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On consecutive quadratic non-residues: a conjecture of Issai Schur

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Abstract

Issai Schur once asked if it was possible to determine a bound, preferably using elementary methods, such that for all prime numbers p greater than the bound, the greatest number of consecutive quadratic non-residues modulo p is always less than $p^{1/2}$. This paper uses elementary methods to prove that 13 is the only prime number for which the greatest number of consecutive quadratic non-residues modulo p exceeds $p^{1/2}$.

Keywords: Quadratic non-residues; Schur's conjecture

Issai Schur once asked if it was possible to determine a bound, preferably using elementary methods, such that for all prime numbers p greater than the bound, the greatest possible number of consecutive quadratic non-residues modulo p is always less than $p^{1/2}$. (One can find a brief discussion of this problem in R. K. Guy's book [4]). Schur also pointed out that the greatest number of consecutive quadratic non-residues exceeds $p^{1/2}$ for $p = 13$, since 5, 6, 7, and 8 are all quadratic non-residues $(\text{mod}\, p)$. This paper uses elementary methods to prove the following:

Theorem. *$p = 13$ is the only prime number for which the greatest number of consecutive quadratic non-residues modulo p exceeds $p^{1/2}$.*

This problem has been attacked previously using both analytic and elementary methods. We shall briefly consider the results given to us by analytic

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number theory, and then focus on the elementary methods for the remainder of the paper.

In [3], Burgess proves the following:

Theorem (Burgess [3]). *If χ is any non-trivial Dirichlet character of prime modulus p and $\chi(N+1) = \chi(N+2) = \cdots = \chi(N+H)$, then $H = O(p^{1/4}\log p)$.*

From this, it follows that there must be some M, such that for all $p > M$, the greatest number of consecutive quadratic non-residues modulo p is less than $p^{1/2}$. In [7], Norton asserts that he can refine Burgess's method to obtain the following result:

Theorem (Norton [7]). *In Burgess's theorem, $H < 4.1p^{1/4}\log p$ for all p. If $p > e^{15} \approx 3.27 \times 10^6$, then $H < 2.5p^{1/4}\log p$.*

This result implies that $M = e^{15} \approx 3.27 \times 10^6$ is a suitable value for the aforementioned constant. Unfortunately, however, Norton does not prove this result in his paper, and without a value such as 4.1 for the implied constant in Burgess's theorem, we cannot use Burgess's theorem to find a suitable constant, M, in Schur's conjecture.

Now we consider the work that has been done on the problem using elementary methods. Brauer [2] has proved the following theorem:

Theorem (Brauer [2]). *For prime numbers p, of the form $4n-1$, the maximum length l, of sequences of quadratic residues and non-residues satisfies $l < p^{1/2}$.*

Hudson then considers the case $p \equiv 1 \pmod 4$ by breaking it up into several cases. In [6], he proves that the maximum number of consecutive quadratic non-residues $(\text{mod}\, p)$ is less than $p^{1/2}$ if $p \equiv 1 \pmod{24}$. In [5], he demonstrates that this also holds if $p \equiv 5$ or $p \equiv 17 \pmod{24}$. Putting these together, Hudson obtains the following beneficial result:

Theorem (Hudson [5]). *If p is a prime, and the greatest number of consecutive quadratic non-residues modulo p exceeds $p^{1/2}$, then $p \equiv 13 \pmod{24}$.*

In the same paper, Hudson proposes a proof that the greatest number of consecutive quadratic non-residues modulo p, is less than $p^{1/2}$ for $p > 2^{332}$ and $p \equiv 13 \pmod{24}$. If his proof of this result were correct, it would complete an elementary proof of Schur's conjecture. But upon a careful reading, one sees that the proof of his assertion is flawed. In particular, Hudson claims that the existence of a quadratic non-residue in the interval $(\frac{p^{1/2}}{128} - 2^{3/2}p^{1/8}, \frac{p^{1/2}}{128})$ implies a quadratic *non-residue* is contained in the interval $(p^{1/2} - 2^{17/2}p^{1/8}, p^{1/2})$, whereas it really implies a quadratic *residue* is contained in the interval $(p^{1/2} - 2^{17/2}p^{1/8}, p^{1/2})$, since 2 is a quadratic non-residue modulo p if $p \equiv 13 \pmod{24}$.

