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Products of straight spaces

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A metric space X is straight if for each finite cover of X by closed sets, and for each real valued function f on X, if f is uniformly continuous on each set of the cover, then f is uniformly continuous on the whole of X. A locally connected space is straight iff it is uniformly locally connected (ULC). It is easily seen that ULC spaces are stable under finite products. On the other hand the product of two straight spaces is not necessarily straight. We prove that the product $X \times Y$ of two metric spaces is straight if and only if both X and Y are straight and one of the following conditions holds:

(a) both X and Y are precompact;
(b) both X and Y are locally connected;
(c) one of the spaces is both precompact and locally connected.

In particular, when X satisfies (c), the product $X \times Z$ is straight *for every* straight space Z. Finally, we characterize when infinite products of metric spaces are ULC and we completely solve the problem of straightness of infinite products of ULC spaces.

1. Introduction

All spaces in the sequel are metric. Given a space X, $C(X)$ denotes the set of all continuous functions $f : X \to \mathbb{R}$. The following notion, already studied in [2,3], will be the object of investigation of this paper.

Definition 1.1. A space X is called **straight** if whenever X is the union of finitely many closed sets, then $f \in C(X)$ is uniformly continuous (briefly, u.c.) if and only if its restriction to each of the closed sets is u.c.

Example 1.2. Every compact space is obviously straight. For the same reason every UC-space is straight (a space X is UC if every $f \in C(X)$ is uniformly continuous [1]).

More examples are obtained from the following stronger property.

Definition 1.3. ([6, 3-2]) A metric space X is **uniformly locally connected** (ULC), if for every $\varepsilon > 0$ there is $\delta > 0$ such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$.

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It is easy to see that a ULC space is locally connected, namely each point has a basis of connected neighbourhoods. Therefore a compact space need not be ULC.

Theorem 1.4. *([2, Theorem 3.9]) A locally connected metric space is straight if and only if it is uniformly locally connected.*

In particular $\mathbb{R}$ is straight and every topological vector space is straight. The circle minus a point is not straight (it is locally connected but not uniformly so). As far as non-locally connected spaces are concerned, $\mathbb{Q}$ is not straight. More generally a totally disconnected space is straight if and only if it is a UC-space [2, Theorem 4.6].

One of the main results of [3] is a characterization of the complete straight spaces in terms of the properties of the quasi-components of the space and its subspaces (see Corollary 2.16 and Definition 2.14 below).

If a product $X \times Y$ is straight then both X and Y are straight, but the converse is not true in general (i.e., the class of straight spaces is not closed under finite products). One of the main goals of the paper is to establish precisely when straightness is preserved under products.

As a first step we show that the class of ULC spaces behaves better in this respect: it is included in the class of straight spaces and it is stable under finite products (Lemma 3.8). Moreover, if X is a precompact ULC space, then $X \times Y$ is straight for every straight space Y (and this property characterizes the precompact ULC space, cf. Corollary 5.10), i.e., the precompact ULC spaces have the best possible behavior with respect to productivity.

The failure of the corresponding property for straight spaces can be witnessed as follows: if K is a totally disconnected compact space (e.g. the Cantor space), then $\mathbb{R} \times K$ is not straight although both factors are straight. This follows from the following curious *dichotomy*: if a product $X \times Y$ is straight, then either X is precompact or Y is uniformly locally connected (Corollary 4.3). This implies the "only if" direction in the following theorem that completely describes when straightness is available for a product of two spaces.

Theorem A. *The product $X \times Y$ of two metric spaces is straight if and only if both X and Y are straight and one of the following conditions holds*:

(a) *both X and Y are precompact*;
(b) *both X and Y are ULC*;
(c) *one of the spaces is both precompact and ULC.*

The sufficiency of (b) and (c) was already commented above. To the proof of the sufficiency of (a) is dedicated the entire Section 5.1. The proofs use essentially criterions (Theorem 2.7 and Lemma 2.9) for straightness of dense subspaces based on the notion of a *tight extension* (this is specific form of dense embedding introduced in [3], see Definition 2.6). The class of tight embeddings has many nice properties that could be useful in other situations (see Theorems 5.1 and 6.11, as well as the comment in the last section). In Theorem 5.1 we establish first a natural general property of the class of tight maps: they are closed under finite products. As a corollary we obtain the sufficiency of (a) (Theorem 5.5). To resume, the proof of Theorem A is contained in Corollary 4.3, Lemma 3.8, Theorem 5.5 and Corollary 5.10. This theorem was announced without proof in [3].

For reader's convenience we formulate explicitly the following immediate corollary from Theorem A:

Corollary 1. *Let $X_1, \dots, X_n$ be metric spaces. Then $X = \prod_{i=1}^{n} X_i$ is straight if and only if all spaces X_i are straight and one of the following conditions holds*:

(a) *all spaces X_i are precompact*;
(b) *all spaces X_i are ULC*;
(c) *all but one of the spaces are both precompact and ULC.*

The following fact on straightness of products of two spaces is established also by Nishijima and Yamada [8] if $X \times (\omega+1)$ is straight for some metric space X, then X is precompact and $X \times K$ is straight for every compact space K (see Example 4.6 and Corollary 5.7 for more details).

Finally, in Section 6 we face the problem of straightness of infinite products of spaces and we completely solve the problem of straightness of infinite products of ULC spaces:

Theorem B. *Let X_n be a ULC space for each $n \in \mathbb{N}$ and $X = \prod_n X_n$.*

(a) *X is ULC iff all but finitely many X_n are connected.*
(b) *The following are equivalent*:
  (b_1) *X is straight*;
  (b_2) *either X is ULC or each X_n is precompact.*

This theorem is proved at the end of Section 6. In particular, the theorem completely settles the case of infinite powers of ULC space:

Corollary 2. *Let X be ULC. Then*

(a) *X^ω is ULC iff X is connected*;
(b) *X^ω straight iff X is either connected or precompact.*

As far as straightness of infinite products $X = \prod_n X_n$ is concerned, we prove in Proposition 6.7 that straightness of X implies the straightness of each space as well as the disjunction of the condition (b_2) (from Theorem B) and the following one:

(i) all but one of the spaces are both precompact and ULC and all but finitely many of the spaces are connected.

While (i) is easily seen to be also sufficient (see Remark 6.8), we do not know whether a product of infinitely many precompact straight spaces is straight (see Question 7.2).

2. Background

Notations. 1. We identify $\omega + 1$ with a compact subset of $\mathbb{R}$ of order type $\omega + 1$, namely with an increasing converging sequence together with its limit point.

2. Usually a metric space X with metric d will be denoted by (X, d). In the presence of more spaces X, Y, we will use subscripts d_X, d_Y to avoid confusion.

3. As we are interested in the uniform properties of metric spaces, we can assume that metrics are bounded by 1 to avoid unnecessary difficulties.

4. Unless otherwise stated, the metric $d(x, y)$ on a product $\prod_{i=1}^n X_i$ of finitely many metric spaces (X_i, d_i) is defined as the sum $\sum_i d_i(x_i, y_i)$, where x_i, y_i are the coordinates of x, y. In the case of an infinite (countable) product $\prod_{i=1}^\infty X_i$, one has to start with uniformly bounded metrics d_n (see the remark in the previous item) and define $d(x, y) = \sum_n \frac{1}{2^n} d_n(x_n, y_n)$ where x and y are points from the product $\prod_{i=1}^n X_n$ and x_i and y_i are corresponding coordinates.

5. We will frequently use subscripts like C_ε^+, C_ε^- and variants of it (e.g. $A_\varepsilon, B_\varepsilon$ where A, B is a given binary cover of a space). Such notation refers to Definition 2.2.

6. The ball of center x and radius ε in a metric space (X, d) is denoted by $B_\varepsilon(x)$. If the metric is not clear from the context we also use the notation $B_\varepsilon^d(x)$. For a metric space M, we use also $B_\varepsilon^M(x)$; it can be convenient when we deal with a space and its subspaces.

We recall here some non-trivial facts from [2] which will be often used in the sequel.

In the definition of "straight" it suffices to consider only binary unions:

Theorem 2.1. *([2]) A space X is straight if and only if whenever X is the union of two closed sets, then $f \in C(X)$ is u.c. if and only if its restriction to each of the closed sets is u.c.*

Using this characterization one can prove the following necessary and sufficient condition for straightness. We need first a definition.

Definition 2.2. Let (X, d) be a metric space. A pair C^+, C^- of closed sets of X is *u-placed* if $d(C_\varepsilon^+, C_\varepsilon^-) > 0$ holds for every $\varepsilon > 0$, where $C_\varepsilon^+ = \{x \in C^+ \colon\ d(x, C^+ \cap C^-) \geqslant \varepsilon\}$ and $C_\varepsilon^- = \{x \in C^- \colon\ d(x, C^+ \cap C^-) \geqslant \varepsilon\}$.

