldots\}$. This follows from the fact that $[i]$ reduces the ith partial sum by 1, regardless of when it is applied. Thus the final partial sum sequence is uniquely determined, and this determines θ. Let p be the smallest index which appears in the list $i_1, i_2, \ldots$ i.e. p is the index of the largest 'active' part. Let $q \geqslant p$ be the largest index with the property that $\lambda_i - \lambda_{i+1} \leqslant 1$ for $i = p, p+1, \ldots, q-1$. In other words, λ_q is the last part in the 'run' (=sequence of consecutive parts differing by at most one) which follows λ_p (see Figure 4). The significance of q lies in the following claim: *q must appear in the list $i_1, i_2, \ldots$, and must precede all occurrences of any index $k < q$.*

This follows from the fact that for $p \leqslant k < q$, $[k]$ is not 'legal' until λ_q, and perhaps other parts, have been reduced in size. Note that $\lambda_q > \lambda_{q+1} + 1$, since otherwise $\lambda_q = 1$ and no transfers at all are legal. In fact, q is uniquely determined by λ and μ: *q is the index of the largest part λ_q such that $\lambda_q > \lambda_{q+1} + 1$, and such that $\lambda[q] \geqslant \mu$.*

If q is not active, then $[q]$ always remains legal, and hence θ cannot be H-minimal. As indicated above, the first $k \leqslant q$ to be active must be q itself. Next observe that the first occurrence of $[q]$ can be permuted to the beginning of the list $[i_1][i_2]\ldots$. This is based on the following general assertion: *For any a and b, we have $[a][b] = [b][a]$ provided both of the operations $[a][b]$ and $[b][a]$ can be carried out. In particular, if $[a][b]$ can*

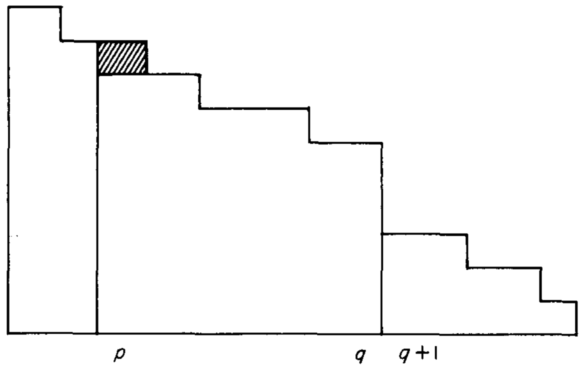


FIGURE 4.

be (legally) applied to τ, and $\tau_b \neq \tau_{b+1}+1$, then $[b][a]$ can be applied to τ, and $\tau[b][a] = \tau[a][b]$.

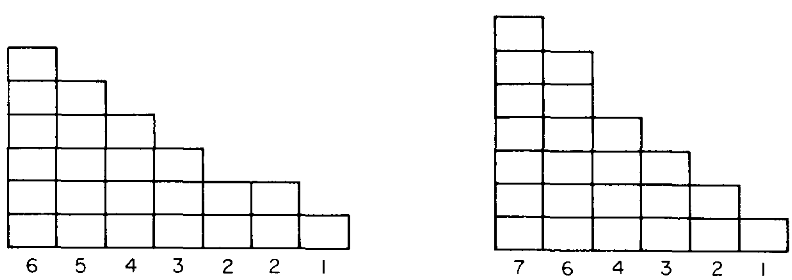
As noted before, the partial sums of $\tau[a][b]$ and $\tau[b][a]$ are identical, so it suffices to check that $\tau[b]$ is legal if $\tau_b \neq \tau_{b+1}+1$. The case $\tau_b = \tau_{b+1}$ is impossible, since otherwise $\tau[a][b]$ is not legal. Hence $\tau_b > \tau_{b+1}+1$, and the statement follows.

If $\theta = \lambda[i_1][i_2]\ldots$, and q is defined as above, then the first occurrence of $[q]$ can be permuted to the beginning of the list, i.e. $\theta = \lambda[q][i_1][i_2]\ldots$, and the latter is a legal sequence of operations.

This follows from the fact that the inequality $\lambda_q > \lambda_{q+1}+1$ is true initially, and remains true as long as operators $[i]$ are applied with $i > q$.