The research presented here tackles the case $p \equiv 13 \pmod{24}$ by using an alternate method. In addition to completing an elementary proof of Schur's conjecture, this paper also shows that $p = 13$ is the only prime number for which the greatest number of consecutive quadratic non-residues $(\bmod\, p)$ exceeds $p^{1/2}$.

The argument in this paper breaks down into two parts. In the first part, it is shown that if $p \equiv 13 \pmod{24}$ is sufficiently large and if there exists an interval containing more than $p^{1/2}$ integers, all of which are quadratic non-residues modulo p, then there must exist such an interval J satisfying

$$J \subset \left(\frac{p+3+p^{1/2}}{2}, \frac{p}{2} + 2^{1/2}p^{3/4} - p^{1/2}\right).$$

Then, in the second part, it is shown that this cannot hold for p sufficiently large by demonstrating that one can find two integers, $a, b \in J$ such that if $R \equiv ab \pmod{p}$ and $0 \leqslant R \leqslant p-1$, then $R \in J$. This implies that J contains a quadratic residue R, so J could not have existed in the first place. This implies that the greatest number of consecutive quadratic non-residues modulo p, is less than $p^{1/2}$ when p is sufficiently large. The remaining cases, where p is less than a given bound, are handled by computer.

Lemma 1. *If p is a prime number such that $p \equiv 13 \pmod{24}$, $p > 38\,659$, and there is a sequence of more than $p^{1/2}$ consecutive quadratic non-residues $(\bmod\, p)$, there must be such a sequence in the interval $(\frac{p+3+p^{1/2}}{2}, \frac{p}{2} + 2^{1/2}p^{3/4} - p^{1/2})$.*

Proof. $p \equiv 13 \pmod{24}$ implies that every number of the form $2a^2$ is a quadratic non-residue. This means that there must exist a quadratic non-residue, say N, in the interval

$$(p^{1/2} - 2^{3/2}p^{1/4} + 2, p^{1/2}),$$

because if c is the smallest positive integer such that $2c^2 > p^{1/2}$, then $2(c-1)^2 > p^{1/2} - 2^{3/2}p^{1/4} + 2$.

Suppose J is an integer interval containing more than $p^{1/2}$ consecutive quadratic non-residues. Multiplying each member of J by N and reducing $(\bmod\, p)$, we obtain a collection of quadratic residues in which each quadratic residue differs from the next by $N < p^{1/2}$. This collection must span more than

$$(p^{1/2} - 1)(p^{1/2} - 2^{3/2}p^{1/4} + 2) > p - 2^{3/2}p^{3/4} + p^{1/2}$$

integers. Since -1 is a quadratic residue $(\bmod\, p)$, $p - b$ must be a quadratic non-residue whenever b is. Therefore, if our collection of quadratic residues is to lie entirely outside a sequence of more than $p^{1/2}$ consecutive quadratic non-residues,

J must either be contained in

$$\left(1, 2^{1/2}p^{3/4} - \frac{p^{1/2}}{2}\right),$$

$$\left(p - 2^{1/2}p^{3/4} + \frac{p^{1/2}}{2}, p - 1\right)$$

or

$$\left(\frac{p}{2} - 2^{1/2}p^{3/4} + \frac{p^{1/2}}{2}, \frac{p}{2} + 2^{1/2}p^{3/4} - \frac{p^{1/2}}{2}\right). \tag{1}$$

But J cannot be fully contained in the first of these intervals because the difference between the square numbers in $(1, 2^{1/2}p^{3/4} - \frac{p^{1/2}}{2})$ is less than $p^{1/2}$. Similarly, J cannot be contained in the second of these intervals because any sequence of the form given by J in $(p - 2^{1/2}p^{3/4} + \frac{p^{1/2}}{2}, p - 1)$ would have to correspond to a similar sequence in $(1, 2^{1/2}p^{3/4} - \frac{p^{1/2}}{2})$ So such a J can only be contained in the interval given by (1).