In other words C^+, C^- is u-placed if for every pair of sequences $x_n \in C^+$ and $y_n \in C^-$ with $d(x_n, y_n) \to 0$, we have $d(x_n, C^+ \cap C^-) \to 0$ (for $n \to \infty$). In particular, if $C^+ \cap C^- = \emptyset$, then C^+, C^- is u-placed iff $d(C^+, C^-) > 0$.

Theorem 2.3. *([2, Corollary 2.10]) A metric space (X, d) is straight if and only if every pair of closed subsets, which form a cover of X, is u-placed.*

Corollary 2.4. *([3]) If a metric space (X, d) is straight and a proper subset $H \subset X$ is clopen, then the distance between H and $X \setminus H$ is positive.*

Now we will need the following equivalent description of ULC spaces:

Lemma 2.5. *A metric space (X,d) is ULC if and only if for each $\varepsilon > 0$ there is a positive δ such that for each $x \in X$, there is an open connected set W_x such that*

$$B_\delta(x) \subseteq W_x \subseteq B_\varepsilon(x). \tag{1}$$

Without the requirement of openness of W_x this is [3, Lemma 3.1]. It remains to observe that once a connected set W_x with the above property is found, one can use local connectedness of the space to replace W_x by a larger set W_x^* still contained in $B_\varepsilon(x)$ that is both connected and open.

Another group of results from [3] we are going to use here concerns preservation of straightness under extensions. The following property of extensions will be crucial.

Definition 2.6. ([3]) An extension $X \subseteq Y$ of topological spaces is called **tight** if for every closed binary cover $X = F^+ \cup F^-$ one has

$$\overline{F^+}^Y \cap \overline{F^-}^Y = \overline{F^+ \cap F^-}^Y. \tag{2}$$

Let us note that even a one-point extension can easily fail to be tight: take $X = \{1/n\colon\ n \in \mathbb{N}\}$, $Y = X \cup \{0\}$ and as F^+, F^- the subsequences with even and odd indices respectively. Examples of tight extensions are provided by the following

Theorem 2.7. *([3]) Let X, Y be metric spaces, $X \subseteq Y$ and let X be dense in Y. Then X is straight if and only if Y is straight and the extension $X \subseteq Y$ is tight.*

Since the tightness of an extension $X \subseteq Y$ is equivalent to the joint tightness of all one-point extensions $X \subseteq X \cup \{y\}$, $y \in Y \setminus X$, the theorem implies that an extension Y of X is straight iff the one-point extensions $X \cup \{y\}$ are straight for all $y \in Y \setminus X$.

By the theorem (and the corollary below) every non-complete straight space has a proper tight extension.

Corollary 2.8. *Let X be a metric space. Then X is straight if and only if its completion $\widetilde{X}$ is straight and $\widetilde{X}$ is a tight extension of X.*

Let us recall some facts from [3] for easier reference:

Lemma 2.9. *Let $X \subseteq Y$ be dense in Y.*

1. *If X is ULC, then Y is ULC as well (and Y is a tight extension of X). In particular the completion of a ULC space is ULC.*
2. *If Y is ULC, then the following are equivalent:*
   (i) *X is ULC;*
   (ii) *X is straight;*
   (iii) *Y is a tight extension of X.*

The next construction shows that the property ULC can be easily lost under passage to closed subspaces.

Example 2.10. Let (X,d) be a metric space. Then X is homeomorphic to a closed subspace of a ULC space.

Proof. We will construct a space $X' \supset X$ such that each pair of points $x, y \in X$ lie in a connected set $I_{x,y} \subset X'$ of diameter $d(x,y)$. This is easy to do as follows. Fix a linear ordering of X and for each pair of points $x < y$ in X consider a space $I_{x,y}$ isometric to a closed interval of $\mathbb{R}$ of length $d(x,y)$. Let X' be the topological space $(X \cup \bigcup_{x<y} I_{x,y})/E$ where E is the equivalence relation on the disjoint union $X \cup \bigcup_{x<y} I_{x,y}$ which identifies one of the extremes of $I_{x,y}$ with x and the other with y. In this way X is naturally identified with a subspace of X' via $x \mapsto [x]$, where $[x]$ is the class of x modulo E. Moreover X is a closed subspace of X' because $X' \setminus X$ is homeomorphic to a disjoint union of open intervals which are open subsets of X'. The metric on X' is the biggest possible compatible with the fact that the inclusion $X \subset X'$ is an isometry and that $I_{x,y}$ is isometric to an interval of $\mathbb{R}$ of length $d(x,y)$. To finish the proof we show that X' is ULC. Let $u, v \in X'$. Then for some $x, y, x', y' \in X$ we have $u \in I_{x,y}$ and $v \in I_{x',y'}$ (with the natural identification of $I_{x,y}$, $I_{x',y'}$ as subsets of X'). The set $W = I_{x,y} \cup I_{x',y'} \cup I_{x,x'} \cup I_{y,y'}$ is connected, contains u and v, and has diameter $\leqslant d(x,y) + d(x',y') + d(x,x') + d(y,y')$. A case analysis shows that W contains a connected subset, still containing u, v, and of diameter $d(u,v)$. This proves that X' is ULC. □

Definition 2.11. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is **discrete** if it has no accumulation points in X, and it is **uniformly discrete** if there is a non-zero lower bound to the set of all the distances $d(x_n, x_m)$ for $n \neq m$. Two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are **adjacent** if $d(a_n, b_n) \to 0$ for $n \to \infty$.

Example 2.12. In terms of the above definition a space X is UC iff for every pair $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of adjacent sequences in X, with $a_n \neq b_n$ for all $n \in \mathbb{N}$, there exists a (common) accumulation point in X.

According to [3], a space (X, d) is **weakly uniformly locally connected** (WULC) if for each pair of *discrete* adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in X, there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of connected subsets of X and $k \in \mathbb{N}$ such that $\lim_{n\to\infty} \operatorname{diam} C_n = 0$, and $a_{n+k} \in C_n$, $b_{n+k} \in C_n$ for every $n \in \mathbb{N}$. It follows from the definitions that ULC implies WULC.

Now we recall another notion of connectedness introduced in [3] weaker than WULC but strong enough to imply straightness. To this end we recall that the **quasi-component** of a point $x \in X$ is the intersection of all clopen sets containing x. Hence x is in the same quasi-component of y in X if x cannot be separated from y, i.e. for every partition $X = A \cup B$ with A, B open, x and y lie both in A or both in B. One can define a metric $\hat{d}$ as follows:

Definition 2.13. ([3]) Given a metric space (X, d) and $x, y \in X$ we say that $I \subset X$ **quasi-connects** x and y if x, y belong to I and are in the same quasi-component of I. We define $\hat{d}(x, y)$ as the minimum between 1 and the infimum of the diameters of the subsets I of X which quasi-connect x and y. So $\hat{d}(x, y) = 1$, if there is no set I quasi-connecting x and y.

The next definition introduces a notion of connectedness between WULC and straightness.

Definition 2.14. ([3]) A metric space (X, d) is **approximatively locally connected** (ALC) if for each pair of *discrete* adjacent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} \hat{d}(a_n, b_n) = 0$.

Clearly, every compact space is ALC since it does not contain discrete sequences.

Theorem 2.15. *([3]) UC ⇒ WULC ⇒ ALC ⇒ straight.*

The next statement shows the importance of the ALC property:

Corollary 2.16. *A complete space X is straight if and only if it is ALC.*

Even if we are not going to use it in the sequel, let us note that if a dense subspace X of a space Y is ALC, then Y itself is ALC [3]. In particular, the completion of an ALC space is ALC.

3. First properties related to products

We show in this section and the next one the important role played by products in questions related to straightness. To mention at least one group of results, the behavior of ULC, ALC and straightness with respect to finite powers is treated in Lemma 3.8, Corollaries 4.7 and 4.4 respectively.

Definition 3.1. A u.c. map $f : X \to Y$ of metric spaces is said to **allow lifting of adjacent sequences** if for every pair of adjacent sequences (x_n) and (y_n) in Y, there exist subsequences (x_{n_k}) and (y_{n_k}) and two adjacent sequences x'_k and y'_k in X such that $f(x'_k) = x_{n_k}$ and $f(y'_k) = y_{n_k}$ for every k.

One can easily prove:

Lemma 3.2. *Let $f : X \to Y$ be a map of metric spaces that allows lifting of adjacent sequences. Then a function $g : Y \to \mathbb{R}$ is u.c. iff the function $g \circ f$ is u.c.*

Example 3.3. There are two relevant instances of maps allowing lifting of adjacent sequences: (i) projections in products; (ii) continuous open homomorphisms between metric topological groups [4].

