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On Borel fixed ideals generated in one degree

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Abstract

We construct a shellable polytopal cell complex that supports a minimal free resolution of a Borel fixed ideal, which is Borel generated by just one monomial in $S = \Bbbk[x_1, x_2, \ldots, x_n]$; this includes the case of powers of the homogeneous maximal ideal $(x_1, x_2, \ldots, x_n)$ as a special case. In our most general result we prove that for any Borel fixed ideal I generated in one degree, there exists a polytopal cell complex that supports a minimal free resolution of I.

Keywords: Monomial ideal; Minimal free resolution; Cellular resolution; LCM-lattice

1. Introduction

We study resolutions over the polynomial ring $S = \Bbbk[x_1, x_2, \ldots, x_n]$, where $\Bbbk$ is a field. The idea to encode the structure of the resolution of a monomial ideal in the combinatorial structure of a simplicial complex was introduced in [3] (see also [9]). The idea was generalized later in [4], where resolutions supported on a *regular* cell complex were introduced. The generalization continued in [2] and [14], where monomial resolutions supported on a CW-complex were introduced and studied. An example of a monomial ideal whose minimal free resolution is supported on a CW-complex, but cannot be supported on a regular cell complex is given in [19]. More importantly, a large class of monomial ideals whose resolution cannot be supported on a CW-complex is also given in [19].

In this paper we study d-generated Borel fixed ideals, i.e. Borel fixed ideals generated in the same degree d. The Eliahou–Kervaire resolution of a d-generated Borel fixed ideal is min-

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imal and can be supported on a CW-complex, as proved in [2] using discrete Morse theory. As a worked example in [2], the authors give the Morse complex that supports a minimal free resolution for powers of the homogeneous maximal ideal $(x_1, \dots, x_n)$ of the polynomial ring $S = \Bbbk[x_1, x_2, \dots, x_n]$. More generally, the Morse complex that supports a minimal free resolution of a Borel principal ideal is given in [14]. It is not clear whether any of those Morse complexes are regular. Thus a natural question is the following:

Does there exists a *regular* cell complex that supports a minimal resolution of a d-generated Borel fixed ideal?

We answer the above question positively in this chapter, which is organized as follows:

In Section 2, we give the basic notation and preliminaries for the rest of this paper and we refer to the literature for more details.

In Section 3, we answer the above natural question by constructing inductively a shellable polytopal cell complex that supports the minimal free resolution of a Borel principal ideal in $S = \Bbbk[x_1, x_2, \dots, x_n]$; this includes the case of powers of the homogeneous maximal ideal as a special case. Our most general result is Theorem 20, where we prove that for any d-generated Borel fixed ideal I, there exists a polytopal cell complex that supports a minimal free resolution of I. It should be noted that the basis we use in the minimal free resolution is different than the one used in the Eliahou–Kervaire resolution (see [10]).

Finally, in Section 4, we consider the lcm-lattice of a d-generated Borel fixed ideal. In particular, in Proposition 22, we show that it is ranked.

2. Notation-preliminaries

2.1. Monomial ideals

We work over the polynomial ring $S = \Bbbk[x_1, x_2, \dots, x_n]$ with $\operatorname{char}(\Bbbk) = 0$. In examples, we often use the letters $a, b, c, d, \dots,$ instead of $x_1, x_2, x_3, x_4, \dots,$ respectively. For a monomial $\mathbf{m} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in S, we define the exponent vector to be $e(\mathbf{m}) = (a_1, a_2, \dots, a_n)$ and we set $\max(\mathbf{m})$ to be the largest index of a variable that divides $\mathbf{m}$.

All ideals in this paper are considered to be *monomial* ideals. If I is a monomial ideal, then it is trivial to determine whether a monomial $\mathbf{m}$ is in I: $\mathbf{m} \in I$ if and only if $\mathbf{m}$ is divisible by a monomial generator $\mathbf{n}$ of I. Note that by removing any monomial generators of I divisible by other generators, every monomial ideal $I \subseteq S$ has a *unique* minimal finite set $G(I)$ of monomial generators.