We now refer to the following theorem of Brauer's:

Theorem (Brauer [1]). *The least odd quadratic non-residue u modulo a prime p satisfies $u < 2^{3/5}p^{2/5} + 2^{-(6/5)} \cdot 25p^{1/5} + 3$ for $p = 8n + 5$.*

This implies that there exists an odd quadratic non-residue u, less than $p^{1/2}$ if $p > 38\,659$. Then, since $\frac{p+1}{2}$ is a quadratic non-residue $(\text{mod}\, p)$, $u(\frac{p+1}{2}) \equiv \frac{p+u}{2}$ is a quadratic residue $(\text{mod}\, p)$. Therefore, there exists a quadratic residue in the interval $(\frac{p}{2}, \frac{p+p^{1/2}}{2})$, so there must exist a corresponding quadratic residue in the interval $(\frac{p-p^{1/2}}{2}, \frac{p}{2})$, which means that J cannot pass through $\frac{p}{2}$.

Combining this with the fact that $-b$ is a quadratic non-residue whenever b is, we know that if such a J exists, there must be at least one such J in the interval

$$\left(\frac{p}{2}, \frac{p}{2} + 2^{1/2}p^{3/4} - \frac{p^{1/2}}{2}\right). \tag{2}$$

Now note that 3 is a quadratic residue $(\text{mod}\, p)$. Therefore, for odd m, $\frac{p+3m}{2}$ must be a quadratic residue $(\text{mod}\, p)$ if $\frac{p+m}{2}$ is. Combined with the fact that there exists a quadratic residue in the interval $(\frac{p}{2}, \frac{p+p^{1/2}}{2})$, we find that if such a J lies in the interval given by (2), that same J must also lie in the interval

$$\left(\frac{p + 3 + p^{1/2}}{2}, \frac{p}{2} + 2^{1/2}p^{3/4} - \frac{p^{1/2}}{2}\right). \tag{3}$$

To see why, suppose that $\frac{p+3}{2}+x$ is the first entry in J. Then, we can assume that $\frac{p+1}{2}+x$ is a quadratic residue, meaning $\frac{p}{2}+3(\frac{1}{2}+x)=\frac{p+3}{2}+3x$ is also a quadratic residue. Therefore we must have $\frac{p+3}{2}+3x-(\frac{p+3}{2}+x)>p^{1/2}$, or $x>\frac{p^{1/2}}{2}$. $\square$

Lemma 2. *Suppose $p>38\,659$ is a prime congruent to* 13 *modulo* 24 *and $\frac{p+1}{2}+k$ is a quadratic non-residue, where $k>0$ is some fixed integer. Then, if there exists a such that $\frac{1}{4}\leqslant a\leqslant\frac{15}{32}$, and $(ap^{1/2}-2)^2>2k+2(1-a)p^{1/2}+2-\lfloor p^{1/2}\rfloor$, and the difference between $(\frac{p+1}{2})^2+k+k^2+2kx+x+x^2$ at $x=\lfloor(1-a)p^{1/2}\rfloor$ and $x=\lfloor ap^{1/2}\rfloor-2$ is greater than p, $\frac{p+1}{2}+k$ is not the smallest number in a sequence of more than $p^{1/2}$ consecutive quadratic non-residues.*

Proof. Suppose that all the integers of the form

$$\frac{p+1}{2}+k+m$$

are quadratic non-residues, where m is an integer ranging from 0 to $\lfloor p^{1/2}\rfloor$.

Note that the product of two integers of this form, say $\frac{p+1}{2}+k+m$ and $\frac{p+1}{2}+k+n$, is a quadratic residue, and equals

$$\left(\frac{p+1}{2}\right)^2+(p+1)k+k^2+(m+n)\left(k+\frac{p+1}{2}\right)+mn \tag{4}$$

which if m and n both equal the same value, say x, reduces to

$$\left(\frac{p+1}{2}\right)^2+(p+1)k+k^2+(2x)\left(k+\frac{p+1}{2}\right)+x^2$$

which is congruent to

$$\left(\frac{p+1}{2}\right)^2+k+k^2+2kx+x+x^2 \tag{5}$$

modulo p.