Lemma 3.4. *If $f : X \to Y$ is a map of metric spaces that allows lifting of adjacent sequences and X is straight, then also Y is straight.*

Proof. Assume $Y = F^+ \cup F^-$ is a closed binary cover of Y. Then $X = f^{-1}(F^+) \cup f^{-1}(F^-)$ is a closed binary cover of X. Now if $g : Y \to \mathbb{R}$ is a continuous function such that $g|_{F^+}$ and $g|_{F^-}$ are u.c., then $f_1 = g \circ f : X \to \mathbb{R}$ is continuous and $f_1|_{f^{-1}(F^+)}$ and $f_1|_{f^{-1}(F^-)}$ are u.c. as compositions of u.c. functions. Then f_1 is u.c. since X is straight. Now g is u.c. by Lemma 3.2. □

The next corollaries follow from Lemma 3.4 and Example 3.3.

Corollary 3.5. *Let X, Y be metric spaces. If $X \times Y$ is straight, then both X and Y are straight.*

A subspace Y of a metric space X is said to be a **uniform retract** of X if there exists a u.c. map $r: X \to Y$ such that $r\restriction_Y = id_Y$.

Corollary 3.6. *Uniform retracts of a straight space are straight.*

Proof. Let $r: X \to Y$ be a uniform retraction. Then r allows lifting of adjacent sequences so that Lemma 3.4 applies. □

Here *uniform retract* cannot be replaced by the weaker property *C_u-embedded subspace* (i.e., a subspace Y of X such that every u.c. $f: Y \to \mathbb{R}$ can be extended to a u.c. function $X \to \mathbb{R}$). Take $X = \mathbb{R}_+ \times \mathbb{R}$ and $Y =$ the two branches of the hyperbola $\pm xy = 1$ in X.

The next corollary follows directly from Corollary 3.6 since uniformly clopen subspaces are uniform retracts. Moreover, each clopen proper subset of a straight space must have a positive distance of its complement (see Corollary 2.4), hence each clopen subset of a straight space is uniformly clopen.

Corollary 3.7. *Clopen subspaces of straight spaces are straight.*

The ULC spaces form a class of straight spaces stable under products:

Lemma 3.8. *A product $X \times Y$ is ULC if and only if both X and Y are ULC.*

Proof. If the product $X \times Y$ is ULC, then by Corollary 3.6 and Theorem 1.4 X and Y are ULC (as local connectedness is preserved under the projections of the product). On the other hand, assume that $a_n = (x_n, y_n) \in X \times Y$ and $b_n = (x'_n, y'_n) \in X \times Y$ are adjacent sequences in $X \times Y$, i.e., $d(a_n, b_n) \to 0$. Find connected sets C_n and B_n in X and Y respectively, containing $\{x_n, x'_n\}$ and $\{y_n, y'_n\}$ respectively, with $\operatorname{diam}(B_n) \to 0$ and $\operatorname{diam}(C_n) \to 0$. Then $C_n \times B_n$ is connected and $\operatorname{diam}(C_n \times B_n) \to 0$, witnessing that $X \times Y$ is ULC. □

We conclude the section proving a fact (Lemma 3.9 below) which provides a wide supply of tight extensions. We shall prove a more general result in Theorem 5.1, nevertheless, we prefer to give a direct (shorter) proof in this particular case.

Lemma 3.9. *Let Y be a metric space and let $X \subseteq Y$ be dense in Y. Suppose X is ULC. Then for any metric space Z, $Y \times Z$ is a tight extension of $X \times Z$.*

Proof. Take a closed binary cover $X \times Z = F^+ \cup F^-$. We should prove that $\overline{F^+}^{Y\times Z} \cap \overline{F^-}^{Y\times Z} \subseteq \overline{F^+ \cap F^-}^{Y\times Z}$. Suppose the contrary. Then there is a point

$$(y,z) \in \overline{F^+}^{Y\times Z} \cap \overline{F^-}^{Y\times Z} \tag{3}$$

and a neighborhood W of (y,z) such that

$$W \cap \left(F^+ \cap F^-\right) = \emptyset. \tag{4}$$

Without loss of generality, we may assume that $W = U \times V$. Let (x_n^+, z_n^+) be a sequence in F^+ converging to (y,z) and let (x_n^-, z_n^-) be a sequence in F^- converging to (y,z). Choose $\varepsilon > 0$ such that for all sufficiently large n we have $B_\varepsilon^X(x_n^+) \subset U$ and $B_\varepsilon^X(x_n^-) \subset U$. By taking a subsequence we may assume that these inclusions hold for every n.

Since X is ULC, by Lemma 2.5 there is $\delta > 0$ and connected sets $W_{x_n^+}$ and $W_{x_n^-}$ with $B_\delta^X(x_n^+) \subset W_{x_n^+} \subset B_\varepsilon^X(x_n^+) \subset U$ and $B_\delta^X(x_n^-) \subset W_{x_n^-} \subset B_\varepsilon^X(x_n^-) \subset U$. For n large enough z_n^+ and z_n^- lie in V. The connected sets $W_{x_n^+} \times \{z_n^+\}$ and $W_{x_n^-} \times \{z_n^-\}$ are disjoint from $F^+ \cap F^-$ and therefore $W_{x_n^+} \times \{z_n^+\} \subset F^+$ and $W_{x_n^-} \times \{z_n^-\} \subset F^-$ for every sufficiently large n. On the other hand for all sufficiently large n we have $W_{x_n^+} \cap W_{x_n^-} \supset B_\delta^X(x_n^+) \cap B_\delta^X(x_n^-) \supset B_{\delta/2}^X(y) \neq \emptyset$. So there is $x \in X$ such that, for all large n, $x \in W_{x_n^+} \cap W_{x_n^-}$. Now $(x, z_n^+) \in F^+$ tends to (x,z) and $(x, z_n^-) \in F^-$ tends to (x,z). So $(x,z) \in F^+ \cap F^-$. This contradicts the fact that $(x,z) \in W$. □

Remark 3.10. Let X be a UC space and let Y be a compact ULC space. Then $X \times Y$ is a WULC. Indeed, let $(x_n, y_n) \in X \times Y$ and $(x'_n, y'_n) \in X \times Y$ be two discrete adjacent sequences. Since Y is compact it follows that $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ are discrete adjacent sequence in X. Since X is UC, by Example 2.12 we have $x_n = x'_n$ for all but finitely many n. Now using the assumption that Y is ULC we get a sequence $(C_n)_{n\in\mathbb{N}}$ of connected subsets of Y and $k \in \mathbb{N}$ such that $\lim_{n\to\infty} \operatorname{diam} C_n = 0$, and $a_{n+k} \in C_n, b_{n+k} \in C_n$ for every $n \in \mathbb{N}$. Since $x_n = x'_n$ for all big enough n, the connected sets $\{x_n\} \times C_n$ witness WULC for $X \times Y$.

A stronger result will be given below (see the WULC option of Theorem 4.2).

4. Necessary conditions for straightness of finite products

4.1. The ULC/precompact dichotomy of binary products

Theorem 4.1. *If $X \times Y$ is ALC, then X is ULC or Y is compact.*

Proof. Assume that Y is not compact. We shall prove that X is locally connected. Then, being straight, it is also ULC.

Let $z \in X$ and let U be a neighbourhood of z. We need to find a connected neighbourhood Q of z contained in U. Let $Q = Q_U(z)$ be the quasi-component of z in U, namely the intersection of all the (relatively) clopen subsets of U containing z. It suffices to prove that Q is open. Indeed, in such a case Q will be a minimal clopen subset of U, and therefore it will be connected. Assume for a contradiction that Q is not a neighbourhood of some $x \in Q$. Then there exists a sequence $x_n \to x$ in U such that $x_n \notin Q$ for every $n \in \omega$. Since $Q = Q_U(z)$ is the quasi-component of x as well, this implies that x_n and x cannot be quasi-connected by a set contained in U. It follows that there is $\delta > 0$ such that $\hat{d}(x_n, x) > \delta$ for every n, where $d = d_X$ is the metric on X (it suffices to take δ smaller than the distance between x and the complement of U). Now since Y is not compact it contains a discrete sequence $(r_n)_{n\in\mathbb{N}}$. Let $u_n = (x_n, r_n)$ and $v_n = (x, r_n)$. From $\hat{d}(x_n, x) > \delta$ we deduce:

Claim. $\hat{d}_{X\times Y}(u_n, v_n) > \delta$.

In fact suppose for a contradiction that $I \subset X \times Y$ is a set of diameter $\leqslant \delta$ which quasi-connects u_n and v_n. Its projection $\pi(I)$ on X has diameter $\leqslant \delta$, so cannot quasi-connect x_n and x so $\pi(I)$ can be partitioned into clopen sets A, B containing x_n and x respectively. But then $\pi^{-1}(A) \cap I$ and $\pi^{-1}(B) \cap I$ form a clopen partition of I separating u_n and v_n. This contradiction proves the claim. □

Since u_n, v_n are discrete adjacent sequences, we conclude with Claim that $X \times Y$ is not ALC, contradicting the assumptions. □

The next theorem gives an easy criterion for a finite product of metric spaces to be ALC (resp., WULC).