Now we can complete the proof of Lemma 4. Suppose that $\tau = \lambda[j_1][j_2]\ldots$ is any other partition which is H-minimal in the interval $\mu \leqslant \lambda$. Then by the arguments given above, q appears in the list $j_1, j_2, \ldots$, and the first occurrence of q can be permuted to the beginning of the list. Thus $\tau = \lambda[q][j_1][j_2]\ldots$, and both θ and τ are H-minimal in the interval $\mu \leqslant \lambda[q]$. The Lemma now follows by induction on $h(\mu, \lambda)$.

Denote the H-minimal and V-maximal partitions in $\mu \leqslant \lambda$ by $\lfloor \lambda \rfloor_\mu$ and $\lceil \mu \rceil^\lambda$, respectively. Write $\lfloor \lambda \rfloor_{\{1^n\}} = \underline{\lambda}$ and $\lceil \mu \rceil^{\{n\}} = \bar{\mu}$. An explicit description of these partitions is complicated in general. However, $\underline{\{n\}}$ and $\overline{\{1^n\}}$ are extremely easy to describe. Write $n = \binom{m+1}{2} + r$, $0 \leqslant r < m$. Then $\underline{\{n\}}$ is obtained by adding one additional copy of r to the partition $\{m, m-1, \ldots, 2, 1\}$, and $\overline{\{1^n\}}$ is the conjugate of $\underline{\{n\}}$. Figure 5 illustrates the case $n = 23$:



$\underline{\{23\}}$ $\overline{\{1^{23}\}}$

FIGURE 5.

For any partition μ, $\bar{\mu}$ has distinct parts (otherwise it would not be V-maximal), and $\bar{\mu}=\mu$ if and only if μ has distinct parts. Dually, $\underline{\lambda}$ always has the property that adjacent parts differ by at most 1. Let us call such a partition a *run*: thus λ is a run iff $\underline{\lambda}=\lambda$ iff λ^* has distinct parts.

LEMMA 5. *(a) The map $\mu \to \bar{\mu}$ is a closure operator on P_n, mapping P_n onto the set of partitions with distinct parts.*
(b) $\bar{\mu}$ is the unique smallest partition into distinct parts which majorizes μ.
(c) The map $\lambda \to \underline{\lambda}$ is a closure operator on the dual of P_n, mapping P_n onto the set of runs.
(d) $\underline{\lambda}$ is the unique largest run majorized by λ.

PROOF. We will prove parts (c) and (d). Parts (a) and (b) then follow by duality. Recall that a *closure operator* on a poset P is a map $x \to \bar{x}$ which satisfies (1) $\bar{x} \geq x$, (2) $x \leq y$ implies $\bar{x} \leq \bar{y}$, (3) $\bar{\bar{x}}=\bar{x}$. The first and third conditions are obvious; the second requires proof. To prove (2) it suffices to prove the following statement: *Suppose that $\theta \leq \tau$ and $\underline{\theta}=\theta$. If $\tau' < \tau$ is obtained from τ by a single H-step, then $\theta \leq \tau'$.*

To see this, suppose that $\tau'=\tau[i]$, i.e. τ' is obtained from τ by transferring a single cell from τ_i to τ_{i+1}. Necessarily $\tau_i \geq \tau_{i+1}+2$. Clearly $\tau' \geq \theta$ unless $\sum_1^i \theta_j=\sum_1^i \tau_j$. However, if this occurs, we also have $\sum_1^{i-1} \theta_j \leq \sum_1^{i+1} \tau_j$ and $\sum_1^{i+1} \theta_j \leq \sum_1^{i+1} \tau_j$. Subtracting gives $\theta_{i+1} \leq \tau_{i+1} \leq \tau_i \leq \theta_i$. But this implies $\theta_i-\theta_{i+1} \geq \tau_i-\tau_{i+1} \geq 2$, which is a contradiction, since θ was assumed to be a run. This proves that $\lambda \mapsto \underline{\lambda}$ is a closure operator, and also verifies statement (d).

One of the arguments just made will be useful in other situations, so we isolate it as a lemma:

LEMMA 6. *Suppose that $\theta \leq \tau$ and $\sum_1^i \theta_j=\sum_1^i \tau_j$. Then $\theta_{i+1} \leq \tau_{i+1} \leq \tau_i \leq \theta_i$.*

Whenever a closure operator $x \to \bar{x}$ is defined on a lattice L, it is easy to see that the closed elements are closed under $\wedge$. Thus we obtain

COROLLARY 7. *The set of partitions in P_n with distinct parts is closed under $\wedge$. The set of runs in P_n is closed under $\vee$.*

This fact can actually be given a short direct proof. Consider partitions λ and τ, with partial sums $s_1, s_2, \ldots$ and $t_1, t_2, \ldots$, respectively. Then λ and τ have distinct parts if and only if their partial sum sequences are *strictly concave*, i.e. $s_i+s_{i+2}<2s_{i+1}$ and $t_i+t_{i+2}<2t_{i+1}$ for $i=1,2,\ldots m-2$, where m in each case is the number of parts. Further, $\lambda \wedge \tau$ has partial sum sequence $\min\{s_1, t_1\}$, $\min\{s_2, t\}, \ldots$. It is easy to see that the min of two strictly concave sequences is strictly concave.