If there exists an a such that $\frac{1}{4}\leqslant a\leqslant\frac{15}{32}$ and the difference between $(\frac{p+1}{2})^2+k+k^2+2kx+x+x^2$ at $x=\lfloor(1-a)p^{1/2}\rfloor$ and $x=\lfloor ap^{1/2}\rfloor-2$ is greater than p, we note that we can find an integer x contained in the interval $(ap^{1/2}-2,(1-a)p^{1/2}]$, and an integer c such that

$$\left(\frac{p+1}{2}\right)^2+k+k^2+2kx+x+x^2>\left(c+\frac{1}{2}\right)p+\tfrac{1}{2}+k+\lfloor p^{1/2}\rfloor$$

and

$$\left(\frac{p+1}{2}\right)^2 + k + k^2 + 2k(x-1) + x - 1 + (x-1)^2 \leqslant \left(c + \frac{1}{2}\right)p + \tfrac{1}{2} + k + \lfloor p^{1/2} \rfloor. \quad (6)$$

Now suppose that

$$\left(\frac{p+1}{2}\right)^2 + k + k^2 + 2k(x-1) + x - 1 + (x-1)^2 \geqslant \left(c + \frac{1}{2}\right)p + \tfrac{1}{2} + k.$$

Then, combining this with (6), we reach the absurd conclusion that a quadratic residue equals a quadratic non-residue. Therefore, we have

$$\left(\frac{p+1}{2}\right)^2 + k + k^2 + 2k(x-1) + x - 1 + (x-1)^2 < \left(c + \frac{1}{2}\right)p + \frac{1}{2} + k. \quad (7)$$

Now consider (4) again. Let m and n vary so that $m = x - y$ and $n = x + y$, where x is an integer that satisfies the above conditions, and y is an integer ranging from 0 to the smallest integer larger than $ap^{1/2} - 2$. Since x lies in $(ap^{1/2} - 2,\ (1-a)p^{1/2}]$, we continue to meet the condition that m and n are both integers between 0 and $\lfloor p^{1/2} \rfloor$ inclusive, because $y < ap^{1/2} - 1$, and $\lfloor p^{1/2} \rfloor - (1-a)p^{1/2} > ap^{1/2} - 1$. If m and n vary this way, the only part of (4) that changes is the product mn. Also note that

$$0 < mn - (m-1)(n+1) = n - m + 1 < p^{1/2}$$

when $0 \leqslant n - m < p^{1/2} - 1$, which holds when y varies as above. So we have a collection of quadratic residues in which no quadratic residue exceeds the next by more than $p^{1/2}$. This collection spans an interval of

$$x^2 - (x - ap^{1/2} + 2)(x + ap^{1/2} - 2) = (ap^{1/2} - 2)^2. \quad (8)$$

Note that increasing x by 1 in (5) increases the value of the expression by $2k + 2x + 2$. Combining this with (7) and (8), we find that one of the quadratic residues in the aforementioned collection is congruent $(\bmod\, p)$ to an integer in the interval, $(\frac{p+1}{2} + k,\ \frac{p+1}{2} + k + \lfloor p^{1/2} \rfloor)$ if

$$(ap^{1/2} - 2)^2 > 2k + 2x + 2 - \lfloor p^{1/2} \rfloor,$$

and since $x \leqslant (1-a)p^{1/2}$, we have

$$(ap^{1/2} - 2)^2 > 2k + 2(1-a)p^{1/2} + 2 - \lfloor p^{1/2} \rfloor, \quad (9)$$

which proves the lemma. □

Theorem. $p = 13$ *is the only prime number for which the greatest number of consecutive quadratic non-residues modulo p exceeds* $p^{1/2}$.

Proof. Suppose $p > 38\,659$, and suppose there exists a sequence of more than $p^{1/2}$ consecutive quadratic non-residues (mod p). As noted earlier, this implies that $p \equiv 13 \pmod{24}$.

Now suppose that

$$\frac{p+1}{2} + k \tag{10}$$

is a quadratic non-residue, where k is a fixed integer. With Lemma 1 in mind, we need only prove that this is not the least quadratic non-residue in a sequence of more than $p^{1/2}$ consecutive quadratic non-residues when

$$\frac{p^{1/2}}{2} + 1 < k < 2^{1/2}p^{3/4} - p^{1/2}.$$

Now consider three cases:

Case 1: $k < 2p^{1/2}$. Note that the difference between (5) at $x = \lfloor \frac{3p^{1/2}}{4} \rfloor$ and $x = \lfloor \frac{p^{1/2}}{4} \rfloor - 2$ is greater than p because it equals

$$\begin{aligned}
&\left((2k+1)\left\lfloor \frac{3p^{1/2}}{4} \right\rfloor + \left\lfloor \frac{3p^{1/2}}{4} \right\rfloor^2 \right) - \left((2k+1)\left(\left\lfloor \frac{p^{1/2}}{4} \right\rfloor - 2 \right) + \left(\left\lfloor \frac{p^{1/2}}{4} \right\rfloor - 2 \right)^2 \right) \\
&\quad = 2k\left(\left\lfloor \frac{3p^{1/2}}{4} \right\rfloor - \left\lfloor \frac{p^{1/2}}{4} \right\rfloor \right) + \left\lfloor \frac{3p^{1/2}}{4} \right\rfloor + 4k + \left\lfloor \frac{3p^{1/2}}{4} \right\rfloor^2 - \left\lfloor \frac{p^{1/2}}{4} \right\rfloor^2 + 3\left\lfloor \frac{p^{1/2}}{4} \right\rfloor - 2 \\
&\quad > 2k\left(\frac{p^{1/2}}{2} - 1 \right) + \frac{3p^{1/2}}{4} - 1 + 4k + \frac{9p}{16} - \frac{3p^{1/2}}{2} + 1 - \frac{p}{16} + 3\left(\frac{p^{1/2}}{4} - 1 \right) - 2 \\
&\quad = kp^{1/2} + 2k + \frac{p}{2} - 5 > \frac{p}{2} + p^{1/2} + p^{1/2} + 2 + \frac{p}{2} - 5 > p,
\end{aligned}$$

since $k > \frac{p^{1/2}}{2} + 1$. With Lemma 2 in mind, we find that (10) is not the least quadratic non-residue in a sequence of more than $p^{1/2}$ consecutive quadratic non-residues if

$$\left(\frac{p^{1/2}}{4} - 2 \right)^2 > 2k + \frac{3p^{1/2}}{2} + 2 - \lfloor p^{1/2} \rfloor. \tag{11}$$

Since $k < 2p^{1/2}$, (11) holds whenever

$$\left(\frac{p^{1/2}}{4} - 2 \right)^2 > \frac{9p^{1/2}}{2} + 3,$$

which holds for $p > 7711$.

Case 2: $2p^{1/2} < k < 8p^{1/2}$. Note that the difference between (5) at $x = \lfloor \frac{5p^{1/2}}{8} \rfloor$ and $\lfloor \frac{3p^{1/2}}{8} \rfloor - 2$ is greater than p because it equals

$$\left((2k+1)\left\lfloor\frac{5p^{1/2}}{8}\right\rfloor + \left\lfloor\frac{5p^{1/2}}{8}\right\rfloor^2\right) - \left((2k+1)\left(\left\lfloor\frac{3p^{1/2}}{8}\right\rfloor - 2\right) + \left(\left\lfloor\frac{3p^{1/2}}{8}\right\rfloor - 2\right)^2\right)$$
$$= 2k\left(\left\lfloor\frac{5p^{1/2}}{8}\right\rfloor - \left\lfloor\frac{3p^{1/2}}{8}\right\rfloor\right) + \left\lfloor\frac{5p^{1/2}}{8}\right\rfloor + 4k + \left\lfloor\frac{5p^{1/2}}{8}\right\rfloor^2 - \left\lfloor\frac{3p^{1/2}}{8}\right\rfloor^2 + 3\left\lfloor\frac{3p^{1/2}}{8}\right\rfloor - 2$$
$$> 2k\left(\frac{p^{1/2}}{4} - 1\right) + \frac{5p^{1/2}}{8} - 1 + 4k + \frac{25p}{64} - \frac{5p^{1/2}}{4} + 1 - \frac{9p}{64} + 3\left(\frac{3p^{1/2}}{8} - 1\right) - 2$$
$$= \frac{kp^{1/2}}{2} + 2k + \frac{p}{4} + \frac{p^{1/2}}{2} - 5 > p + 4p^{1/2} + \frac{p}{4} + \frac{p^{1/2}}{2} - 5 > p,$$

since $k > 2p^{1/2}$. With Lemma 2 in mind, we find that (10) is not the least quadratic non-residue in a sequence of more than $p^{1/2}$ consecutive quadratic non-residues if

$$\left(\frac{3p^{1/2}}{8} - 2\right)^2 > 2k + \frac{5p^{1/2}}{4} + 2 - \lfloor p^{1/2} \rfloor. \qquad (12)$$

Since $k < 8p^{1/2}$, (12) holds when

$$\left(\frac{3p^{1/2}}{8} - 2\right)^2 > \frac{65p^{1/2}}{4} + 3,$$

which holds for $p > 15917$.