A monomial ideal I in S is *Borel fixed*, if for every $\mathbf{m}$ in $G(I)$ and every x_t that divides $\mathbf{m}$,

$$\mathbf{m}_{t \to s} := \frac{\mathbf{m}}{x_t} x_s$$

is in I for all $1 \leqslant s < t$. Let $\Gamma = \{\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_r\}$ be a finite set of monomials in S. If I is the smallest Borel fixed ideal such that Γ is a subset of $G(I)$, then we say that I is *Borel generated* by $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_r$ and we write

$$I = \langle \mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_r \rangle.$$

In particular, if $\Gamma = \{\mathbf{m}\}$, then we call I *Borel principal* and we write

$$I = \langle \mathbf{m} \rangle.$$

Example 1. Let $R = \Bbbk[a, b, c]$. The ideal (a^2, ab, b^2, ac, bc) is a Borel fixed ideal, which is also Borel principal, because

$$\left(a^2, ab, b^2, ac, bc\right) = \langle bc\rangle.$$

For more on monomial ideals we refer to [8,9,16].

2.2. *Polytopal complexes*

We assume familiarity with the basic notions of CW-complex, polytopal complex and *regular* cell complex and their differences. Recall that the closures of the cells of a regular CW-complex are homeomorphic with closed balls. For example, any polytopal cell complex is regular.

The underlying set of a polytopal complex X is the point set $|X| := \bigcup_{P \in X} P$. A *subdivision* of a polytope X is a polytopal complex X' with the underlying space $|X| = X'$. This means that X is the union of the polytopes in X'. A subdivision X' of a polytopal complex X is called *regular* if there is a piecewise linear convex function $f : |X| \to \mathbb{R}$ whose domains of linearity are the facets of X'. Such a function f is called a *support function* for the subdivision X' (see, e.g., [7, p. 34]).

Example 3. (See [7, p. 37].) Let $X \subset \mathbb{R}^N$ be a polytope and let $H \subset \mathbb{R}^N$ be a hyperplane given by the equation $a_1x_1 + \cdots + a_Nx_N = b$ $(a_1, \ldots, a_N, b \in \mathbb{R})$. If H cuts X in two parts of the same dimension, say $X = Z \cup W$, then the union of the face lattices of Z and W forms a regular subdivision of X. The function $f : |X| \to \mathbb{R}$, with $f(y) = |a_1y_1 + \cdots + a_Ny_N - b|$, $y = (y_1, \ldots, y_N)$ is a support function of this regular subdivision. Taking intersections of such subdivisions one concludes that an arbitrary finite system of hyperplanes $H_1, \ldots, H_k \subset \mathbb{R}^N$ cuts X into smaller polytopes that define a regular subdivision of X.

Let X be a pure k-dimensional polytopal complex ($k \geqslant 1$). As in [20, p. 233], a *shelling* of X is a linear ordering $F_1, F_2, \ldots, F_s$ of the facets of X, which satisfies the following condition:

For $1 < j \leqslant s$ the intersection of the facet F_j with the previous facets is nonempty and is a beginning of a shelling of the $(k-1)$-dimensional boundary complex of F_j, that is,

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right) = G_1 \cup G_2 \cup \cdots \cup G_r$$

for some shelling $G_1, \ldots, G_r, \ldots, G_t$ of the boundary complex of F_j, and $1 \leqslant r \leqslant t$.

A polytopal complex is *shellable* if it is pure and has a shelling. Note that polytopes are shellable [20, p. 240] and that a regular subdivision of a polytope is shellable [20, p. 243]. For more on polytopal complexes we refer to [20].

2.3. *Cellular resolutions*

As in [4], let X be a *regular* cell complex having $G(I)$, the set of minimal generators of I, as its set of vertices and let ϵ_X be an *incidence function* on X. It is well known that such a function exists (see, e.g., pp. 244–248 in [15]). Next we label each nonempty face F of X by the least common multiple $\mathbf{m}_F$ of the monomials $\mathbf{m}_j$ in $G(I)$, which correspond to the vertices of F. The *degree* $\mathbf{a}_F$ of the face F is defined to be the exponent vector $e(\mathbf{m}_F)$.

Let SF be the free S-module with one generator F in degree $\mathbf{a}_F$. The *cellular complex* $\mathbf{F}_X$ is the $\mathbb{Z}^n$-graded S-module $\bigoplus_{\emptyset \neq F \in X} SF$ with differential

$$\partial F = \sum_{\emptyset \neq F' \in X} \epsilon_X(F, F') \frac{\mathbf{m}_F}{\mathbf{m}_{F'}} F'.$$

For each degree $\mathbf{b} \in \mathbb{Z}^n$ let $X_{\preccurlyeq \mathbf{b}}$ be the subcomplex of X on the vertices of degree $\preccurlyeq \mathbf{b}$. The following results are proved in [4].