The next lemma answers the general question: which partitions $\tau < \lambda$ are H-reachable from λ?

LEMMA 8. *Given $\tau < \lambda$, let $k_0=0, k_1, k_2, \ldots$ denote the indices such that*

$$\sum_{j=1}^{k_i} \tau_j=\sum_{j=1}^{k_i} \lambda_j.$$

Then τ is H-reachable from λ if and only if in each 'block' $\tau_{k_i+1}, \ldots, \tau_{k_{i+1}}$, each run has at most one repeated part, and this part is not repeated more than twice.

PROOF. Clearly, in any chain from λ to τ, there can be no transfer of cells between different 'blocks'. The lemma is an immediate consequence of the following two statements: (1) if τ fails to satisfy the conditions of the lemma, then for every τ' such that $\tau < \tau' \leqslant \lambda$ and $\tau < \tau'$ is an H-step, τ' also fails to satisfy the conditions; (2) if τ satisfies the conditions, then either $\tau = \lambda$ or there exists a τ' such that $\tau < \tau' \leqslant \lambda$, where $\tau < \tau'$ is an H-step and τ' satifies the conditions.

The first assertion is straightforward. Suppose that $\tau = \tau'[i]$, in the notation of Lemma 4. If τ contains a sequence $\tau_j = \tau_{j+1} = \tau_{j+2}$ of equal parts, then so does τ', since i cannot be equal to $j, j+1$, or $j+2$. It is possible that τ' contains one additional 'barrier' k not present in τ, but $k = i$ if it exists. Thus the sequence $\tau_j = \tau_{j+1} = \tau_{j+2}$ lies entirely within a block of τ', and the statement is proved in this case. If τ contains two pairs of repeated parts in a run, an easy argument shows that τ' contains either (a) the same pair, (b) a 'closer' pair, or (c) a sequence of three equal parts. Moreover it is easy to check that these configurations lie within a block of τ'. This proves statement (1).

Suppose next that τ satisfies the conditions of the lemma. We claim first that if some block of τ contains a repeated part $\tau_i = \tau_{i+1}$, then the transformation $[i]^{-1}$ yields a partition τ' with the desired properties. The only nontrivial step is to show that applying $[i]^{-1}$ to τ is legal, i.e. $\tau_{i-1} \geqslant \tau_i + 1 \geqslant \tau_{i+1} - 1 \geqslant \tau_{i+2}$. If τ_{i-1}, τ_i, τ_{i+1}, and τ_{i+2} lie in a block of τ, this is trivial, since τ contains no parts repeated three times. If $\tau_{i-1} > \tau_i$ and $\tau_{i+1} > \tau_{i+2}$, it is also trivial. Suppose that $\tau_{i-1} = \tau_i$ and these two parts lie in different blocks, i.e. $i - 1 = k_\alpha$ for some α. By Lemma 6, $\tau_i \leqslant \lambda_i \leqslant \lambda_{i-1} \leqslant \tau_{i-1}$, which implies $\tau_i = \lambda_i$. But this implies

$$\sum_{j=1}^{i} \tau_j = \sum_{j=1}^{i} \lambda_j,$$

which means that τ_i is a block of size 1, contrary to the assumption that τ_i and τ_{i+1} lie in a block. A similar argument deals with the case when τ_{i+1} and τ_{i+2} are equal and lie in separate blocks. This proves that $\tau' = \tau[i]^{-1}$ exists. It is easy to see that $\tau < \tau' \leqslant \lambda$, and that $[i]^{-1}$ does not introduce any 'bad' configurations within a block (i.e. parts repeated three times or pairs of repeated parts in a run). This completes the proof of statement (2) in this case.