Theorem 4.2. *Let $X_1, \dots, X_n$ be metric spaces. Then $X = \prod_{i=1}^n X_i$ is ALC* (*resp., WULC*) *if and only if one of the following conditions holds*:

(a) *all spaces X_i are compact*;
(b) *all spaces X_i are ULC*;
(c) *one of the spaces is ALC* (*resp., WULC*) *and all other spaces are both compact and ULC.*

Proof. Assume that X is ALC (resp., WULC). Then clearly every space X_i is ALC (resp., WULC). Assume some of the spaces, say X_1, is neither compact nor ULC. Then Theorem 4.1 yields that $\prod_{i=2}^n X_i$ is both compact and ULC. This proves (c). Hence we can assume that each one of the spaces is either compact or ULC. If one of the spaces, say X_1, is non-compact, then it is ULC and by Theorem 4.1 all spaces X_i, $i > 1$, are ULC. Thus (b) holds true. If there exists a space that is compact, but non-ULC, then a similar argument leads to (a).

Since both compact and ULC imply WULC (hence ALC as well), to prove the sufficiency it is enough to consider only the case (c). Assume that all spaces X_i, $i > 1$, are both compact and ULC. Let $Y = \prod_{i=2}^n X_i$. Then Y is a compact ULC space by Lemma 3.8. We shall prove that X is ALC (resp., WULC) when X_1 has the same property.

Let (x_n, y_n) and (x'_n, y'_n) be discrete adjacent sequences in $X = X_1 \times Y$. We can assume without loss of generality that $y_n \to y$ and $y'_n \to y$ for some $y \in Y$ (as (y_n) and (y'_n) are adjacent sequence in the compact space Y). Now discreteness of (x_n, y_n) and (x'_n, y'_n) yields that (x_n) and (x'_n) are discrete adjacent sequences in X. We can find a sequence (I_n) of subsets of X such that

(a) $\operatorname{diam} I_n \to 0$, and
(b_1) I_n quasi-connects x_n and x'_n, in case X is ALC,
(b_2) I_n is connected, in case X is WULC.

Since Y is ULC and $y_n \to y$, $y'_n \to y$, there is a sequence (C_n) of connected subsets of Y such that $y_n, y'_n \in C_n$ and $\operatorname{diam} C_n \to 0$. Let $J_n = I_n \times C_n$. Then $\operatorname{diam} J_n \to 0$ and J_n quasi-connects (x_n, y_n) and (x'_n, y'_n) in case X is ALC, while J_n is connected in case X is WULC. Hence, J_n witness ALC-ness (resp., WULC-ness) of X. □

The next corollary establishes one direction (the necessary condition) of Theorem A.

Corollary 4.3. *If $X \times Y$ is straight, then X is ULC or Y is precompact.*

Proof. Assume that Y is not precompact. We shall prove that X is ULC. The completion $\widetilde{X \times Y} = \widetilde{X} \times \widetilde{Y}$ is straight by Corollary 2.8. Hence by Corollary 2.16 it is ALC. Since $\widetilde{Y}$ is not compact by our hypothesis, Theorem 4.1 yields that $\widetilde{X}$ is ULC. Now Lemma 2.9 implies that X is ULC. □

Corollary 4.4. *If $X \times X$ is straight for a metric space X, then X is either ULC or straight and precompact.*

It was proved in [3, Corollary 5.11] that if $X \times (\omega + 1)$ is ALC, then X is complete. Using Theorem 4.1 we obtain the following stronger result:

Corollary 4.5. *$X \times (\omega + 1)$ is ALC if and only if X is compact.*

Example 4.6. As an application of the above corollary let us verify the following fact proved in [8]: if $X \times (\omega + 1)$ is straight for some metric space X, then X is precompact. Indeed, the completion $\widetilde{X} \times (\omega + 1)$ of $X \times (\omega + 1)$ is straight by Corollary 2.8, so also ALC by Corollary 2.16. Now the above corollary implies that $\widetilde{X}$ is compact, i.e., X is precompact. Note that this fact could be obtained also directly from Corollary 4.3 as $(\omega + 1)$ is not ULC (see also Corollary 4.11).

The next corollary (which should be compared to Corollary 4.4 and Lemma 3.8) shows that the ALC and WULC properties are preserved by non-trivial finite powers only when the starting space is compact or ULC, i.e. only in cases which are trivially true.

Corollary 4.7. *For every metric space X TFAE*:

(a) *$X \times X$ is ALC.*
(b) *X is either compact or ULC.*
(c) *X^n is WULC for every $n \in \mathbb{N}$.*

Proof. (a) → (b) follows directly from Theorem 4.1, (b) → (c) follows from Lemma 3.8 and (c) → (a) follows form Theorem 2.15. □

In particular, if $X \times X$ is straight for a complete metric space X, then X is either ULC or compact. Consequently, all finite powers of X are straight.

Example 4.8. Let X be a UC space. Then X is complete, hence by Corollary 2.16 $X \times X$ is straight if and only if it is ALC. Hence the above proposition implies that $X \times X$ is straight if and only if X is either compact or ULC. Since examples of UC spaces that are neither either compact nor ULC exist in profusion (just take any non-compact totally disconnected UC space), we see that $X \times X$ need not be straight for a UC space X.

4.2. Characterization of ULC spaces via straightness of products

In the next corollary of Corollary 4.3 we characterize the ULC spaces in terms of straightness of various products. Note that by Theorem 1.4 a space is ULC iff it is straight and locally connected.

Corollary 4.9. *For a metric space X TFAE*:

(a) *X is ULC*;
(b) *$\mathbb{R} \times X$ is straight*;
(c) *$\mathbb{N} \times X$ is straight*;
(d) *$Y \times X$ is straight for some non-precompact space Y.*

Proof. (a) implies both (b) and (c) by Lemma 3.8. (d) implies (a) by Corollary 4.3. The implications (b) → (d) and (c) → (d) are obvious. □

Corollary 4.10. *If $f : X \to Y$ is a u.c. map allowing the lifting of adjacent sequences, then Y is ULC whenever X is ULC.*

Proof. It suffices to note that also the map $f \times id_{\mathbb{N}} : X \times \mathbb{N} \to Y \times \mathbb{N}$ allows the lifting of adjacent sequences and then apply Corollary 4.9 and Lemma 3.4. □

Let C be the Cantor space.

Corollary 4.11. *Let X be a metric space. If the product $X \times Y$ is straight for some non-discrete zero-dimensional space Y, then X is precompact and Y is UC. In particular, if either $X \times C$ or $X \times (\omega + 1)$ is straight, then X is precompact.*

Proof. Precompactness of X follows from Corollary 4.3 since Y is not locally connected. Since straight zero-dimensional spaces are UC, the second part follows from Corollary 3.5. □

Remark 4.12. Note that if the space Y is discrete, then Y must be uniformly discrete (by straightness).

Moreover, zero-dimensionality of Y is important in Corollary 4.11. Take the open unit disk D in the plane. Then D is precompact, ULC and it is not UC. However, by Theorem 5.9, $X \times Y$ is straight for each straight space X.

5. Sufficient conditions

5.1. A general property of tight extensions and its consequences

The following theorem shows that tight extensions are preserved under finite products. Later it will be extended to infinite products under the additional assumption that the spaces are ULC (see Theorem 6.11). We do not know whether the additional assumption in the infinite case can be removed.

Theorem 5.1. *Let X, Y be dense tight subspaces of the metric spaces X', Y'. Then $X' \times Y'$ is a tight extension of $X \times Y$.*

Proof. We can assume $Y = Y'$ (since a composition of two tight extensions is tight).

Let $X \times Y = A \cup B$ with A, B closed in $X \times Y$. Let $\overline{A}$, $\overline{B}$ be the closures of A, B in $X' \times Y$ and assume $(\bar{x}, \bar{y}) \in \overline{A} \cap \overline{B}$. We must prove that $(\bar{x}, \bar{y}) \in \overline{A \cap B}$. Suppose this is not the case and let $U \times V$ be an open neighborhood of $(\bar{x}, \bar{y})$ in $X' \times Y$ with

$$A \cap B \cap (U \times V) = \emptyset. \tag{5}$$

Now fix $(x_n, y_n) \in A$ converging to $(\bar{x}, \bar{y})$ for $n \to \infty$, and $(x'_n, y'_n) \in B$ also converging to $(\bar{x}, \bar{y})$. We can assume that these two sequences lie in $U \times V$. So, by (5), $(x_n, y_n) \notin B$ and $(x'_n, y'_n) \notin A$ for every n.