Case 3: $8p^{1/2} < k$. Note that the difference between (5) at $x = \lfloor \frac{17p^{1/2}}{32} \rfloor$ and $\lfloor \frac{15p^{1/2}}{32} \rfloor - 2$ is greater than p because it equals

$$\left((2k+1)\left\lfloor\frac{17p^{1/2}}{32}\right\rfloor + \left\lfloor\frac{17p^{1/2}}{32}\right\rfloor^2\right) - \left((2k+1)\left(\left\lfloor\frac{15p^{1/2}}{32}\right\rfloor - 2\right) + \left(\left\lfloor\frac{15p^{1/2}}{32}\right\rfloor - 2\right)^2\right)$$
$$= 2k\left(\left\lfloor\frac{17p^{1/2}}{32}\right\rfloor - \left\lfloor\frac{15p^{1/2}}{32}\right\rfloor\right) + \left\lfloor\frac{17p^{1/2}}{32}\right\rfloor + 4k + \left\lfloor\frac{17p^{1/2}}{32}\right\rfloor^2 - \left\lfloor\frac{15p^{1/2}}{32}\right\rfloor^2 + 3\left\lfloor\frac{15p^{1/2}}{32}\right\rfloor - 2$$
$$> 2k\left(\frac{p^{1/2}}{16} - 1\right) + \frac{17p^{1/2}}{32} - 1 + 4k + \frac{289p}{1024} - \frac{17p^{1/2}}{16} + 1 - \frac{225p}{1024} + 3\left(\frac{15p^{1/2}}{32} - 1\right) - 2$$
$$= \frac{kp^{1/2}}{8} + 2k + \frac{p}{16} + \frac{7p^{1/2}}{8} - 5 > p + 16p^{1/2} + \frac{p}{16} + \frac{7p^{1/2}}{8} - 5 > p,$$

since $k > 8p^{1/2}$. With Lemma 2 in mind, we find that (10) is not the least quadratic non-residue in a sequence of more than $p^{1/2}$ consecutive quadratic non-residues if

$$\left(\frac{15p^{1/2}}{32} - 2\right)^2 > 2k + \frac{17p^{1/2}}{16} + 2 - \lfloor p^{1/2} \rfloor. \tag{13}$$

Since $k < 2^{1/2}p^{3/4} - p^{1/2}$, (13) holds whenever

$$\left(\frac{15p^{1/2}}{32} - 2\right)^2 > 2^{3/2}p^{3/4} - \frac{31p^{1/2}}{16} + 3,$$

which holds for $p > 27\,250$.

So when $p > 38\,659$, no sequence of more than $p^{1/2}$ consecutive quadratic non-residues exists.

Now all that remains is to consider the case $p \leqslant 38\,659$. This case can be handled by a simple computation. I have run a computer program which compares the largest number of consecutive quadratic non-residues modulo p with $p^{1/2}$ for all primes p, such that $p \equiv 13 \pmod{24}$ and $p \leqslant 38\,659$. From this I was able to check that 13 is the only prime number for which the greatest number of consecutive quadratic non-residues $\pmod p$ exceeds $p^{1/2}$. $\square$

Remark. The data obtained from this program can be viewed by going to the website http://www.math.caltech.edu/people/hummel.html. A sample of some of the data obtained from the program is given below. The numbers in each set represent p, the greatest number of consecutive quadratic non-residues $\pmod p$, and $\lfloor p^{1/2} \rfloor$ in that order. For all but the smallest numbers, $p^{1/2}$ far exceeds the greatest number of consecutive quadratic non-residues.

$\{13, 4, 3\}$, $\{757, 8, 27\}$, $\{3181, 9, 56\}$, $\{5869, 9, 76\}$, $\{7237, 10, 85\}$, $\{9397, 10, 96\}$, $\{12037, 11, 109\}$, $\{14389, 12, 119\}$, $\{16477, 12, 128\}$, $\{18517, 13, 136\}$, $\{20509, 13, 143\}$, $\{22381, 12, 149\}$, $\{24061, 13, 155\}$, $\{26029, 13, 161\}$, $\{28429, 13, 168\}$, $\{30469, 14, 174\}$, $\{32749, 15, 180\}$, $\{34693, 14, 186\}$, $\{36709, 15, 191\}$, $\{38653, 15, 196\}$.

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