Suppose, finally, that the blocks of τ contain no repeated parts. If there is a block containing at least two parts, say τ_i and τ_{i+1}, it follows from the argument given above that $\tau_{i-1} \geqslant \tau_i + 1 \geqslant \tau_{i+1} - 1 \geqslant \tau_{i+2}$. Hence the operator $[i]^{-1}$ is legal, and it is easy to see that $\tau' = \tau[i]^{-1}$ has the desired properties.

The only remaining case is when all of the blocks of τ have size one. But this means $\tau = \lambda$, and the proof is complete.

The preceding argument actually yields another result which will be essential later on:

LEMMA 9. *Suppose that $\tau \leqslant \lambda$, and τ is H-reachable from λ. It τ is not V-maximal in the interval $\tau < \lambda$, there exists a partition τ' with the following properties*: (*a*) $\tau < \tau' \leqslant \lambda$, (*b*) $\tau < \tau'$ *is both an H-step and a V-step*, (*c*) τ' *is H-reachable from* λ.

PROOF. It is easy to see that if τ is not V-maximal in $\tau \leqslant \lambda$, then some block of τ must contain a part repeated more than once. On the other hand, since τ is H-reachable from λ, Lemma 8 shows that this part cannot be repeated more than twice. Say $\tau_i = \tau_{i+1}$, and these two parts lie in a block. In the preceding argument it was shown that $[i]^{-1}$ is a legal transformation, and also that $\tau' = \tau[i]^{-1}$ satifies the conditions for H-reachability from λ. Clearly $\tau < \tau'$ is both an H-step and a V-step. This completes the proof.

The conditions of H-reachability from $\{n\}$ are quite simple:

COROLLARY 10. *τ is H-reachable from $\lambda=\{n\}$ if and only if each run of τ contains at most one repeated part, and this part is not repeated more than twice.*

EXAMPLE. $\{8, 7, 6, 6, 3, 3, 2, 1\}$ is H-reachable from $\{36\}$, but $\{7, 6, 6, 6, 5, 3, 2, 1\}$ and $\{8, 7, 6, 5, 3, 3, 2, 1, 1\}$ are not.

LEMMA 11. *If $\mu \leqslant \lambda$, then $\lfloor\lambda\rfloor_\mu$ is V-reachable from μ, and $\lceil\mu\rceil^\lambda$ is H-reachable* from λ.

PROOF. By definition, $\lfloor\lambda\rfloor_\mu \geqslant \mu$. Thus by Lemma 3, there is an HV chain from $\lfloor\lambda\rfloor_\mu$ to μ. Since $\lfloor\lambda\rfloor_\mu$ is H-minimal in the interval $\mu \leqslant \lambda$, there can be no H-steps in this chain, and $\lfloor\lambda\rfloor_\mu$ is V-reachable from μ. By a similar argument, $\lceil\mu\rceil^\lambda$ is H-reachable from λ, and the lemma is proved.

We can now state our main theorem, which characterizes the height function $h(\mu, \lambda)$ in P_n.

THEOREM 12. *Suppose that $\mu \leqslant \lambda$. Then all HV-chains from λ to μ have the same length, and this length is $h(\mu, \lambda)$. Furthermore,*

$$\begin{aligned} h(\mu, \lambda) &= w_{\mathrm{H}}(\lfloor\lambda\rfloor_\mu) - w_{\mathrm{H}}(\lambda) + w_{\mathrm{V}}(\lfloor\lambda\rfloor_\mu) - w_{\mathrm{V}}(\mu) \\ &= w_{\mathrm{H}}(\lceil\mu\rceil^\lambda) - w_{\mathrm{H}}(\lambda) + w_{\mathrm{V}}(\lceil\mu\rceil^\lambda) - w_{\mathrm{V}}(\mu). \end{aligned}$$

HV-chains can be constructed by the following algorithm: start at λ and proceed toward μ by H-steps until no further H-steps are possible. This occurs when $\lfloor\lambda\rfloor_\mu$ is reached. Then proceed by V-steps to μ. Lemma 4 shows the uniqueness of $\lfloor\lambda\rfloor_\mu$, and Lemma 11 shows the existence of a V-chain from $\lfloor\lambda\rfloor_\mu$ to μ.