Claim 5.2. *We may choose the two sequences so that $y'_n = y_n$ for every n.*

Proof. Fix (x_n, y_n) and (x'_n, y'_n) as above. If either

$$\liminf_n d\big((x_n, y_n), B \cap \big(X \times \{y_n\}\big)\big) = 0 \quad \text{or} \quad \liminf_n d\big(\big(x'_n, y'_n\big), A \cap \big(X \times \big\{y'_n\big\}\big)\big) = 0$$

then it is easy to construct two sequences as desired. So assume that the two limits are not zero. Then there is a positive ε such that for every n

$$d\big((x_n, y_n), B \cap \big(X \times \{y_n\}\big)\big) > \varepsilon \quad \text{and} \quad d\big(\big(x'_n, y'_n\big), A \cap \big(X \times \big\{y'_n\big\}\big)\big) > \varepsilon.$$

By choosing a subsequence we can assume that for all n,

$$d\big((x_n, y_n), (\bar{x}, \bar{y})\big) < \varepsilon/4 \quad \text{and} \quad d\big(\big(x'_n, y'_n\big), (\bar{x}, \bar{y})\big) < \varepsilon/4.$$

Choose $x \in X$ at distance $< \varepsilon/4$ from the common limit $\bar{x} = \lim_n x_n = \lim_n x'_n$. Given n it then follows that (x, y_n) is at positive distance (at least $\varepsilon/2$) from $B \cap (X \times \{y_n\})$ and therefore belongs to A. Similarly $(x, y'_n) \in B$. The two sequences (x, y_n) and (x, y'_n) have a common limit $(x, \bar{y}) \in X \times Y$, and A, B are closed in $X \times Y$. So $(x, \bar{y}) \in A \cap B$. This contradicts (5), since $(x, \bar{y}) \in U \times V$. □

Thanks to the claim we can assume $y_n = y'_n$. Let

$$A_n = \big\{x \in U \,\big|\, (x, y_n) \in A\big\} \quad \text{and} \quad B_n = \big\{x \in U \,\big|\, (x, y_n) \in B\big\}.$$

Then by (5) $\{A_n, B_n\}$ is a clopen partition of $U \cap X$ with

$$x_n \in A_n \quad \text{and} \quad x'_n \in B_n. \tag{6}$$

Claim 5.3. *For any given $x \in U \cap X$, the sets $\{n \mid x \in A_n\}$ and $\{n \mid x \in B_n\}$ cannot both be infinite.*

Proof. Assume that $(x, y_{n_k}) \in A$ and $(x, y_{n_m}) \in B$ for infinitely many k and m. Then $(x, \bar{y}) = \lim_k (x, y_{n_k}) = \lim_m (x, y_{n_m}) \in A \cap B$. This contradicts (6), since $(x, \bar{y}) \in U \times V$. □

Making use of the sequence $\{A_n, B_n\}$ of binary clopen partitions of $U \cap X$ we produce now a clopen partition of $U \cap X$ consisting of appropriate intersections of the clopen sets A_n, B_n. To describe more conveniently these intersections we use infinite binary sequences $f \in 2^\omega$. For a given f let $f|n \in 2^n$ be its initial sequence of length n. Let $2^{<\omega} = \bigcup_n 2^n$ be the set of all finite binary sequences. If $\sigma \in 2^n$ then $n = lh(\sigma)$ is the length of σ. For $\sigma \in 2^{<\omega}$ define $C_\sigma \subset U \cap X$ inductively as follows.

- $C_\emptyset = U \cap X$;
- if $lh(\sigma) = n$, $C_{\sigma 0} = C_\sigma \cap A_n$ and $C_{\sigma 1} = C_\sigma \cap B_n$.

Note that $\{C_\sigma \mid lh(\sigma) = n\}$ is a partition of $U \cap X$ into at most 2^n relatively clopen sets (some C_σ may be empty).

Now for $f \in 2^\omega$ define $C_f = \bigcap_n C_{f|n}$. Clearly C_f is closed in $U \cap X$ and $U \cap X$ is partitioned by the various C_f. Note that for each $x \in C_f$ we have $x \in A_n$ iff $f(n) = 0$. Hence we get immediately by Claim 5.3:

$$C_f \neq \emptyset \implies f \text{ is eventually constant.} \tag{7}$$

Claim 5.4. *For all $f \in 2^\omega$, C_f is open in $U \cap X$.*

Proof. If C_f is not open, then there is a point $z \in C_f$ in the closure of $(X \cap U) \setminus C_f$. Choose $z_k \in (X \cap U) \setminus C_f$ converging to z for $k \to \infty$. By (7) we can assume without loss of generality that f is eventually equal to the constant 0, i.e. there is N such that $\forall n \geqslant N$, $f(n) = 0$. It then easily follows that $C_f \times \{\bar{y}\} \subset A$. Let $\sigma = f|N$, so $f = \sigma 000000\ldots$. Now take $m \geqslant N$. Since $C_{f|m}$ is an open neighborhood of z, for all k sufficiently big we have $z_k \in C_{f|m}$. So for every big k, since $z_k \notin C_f$, there is some $n_k \geqslant m$ such that $z_k \in B_{n_k}$ (let $g \in 2^\omega$ be such that $z_k \in C_g$ and choose n_k so that $g(n_k) \neq f(n_k)$). So $(z_k, y_{n_k}) \in B$. We can arrange so that n_k tends to ∞ (since m above was arbitrary). So $(z_k, y_{n_k}) \to (z, \bar{y})$. But then since B is closed in $X \times Y$, $(z, \bar{y}) \in B$, contradicting $C_f \times \{\bar{y}\} \subset A$. □

So we have proved that $U \cap X$ is partitioned into the clopen sets C_f. Now consider the sequence $x_n \to \bar{x} \in X'$.

Case 1. $\{x_n \mid n\}$ intersects infinitely many C_f. Then by choosing a subsequence we can assume that for $n \neq m$, x_n and x_m belong to different clopen sets C_f and C_g. Let $P = \bigcup\{C_f \mid \exists n\colon\ x_{2n} \in C_f\}$ and let $Q = (X \cap U) \setminus P$. Then $P \cup Q$ is a clopen partition of $X \cap U$, hence $P' = P \cup (X \setminus U)$ and $Q' = Q \cup (X \setminus U)$ is a binary closed cover of X with $P' \cap Q' = X \setminus U$. Moreover,

$$\bar{x} = \lim_n x_{2n} = \lim_n x_{2n+1} \in \overline{P} \cap \overline{Q} \subset \overline{P'} \cap \overline{Q'},$$

where the closures are taken in X'. Since X' is a tight extension of X, $\bar{x} \in \overline{P' \cap Q'}$. This is absurd since U is a neighborhood of $\bar{x}$ disjoint from $\overline{P' \cap Q'}$.

Case 2. $\{x'_n \mid n\}$ intersects infinitely many C_f. Similar to Case 1.

Case 3. If Case 1 does not hold $\{x_n \mid n\}$ intersects finitely many C_f. So there is a single C_f containing infinitely many x_n. Let $I \subset \mathbb{N}$ be the infinite set $I = \{n \mid x_n \in C_f\}$. Assuming that Case 2 does not hold, there is some C_g containing x'_n for n ranging in an infinite subset J of I. Moreover g must be different from f by (6). Let

$$P = C_f \cup (X \setminus U) \quad \text{and} \quad Q = \bigcup_{h \neq f} C_h \cup (X \setminus U).$$

Then P, Q form a closed binary cover of X and $\bar{x} = \lim_n x_n = \lim_n x_{n'} \in \overline{P} \cap \overline{Q}$, where the closures are taken in X'. Since X' is a tight extension of X, $\bar{x} \in \overline{P \cap Q}$. This is absurd since U is a neighborhood of $\bar{x}$ disjoint from $\overline{P \cap Q}$. □

A direct application of this theorem and Theorem 2.7 implies that finite products of precompact straight spaces are straight:

Theorem 5.5. *Let X, Y be precompact straight spaces. Then $X \times Y$ is precompact straight, too.*

This establishes the sufficiency of (a) in Theorem A. In particular, Theorem 5.5 gives

Corollary 5.6. *All finite powers of a straight space X are straight whenever X is precompact or ULC.*

This should be compared with the limits for multiplicativity of the ALC property, given in Corollary 4.7: $X \times X$ is very rarely ALC as X must be compact or ULC to have this property.

As another immediate corollary we obtain a proof of the following criterion due to Nishijima and Yamada:

Corollary 5.7. *([8]) Let X be a straight space. Then $X \times K$ is straight for each compact space K if and only if $X \times (\omega + 1)$ is straight.*

I proof. Assume $X \times (\omega + 1)$ is straight. Then X must be precompact by Corollary 4.3. Now Theorem 5.5 applies. □

We give also a second proof that does not make recourse to Theorem 5.5:

II proof. Suppose X is straight, K is compact and $X \times K$ is not straight. Take a binary closed cover C^+, C^- of $X \times K$ witnessing it, i.e. there are $\varepsilon > 0$ and adjacent sequences $(u_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$ such that $\{u_i\}_{i\in\mathbb{N}} \subseteq C^+_\varepsilon$ and $\{v_i\}_{i\in\mathbb{N}} \subseteq C^-_\varepsilon$. As K is compact, we may suppose (choosing a subsequence, if necessary) that sequences $(\pi_K u_i)$ and $(\pi_K v_i)$ converge in K; the limits of these sequences have to coincide. Denote this limit as k. Define the subspace $Z = X \times (\{k\} \cup \{\pi_K u_i\} \cup \{\pi_K v_i\})$. Z is uniformly homeomorphic to $X \times (\omega + 1)$ and witnesses non-straightness of $X \times K$. □

5.2. Characterization of precompact ULC spaces

In the sequel we prove the sufficiency of item (c) of our Main Theorem. In doing this we obtain also a characterization of the precompact ULC spaces in terms of straightness of products.