To prove the theorem, suppose that $\lambda=\lambda^{(0)} > \lambda^{(1)} > \cdots > \lambda^{(L)} = \mu$ is any HV-chain between λ and μ, where $\lambda^{(0)} > \lambda^{(1)} > \cdots > \lambda^{(i)}$ is an H-chain and $\lambda^{(i)} > \lambda^{(i+1)} > \cdots > \lambda^{(L)}$ is a V-chain. Since $\lambda^{(i)}$ is H-reachable from λ and V-reachable from μ, we have $\lfloor\lambda\rfloor_\mu \leqslant \lambda^{(i)} \leqslant \lceil\mu\rceil^\lambda$. By Lemma 9, if $\lambda^{(i)} \neq \lceil\mu\rceil^\lambda$, there exists a partition τ such that $\lambda^{(i)} < \tau \leqslant \lambda$, τ is H-reachable from λ, and $\lambda^{(i)} < \tau$ is both an H-step and a V-step. Since all H-paths from $\lambda^{(i)}$ to λ have the same length, we can replace the original H-path by one of the same length which passes through τ. Since $\lambda^{(i)} < \tau$ is a V-step, we can view τ as the 'transition point' instead of $\lambda^{(i)}$ in this path. This argument can be repeated until $\lambda^{(i)} = \lceil\mu\rceil^\lambda$. Now the total length of the chain can be calculated, and the result is

$$w_{\mathrm{H}}(\lceil\mu\rceil^\lambda) - w_{\mathrm{H}}(\lambda) + w_{\mathrm{V}}(\lceil\mu\rceil^\lambda) - w_{\mathrm{V}}(\mu).$$

We have shown that all HV-chains must have this length, including the chain of maximum length whose existence is guaranteed by Lemma 3. A dual argument shows that all HV-chains also have length

$$w_{\mathrm{H}}(\lfloor\lambda\rfloor_\mu) - w_{\mathrm{H}}(\lambda) + w_{\mathrm{V}}(\lfloor\lambda\rfloor_\mu) - w_{\mathrm{V}}(\mu).$$

and the proof is complete.

We conclude by calculating the length of the longest chain in P_n explicitly. Let $\mu=\{1^n\}$ and $\lambda=\{n\}$, and let $h(P_n) = h(\mu, \lambda)$. Write

$$n = \binom{m+1}{2} + \binom{r}{1}$$

where $0 \leqslant r \leqslant m$. Then $\lfloor \lambda \rceil_\mu$ is the partition $\{m, m-1, \ldots, r, r, \ldots, 2, 1\}$, and it is straightforward to compute

$$w_{\mathrm{H}}(\lfloor \lambda \rfloor_\mu) = \binom{m+2}{3} - \binom{m+1-r}{2} \qquad w_{\mathrm{V}}(\lfloor \lambda \rfloor_\mu) = \binom{m+1}{3} + \binom{r}{2}.$$

By Theorem 12, $h(P_n)$ is equal to the sum of these two values, since $w_{\mathrm{H}}(\lambda) = w_{\mathrm{V}}(\mu) = 0$. The leading term in this sum is $m^3/3$. Since m is asymptotic to $\sqrt{2n}$, this shows that

$$h(P_n) \sim \frac{(2n)^{3/2}}{3}.$$

There is an easy way to compute $h(P_n)$ for any particular value of n: form the Ferrers diagram of $\{m, m-1, \ldots, r, r, \ldots, 2, 1\}$, and label the cell in the ith row and jth column with the number $(i-1)+(j-1)$. Then $h(P_n)$ is the sum of all of these numbers. Figure 6 illustrates the computation when $n = 18$:

4					
3	4				
2	3	4	5		
1	2	3	4	5	
0	1	2	3	4	5

$h(P_{18}) = 55$

FIGURE 6.

It is clear from this construction that the successive differences of the function $h(P_n)$ are

$$0, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, \ldots$$

from which the values can be readily computed as given in Table 1.

TABLE 1

n	$h(P_n)$	Δ_n	n	$h(P_n)$	Δ_n	n	$h(P_n)$	Δ_n	n	$h(P_n)$	Δ_n	n	$h(P_n)$	Δ_n
1	0	1	7	11	3	13	32	4	19	60	5	25	94	6
2	1	1	8	14	3	14	36	4	20	65	5	26	100	6
3	2	2	9	17	3	15	40	5	21	70	6	27	106	6
4	4	2	10	20	4	16	45	5	22	76	6	28	112	7
5	6	2	11	24	4	17	50	5	23	82	6	29	119	7
6	8	3	12	28	4	18	55	5	24	88	6	30	126	

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CURTIS GREENE
Department of Mathematics, Haverford College,
Haverford, PA 19041, U.S.A.
and
DANIEL J. KLEITMAN
Department of Mathematics, Massachusetts Institute of Technology,
Cambridge, MA 02139, U.S.A.