Lemma 5.8. *Let X be a compact ULC metric space. Then $X \times Y$ is straight for every straight space Y.*

Proof. Suppose for a contradiction that $X \times Y$ is not straight for some straight space Y.

Since $X \times Y$ is not straight, by Theorem 2.3 there are closed sets $C^+, C^- \subseteq X \times Y$ such that $C^+ \cup C^- = X \times Y$ and C^+, C^- are not u-placed. So there is $\eta > 0$ and a pair of adjacent sequences $(x^n_1, y^n_1) \in C^+$ and $(x^n_2, y^n_2) \in C^-$ such that

$$\operatorname{dist}\big((x^n_i, y^n_i), C^+ \cap C^-\big) \geqslant \eta_0, \quad i = 1, 2. \tag{8}$$

Since X is ULC there is $\lambda > 0$ such that for all $x, x' \in X$ with $\rho(x, x') < \lambda$ there exists a connected subset $C_{x,x'}$ of X such that $\{x, x'\} \subseteq C_{x,x'}$ and $\operatorname{diam}(C_{x,x'}) \leqslant \frac{\eta}{4}$.

We claim that $\forall y \in B^\sigma_\lambda(y^n_1)$, $B^\rho_\lambda(x^n_1) \times \{y\}$ cannot intersect both C^+ and C^-.

In fact, if for a contradiction there were $(w, y) \in C^+$ and $(v, y) \in C^-$ with $\{v, w\} \subseteq B^\rho_\lambda(x^n_1)$, then $C_{w,v} \times \{y\} \cap (C^+ \cap C^-) \neq \emptyset$, contradicting (8) and proving the claim.

We can conclude that for each n sufficiently large there are $y_{C^+}, y_{C^-} \in B^\sigma_\lambda(y^n_1)$ such that $B^\rho_\lambda(x^n_1) \times y_{C^+} \subseteq C^+$ and $B^\rho_\lambda(x^n_1) \times y_{C^-} \subseteq C^-$ (take $y_{C^+} = y^n_1$, $y_{C^-} = y^n_2$).

As X is compact, the sequence (x^n_1) has a subsequence converging to some $x \in X$ (the corresponding subsequence of (x^n_2) also converges to x). Then $(\{x\} \times Y) \cap C^+$ and $(\{x\} \times Y) \cap C^-$ are closed sets which are not u-placed, contradicting the straightness of Y. □

According to Corollary 4.3 and Lemma 3.8, if X is a non-precompact ULC space, then $X \times Y$ is straight iff Y is ULC. The next theorem shows that adding precompactness changes completely the situation:

Theorem 5.9. *If X is precompact and ULC, then $X \times Y$ is straight for every straight space Y.*

Proof. By Lemma 2.9 the (compact) completion $\widetilde{X}$ of X must be ULC. By Lemma 5.8, $\widetilde{X} \times Z$ is straight for every straight space Z. By Lemma 3.9 $\widetilde{X} \times Z$ is a tight extension of $X \times Z$. So by Theorem 2.7 $X \times Z$ is straight. □

Theorem 5.5 complements Theorem 5.9 as it relaxes the hypothesis on the first space: instead of precompact ULC, only precompact straight is used, however the second factor in Theorem 5.5 has to be not only straight, but also precompact.

In the next corollary we characterize the precompact ULC spaces as those spaces X such that $X \times Y$ is straight for every straight space Y.

Corollary 5.10. *For every metric space X the following are equivalent*:

(a) *$X \times Y$ is straight for every straight space Y;*
(b) *$X \times Y$ is straight for every complete straight space Y;*
(c) *$X \times C$ and $X \times \mathbb{N}$ are straight (C is Cantor space);*
(d) *X is precompact and ULC.*

Proof. The implications (a) → (b) → (c) are obvious. The implication (c) → (d) follows from Corollary 4.3. The implication (d) → (a) is covered by the above theorem. □

Remark 5.11. One can replace item (c) in the above corollary by the single condition $X \times C \times \mathbb{N}$ is straight. Note that one cannot just remove the condition of straightness on $X \times \mathbb{N}$ by leaving in (c) only "$X \times C$ is straight" (indeed $C \times C$ is straight, but C is not ULC).

We conclude by a characterization of the larger class of precompact straight spaces by means of straightness of finite products. Using Corollary 3.5 and Corollary 4.11, we obtain immediately:

Corollary 5.12. *For a metric space X, the following two properties are equivalent*:

(i) *X is straight and precompact,*
(ii) *$X \times K$ is straight for every compact space K.*

The implication (i) → (ii) follows from the above theorem. To prove the implication (ii) → (i) note that the straightness of the product $X \times (\omega + 1)$ alone implies precompactness of X by Corollary 4.3.

Remark 5.13. Corollary 4.9 (or Corollary 4.3) explains why the restriction to precompact spaces is necessary. Recall that if $X \times Y$ is straight for a non-precompact space Y then X is ULC, so we would face again the assumptions of Theorem 5.9.

Let us recall the following fact from [2]: each straight totally disconnected space is UC. In particular, all precompact straight totally disconnected spaces are compact. Having in mind also Corollary 5.10, we see that Theorem 5.5 says something interesting for spaces which are neither totally disconnected nor locally connected.

6. When infinite products are straight

6.1. When infinite products are ALC, WULC or ULC

We start by describing the stronger ALC, WULC and ULC properties for infinite products. The spaces X such that X^ω is ULC are described below (see Corollary 6.5).

We have seen that a product $X \times Y$ is ULC iff both X and Y are ULC. The next example shows that this fails for infinite products.

Example 6.1. There is a ULC space X without isolated points such that X^ω is not straight (hence not ULC).

Proof. The starting example is $\mathbb{N}$ with the uniformly discrete uniformity. Certainly, $\mathbb{N}$ is ULC. Consider the infinite product $\mathbb{N}^\omega$.

Put $X = \bigoplus_{\mathbb{N}}[0,1]$, i.e. X is a countable discrete sum of unit intervals. Then X is ULC and it has no isolated points. Consider the map $q: X \to \mathbb{N}$ collapsing the nth copy of $[0,1]$ to n for every $n \in \mathbb{N}$. Define a map $r: X^\omega \to \mathbb{N}^\omega$ as $r = q^\omega: (x_i)_{i\in\omega} \mapsto (q(x_i))_{i\in\omega}$. The space $\mathbb{N}^\omega$ is not straight: this is witnessed by a partition into two clopen sets A, B with $d(A,B) = 0$. Then $r^{-1}(A)$ and $r^{-1}(B)$ define a partition of X^ω into closed sets, and by the definition of r it is easy to see that $d(r^{-1}(A), r^{-1}(B)) = 0$. So X^ω is not straight. □

The example suggests the following more general criterion for straightness of infinite products of ULC spaces.

Lemma 6.2. *For a countable family $\{X_i\colon i \in I\}$ of ULC spaces the product is straight only if all but finitely many of them have finitely many connected components.*

Proof. Assume that infinitely many X_i have infinitely many connected components. It is not restrictive to assume that every X_i has infinitely many connected components. Now we need the following

Claim. *If (X,d) is a ULC space, then there exists a positive δ such that any two distinct connected components of X are at distance $\geqslant \delta$.*

Proof. Assume for a contradiction that for every $\delta > 0$ there exists a pair of distinct connected components C, C' of X with $d(C, C') \leqslant \delta$. Since for $x \in C$ and $y \in C'$ there exists no connected set containing both x and y, we conclude that X is not ULC, a contradiction. This proves the claim. □

By Claim every X_i admits a uniformly continuous surjective map $f_i: X_i \to \mathbb{N}$. Let $f: X = \prod_i X_i \to \mathbb{N}^{\mathbb{N}}$ be the product map. Then f is uniformly continuous and allows for lifting of adjacent sequences. Hence by Lemma 3.4 $\mathbb{N}^{\mathbb{N}}$ is straight, a contradiction. □

One can ask whether an infinite product of ULC spaces is straight precisely when the necessary condition from the above lemma is satisfied. It turns out that this fails even for infinite powers. A counter-example to this effect is given in Example 6.12 below.

Note that the property ULC was necessary in order to establish the necessity of the condition in Lemma 6.2. Straightness of an infinite product of *straight* spaces does not lead to the same condition (C^ω is straight, even compact, with infinitely many connected components). Indeed, Claim does not hold for *straight* spaces (an example of an infinite straight precompact group with trivial connected components is given in [4]). (If Question 7.1 has a positive answer, then this condition is not necessary since then the infinite power of every precompact straight space would be straight.)

Note that the next theorem covers item (a) of Theorem B.

Theorem 6.3. *If X_n is a metric space for every n, then for the space $X = \prod_n X_n$ the following are equivalent*:

(a) *X is ULC*;
(b) *each space X_n is ULC and all, but finitely many, spaces are connected.*

Proof. (a) → (b) As X is ULC, then each X_n is necessarily ULC by Lemma 3.8.

Assume that infinitely many spaces X_{n_k} are disconnected. Then there exists a clopen non-trivial partition $X_{n_k} = A_k \cup B_k$. As X_{n_k} is straight, $d(A_k, B_k) > 0$. So for every $k \in \mathbb{N}$ the characteristic function $f_k : X_{n_k} \to \{0,1\}$ of A_k is u.c., so also $f = \prod_k f_k : X' = \prod_k X_{n_k} \to \{0,1\}^\omega$ is u.c. Obviously this map allows lifting of adjacent sequences. On the other hand, X' is a direct summand of the ULC space X, so X' is ULC again by Lemma 3.8. This implies that $\{0,1\}^\omega$ is ULC by Corollary 4.10 (as an image of the ULC space X'), a contradiction. Hence only finitely many X_n can be disconnected.

(b) → (a) We have to show that X is ULC. For every positive ε there exists n_0 such that all X_n with $n \geqslant n_0$ are connected and for $Z = \prod_{k=1}^{n_0} X_k$, $W = \prod_{k>n_0} X_k$, the factorization $X = Z \times W$ and for the projections $p : X \to Z$ and $q : X \to W$ one has $\operatorname{diam}(\{z\} \times W) \leqslant \varepsilon/2$ for each $z \in Z$. We have seen already that Z is ULC. Hence there exists $\delta > 0$, such that for $z, z' \in Z$ with $d_Z(z, z') < \delta$ there exists a connected set C in Z containing both points and having diameter $\leqslant \varepsilon/2$. Let $x = (z, w), x' = (z', w') \in X = Y \times W$. If $d_X(x, x') < \delta$, then also $d_Z(z, z') < \delta$, so there exists a connected set C in Z as above. Then $C' = C \times W$ is a connected set of X containing both points x, x' and having diameter $\leqslant \varepsilon$. □

Remark 6.4. Note that under the assumption of (b) X is locally connected. Hence uniform local connectedness is equivalent to straightness for X.

There exists a compact ULC space X (say $X = [0,1] \cup [2,3]$), such that X^ω is straight (actually, compact), but not ULC. Hence we deduce that straightness alone of X, provided all spaces X_n are ULC, is not sufficient to imply X is ULC in the above theorem.

The following corollary describes the metric spaces having their countably infinite power ULC.

Corollary 6.5. *Let X be a metric space. Then TFAE*:

- *X^ω is ULC*;
- *X is connected and ULC.*

For a connected and locally connected space X the power X^ω is also locally connected and connected. So the straightness of X^ω from Corollary 6.5 is then equivalent to ULC.

If X^ω is straight then X need not be ULC even if X has no isolated points (take the Cantor set).

Theorem 6.6. *Let $X_1, \ldots, X_n, \ldots$ be metric spaces. Then $X = \prod_{i=1}^\infty X_i$ is ALC* (*resp., WULC*) *if and only if one of the following conditions holds*:

(a) *all spaces X_i are compact*;
(b) *all spaces X_i are ULC and all but finitely many of them are connected*;
(c) *one of the spaces is ALC* (*resp., WULC*) *and all other spaces are both compact, ULC and all but finitely many of them are connected.*

Proof. The necessity follows from Theorems 4.2 and 6.3.

For the sufficiency consider three cases. In case (a) X is compact, so WULC (and ALC). If (b) holds true, then X is ULC by Theorem 6.3. Finally, if (c) holds true, then cases (a) and (b) apply along with Theorem 4.2. □

Since UC spaces are both straight and complete, they are ALC by Corollary 2.16. This is why one is tempted to connect the above theorem to the following old result of Atsuji [1]: a product $X = \prod_{i=1}^\infty X_i$ of metric spaces is UC if and only if one of the following conditions holds:

(i) each X_n is compact, or
(ii) all but finitely many X_n are one-point spaces and either all are uniformly isolated or all are finite except for one which is a UC-space.

Of course, the sufficiency is obvious. For the necessity it suffices to note that every non-compact UC space X has an infinite closed uniformly discrete set D. If a metric space Y has a non-isolated point y, then the product $X \times Y$ contains a closed subset, namely $D \times \{y\}$, that is uniformly discrete and contained in the subspace $(X \times Y)'$ of non-isolated points of $X \times Y$. Consequently, $(X \times Y)'$ is not compact and hence $X \times Y$ is not UC. Hence the product $X \times Y$ can be a UC space only when Y is uniformly discrete. Moreover, if X is not discrete this occurs precisely when Y is finite. Since an infinite product can be uniformly discrete precisely when all but finitely many of the spaces are singletons and the remaining (finitely many) spaces are uniformly discrete, this shows the necessity of (ii).

6.2. Straightness of infinite products

We next show that one direction of Theorem A (the necessary condition) remains valid also in the case of countable products:

Proposition 6.7. *Let $\{X_i\colon i \in I\}$ be a countable family of metric spaces. If the product $X = \prod_{i\in I} X_i$ is straight then all spaces X_i are straight and one of the following three cases occurs*:

(a) *all X_i are ULC and all but finitely many spaces are connected* (*i.e., X is ULC*);
(b) *all X_i are precompact*;
(c) *all but one of the spaces are both ULC and precompact, and all but finitely many spaces are connected.*

Proof. For every $j \in I$ let $Y_j = \prod\{X_i\colon i \in I \setminus \{j\}\}$. Then one can write $X = X_j \times Y_j$ (actually, these spaces are uniformly homeomorphic). Assume that X_j is not ULC for some $j \in I$. Then the straightness of X yields Y_j is precompact, by Corollary 4.3. So all spaces X_i, $i \in I \setminus \{j\}$, are precompact. If X_j is precompact too, then we get (b). If X_j is not precompact, then Y_j is ULC by Corollary 4.3. Hence all spaces X_i $(i \in I \setminus \{j\})$ are ULC and all but finitely of them are connected, by Theorem 6.3. Therefore, (c) holds true.

Now assume that both (b) and (c) fail. Then all spaces X_i are ULC by the above argument. Moreover, if X_j $(j \in I)$ is a space that fails to be precompact, then $Y_j = \prod\{X_i\colon i \in I \setminus \{j\}\}$ is ULC by Corollary 4.3. Hence again by Theorem 6.3 all but finitely many spaces are connected. □

Remark 6.8. It was already proved in Theorem 6.3 that item (a) is equivalent to the ULC property of the infinite product. Let us see that (c) is also a sufficient condition for straightness. Indeed, if for some $j \in I$ all spaces X_i, $i \in I \setminus \{j\}$, are both ULC and precompact and all but finitely many spaces are connected, then the space $Y_j = \prod\{X_i\colon i \in I \setminus \{j\}\}$ is precompact and ULC by Theorem 6.3. Now Theorem 5.9 implies that $X = X_j \times Y_j$ is straight.

What remains open is establishing sufficiency of (b) (see Question 7.2).

For powers we have the following:

Corollary 6.9. *If a power X^ω is straight, then either X is precompact or X^ω is ULC.*

Proof. Assume X is not precompact. Then also X^ω is not precompact. Since X^ω is uniformly homeomorphic to $X^\omega \times X^\omega$, we conclude with Corollary 4.4 that X^ω is ULC. □

The following results were found among the hand written notes of our late co-author Jan Pelant after his death:

Theorem 6.10. *Let X_n be ULC and precompact for each n, then $\prod_n X_n$ is straight.*

To prove the theorem we need the following theorem of independent interest.

Theorem 6.11. *Let X_i be a dense ULC subset of Y_i for each $i \in \mathbb{N}$. Then $\prod_i Y_i$ is a tight extension of $\prod_i X_i$.*

Proof. For every m let $\pi'_m : \prod_i Y_i \to \prod_{i\leqslant m} Y_i$ and $\pi''_m : \prod_i Y_i \to \prod_{i>m} Y_i$ be the projections.

Let A, B be closed subsets of $\prod_i X_i$ with $A \cup B = \prod_i X_i$. Let $\overline{A}$, $\overline{B}$ be the closures of A, B in $\prod_i Y_i$. Let $f \in \prod_i Y_i$ be such that $f \in \overline{A} \cap \overline{B}$. We must show that $f \in \overline{A \cap B}$. If this is not the case there is an open neighborhood U of f with $U \cap A \cap B = \emptyset$. Take a smaller open neighborhood $V \subset U$ at positive distance $\varepsilon > 0$ from the complement of U, namely $d(V, \prod_i Y_i \setminus U) = \varepsilon > 0$.

By definition of the product topology we can take V of the form $V = \prod_{i=0}^{\infty} V_i$, where each V_i is open in Y_i and $V_i = X_i$ for all $i > k$.

Choose $g_n \in A$ with $\lim_n g_n = f$ and $h_n \in B$ with $\lim_n h_n = f$. We can assume that $g_n \in V$ and $h_n \in V$ for every n. Let $f_0 = \pi'_0(f)$ and choose $\varepsilon_0 > 0$ such that $B_{2\varepsilon_0}(f_0) \subseteq V_0$. Then according to Lemma 2.5 there exists a positive

$\delta_0 \leqslant \varepsilon_0$ such that for every $x \in X_0$ there exists a connected open set W_x in X_0 such that $B_{\delta_0}(x) \subseteq W_x \subseteq B_{\epsilon_0}(x)$. As $\lim_n \pi'_0(g_n) = \lim \pi'_0(h_n) = f_0$, there exists n_0 such that $d(\pi'_0(g_n), f_0) < \delta_0$ and $d(\pi'_0(h_n), f_0) < \delta_0$ for all $n > n_0$. Hence $d(\pi'_0(g_n), \pi'_0(h_m)) < 2\delta_0$ for all $m, n > n_0$. Consequently, $W_{\pi'_0(g_n)} \cap W_{\pi'_0(h_m)} \neq \emptyset$ and $W_{\pi'_0(g_n)} \cup W_{\pi'_0(h_m)} \subseteq B_{2\varepsilon_0}(f_0) \subseteq V_0 \cap X_0$ for all $m, n > n_0$. Therefore,

$$C_0 = \bigcup_{n > n_0} W_{\pi'_0(g_n)} \cup W_{\pi'_0(h_n)} \subseteq V_0 \cap X_0$$

is an open connected set and $\pi'_0(g_n), \pi'_0(h_n) \in C_0$ for all $n > n_0$. Suppose for some $r \in \mathbb{N}$ we have constructed a sequence of natural numbers $n_0 \leqslant \cdots \leqslant n_r$ and a sequence of connected open sets $C_i \subseteq X_i \cap V_i$ for each $i \leqslant r$ (so, $C_0 \times \cdots \times C_k \subseteq V_0 \times \cdots \times V_r$) such that

$$\pi'_r(g_n), \pi'_r(h_n) \in C_0 \times \cdots \times C_r \quad \text{for all } n \geqslant n_r.$$

For the inductive step, arguing as above, choose $\varepsilon_{r+1} > 0$ such that $B_{2\varepsilon_{r+1}}(f_{r+1}) \subseteq V_{r+1}$ and find using Lemma 2.5 (as X_{r+1} is ULC) a connected open set $C_{r+1} \subseteq V_{r+1} \cap X_{r+1}$ and $n_{r+1} \geqslant n_r$ such that $\pi'_{r+1}(g_n), \pi'_{r+1}(h_n) \in C_0 \times \cdots \times C_{r+1}$ for all $n \geqslant n_{r+1}$. Let $C = \prod_i C_i$. Then $C \subseteq V \subseteq U$.

Let $s \in C$ and without loss of generality suppose that $s \in A$. Then $s \notin B$, so $U \setminus B$ is a neighborhood of s. It then follows by the definition of the product topology that there is $m \geqslant k$ and non-empty open sets $W_i \subseteq C_i$ with $s \in W = W_0 \times \cdots \times W_m \times \prod_{i>m} X_i \subseteq U \setminus B \subseteq A$. Choose $n \geqslant n_m$, hence $h_n \in C_0 \times \cdots \times C_m \times \prod_{i>m} X_i$. Let $t \in \prod_i X_i$ with $\pi'_m(t) = \pi'_m(s)$ and $\pi''_m(t) = \pi''_m(h_n)$. The connected set

$$Q = C_0 \times \cdots \times C_m \times \{\pi''_m(h_n)\} \subseteq \prod_n X_n$$

meets B as $h_n \in Q$. Since $Q \cap W \neq \emptyset$ and $W \subseteq A$, it follows that $Q \cap A \neq \emptyset$ as well. But then Q is the disjoint union of the non-empty relatively closed sets $Q \cap A$ and $Q \cap B$, contradicting the fact that Q is connected. □

Proof of Theorem 6.10. Let X_n be ULC and precompact for each n, then $\prod_n X_n$ is straight by Theorems 6.11 and 2.7. □

Proof of Theorem B. Item (a) of the theorem was proved in Theorem 6.3.

(b) If X is ULC, then it is also straight. If each X_n is precompact, then X is straight by Theorem 6.10. This proves the implication $(b_2) \to (b_1)$.

To prove the implication $(b_1) \to (b_2)$ suppose that X is straight, but not all X_n are precompact. We must show that X is ULC. According to Proposition 6.7 either X is ULC, or item (c) of the proposition holds true, i.e., all but one of the spaces are both ULC and precompact, and all but finitely many spaces are connected. By Theorem 6.3 the product X is ULC. □

This completely settles the case of infinite products of ULC spaces. We end up with an example.

Example 6.12. According to Corollary 2 from Introduction, X^ω is straight for a ULC space X iff X is either connected or precompact. Let $X = R_2 \cup R_2$, where each R_i is a copy of the reals, each R_i carries the usual metric and $d(R_1, R_2) > 0$. Then X is ULC and neither precompact nor connected. Hence X^ω is not straight.

7. Open questions

We have described when infinite products of ULC spaces are again ULC or straight (Theorem 6.3). The case of precompact spaces is still open, so we start with the following still unsolved

Question 7.1. Let X be a precompact straight space. Is the infinite power X^ω necessarily straight?

More generally:

Question 7.2. Let X_n be a precompact straight space for every $n \in \mathbb{N}$. Is the infinite product $\prod_n X_n$ necessarily straight?

It is easy to see that a positive answer to this question is equivalent to a positive answer to item (b) of the following general question (i.e., the version of Theorem 6.11 for products of *precompact* spaces):

Question 7.3. Let the metric space Y_i be a tight extension of X_i for each $i \in \mathbb{N}$.

(a) Is $\prod_i Y_i$ a tight extension of $\prod_i X_i$.
(b) What about *precompact* metric spaces Y_i?

As far as the more general part (a) is concerned we recall the following well-known facts that give another motivation for the question. The class $\mathcal{P}$ of all perfect maps in the category of topological spaces is known to be determined by the property

$$f \in \mathcal{P} \iff f \times id_Y \in \mathcal{P} \quad \text{for every Hausdorff space } Y. \tag{$*$}$$

Moreover,

(a) $\mathcal{P}$ is closed under composition [5, Corollary 3.7.3];
(b) $\mathcal{P}$ is closed under arbitrary products ([5, Theorem 3.7.7], this is the celebrated Frolík's theorem);
(c) $\mathcal{P}$ is left and right cancelable (i.e., if $fg \in \mathcal{T}$, then $f \in \mathcal{P}$ and $g \in \mathcal{P}$ [5, Proposition 3.7.10]).

The class $\mathcal{T}$ of tight embeddings in the category of metric spaces has similar properties. Indeed, obviously $\mathcal{T}$ is closed under composition and $\mathcal{T}$ is left and right cancelable. Moreover, $\mathcal{T}$ is closed under finite products by Theorem 5.1. Using this one can check that $\mathcal{T}$ has also the property $(*)$ for all metric spaces Y (with $\mathcal{P}$ replaced by $\mathcal{T}$). What is not clear is whether the full counterpart of (b) for countably infinite products is available for $\mathcal{T}$ (this is Question 7.3(a); note that countably infinite products are the limit one should stay in while working with metric spaces). According to Theorem 6.11 this is true if the domains of the maps are ULC. This motivates our hope, that in analogy with the class of perfect maps, also $\mathcal{T}$ is closed under *infinite* products, i.e., Question 7.2 has a positive answer.

Theorem 2.7 gives a criterion for straightness of a dense subspace Y of a straight space X in terms of properties of the embedding $Y \hookrightarrow X$ (namely, when X is a tight extension of Y). The counterpart of this question for *closed* subspaces is somewhat unsatisfactory. We saw that uniform retracts (Corollary 3.6), clopen subspaces (Corollary 3.7), as well as direct summands, of straight spaces are always straight (Corollary 3.5). On the other hand, closed subspaces even of ULC spaces may fail to be straight (see Example 2.10). Another instance when a closed subspace of a straight space fails to be straight is given by the following fact proved in [2]: the spaces X in which *every* closed subspace is straight are precisely the UC spaces [2]. Hence every straight space that is not UC has closed non-straight subspaces. This motivates the following general

Problem 7.4. Find a sufficient condition ensuring that a closed subspace Y of a straight space X is still straight.

Question 7.5. Generalize the results on straight spaces from the category of metric spaces to the category of uniform spaces [7].

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