Proposition 1. *The complex $\mathbf{F}_X$ is a free resolution of I if and only if $X_{\preccurlyeq \mathbf{b}}$ is acyclic over $\Bbbk$ for all degrees $\mathbf{b}$. In this case, $\mathbf{F}_X$ is called a cellular resolution of I.*

Corollary 2. *The cellular complex $\mathbf{F}_X$ is a resolution of I if and only if the cellular complex $\mathbf{F}_{X_{\preccurlyeq \mathbf{b}}}$ is a resolution of the monomial ideal $I_{\preccurlyeq \mathbf{b}}$ for all $\mathbf{b} \in \mathbb{Z}^n$.*

Remark 3. A cellular resolution $\mathbf{F}_X$ is minimal if and only if any two comparable faces $F' \subseteq F$ of the same degree coincide.

The above results are presented in [16] for polytopal complexes.

Example 2. Let $I \subset S$ be a monomial ideal with $G(I) = \{x_1^d, x_2^d, \ldots, x_n^d\}$ for a fixed positive integer d. Then the labeled $(n-1)$-simplex $\Delta_{n-1}(x_1^d, x_2^d, \ldots, x_n^d)$ with vertices in $G(I)$ supports a minimal free resolution of I.

Note that in this paper, we use the term "*cellular*" resolution for a resolution supported on a *regular* cell complex.

2.4. Results from algebraic topology

We need the cellular version of Mayer–Vietoris Theorem and the Künneth Theorem with field coefficients. See [13] or [15] for more details.

Theorem 4 *(Mayer–Vietoris). Let X be a CW-complex and let Y_1 and Y_2 be CW subcomplexes of X such that $X = Y_1 \cup Y_2$. Then there is an exact sequence*

$$\cdots \to \tilde{H}_i(Y_1 \cap Y_2; \Bbbk) \to \tilde{H}_i(Y_1; \Bbbk) \oplus \tilde{H}_i(Y_2; \Bbbk) \to \tilde{H}_i(X; \Bbbk) \to \tilde{H}_{i-1}(Y_1 \cap Y_2; \Bbbk) \to \cdots.$$

Theorem 5 *(Künneth). Let X and Y be two CW-complexes. Then there is a natural isomorphism*

$$\bigoplus_j \big(H_j(X; \Bbbk) \otimes_{\Bbbk} H_{i-j}(Y; \Bbbk)\big) \to H_i(X \times Y; \Bbbk).$$

3. Cellular resolutions of d-generated Borel fixed ideals

3.1. Three basic lemmas

Let I and J be two monomial ideals in S and assume that X and Y are regular cell complexes in some $\mathbb{R}^N$ that support a minimal free resolution of I and J, respectively.

Can we say anything about the cellular resolution of $I + J$ and/or the cellular resolution of IJ?

The following three lemmas give some results related to this question, which will be useful in proving our main results.

Lemma 6. *Let I and J be two monomial ideals in S such that $G(I+J) = G(I) \cup G(J)$. Suppose that*

(i) *X and Y are labeled regular cell complexes in some $\mathbb{R}^N$ that support a minimal free resolution $\mathbf{F}_X$ and $\mathbf{F}_Y$ of I and J, respectively, and*
(ii) *$X \cap Y$ is a labeled regular cell complex that supports a minimal free resolution $\mathbf{F}_{X\cap Y}$ of $I \cap J$.*

Then $X \cup Y$ is a labeled regular cell complex that supports a minimal free resolution $\mathbf{F}_{X\cup Y}$ of $I + J$.

Proof. First let $Z := X \cup Y$ and note that Z is a regular cell complex. Since $G(I+J) = G(I) \cup G(J)$, the vertices of Z are labeled by the elements of $G(I+J)$ in a way that extends the labeling of the vertices of X by $G(I)$ and the labeling of the vertices of Y by $G(J)$. From the labeling of X, Y and $X \cap Y$ and our assumptions above, it follows that $G(I \cap J) = G(I) \cap G(J)$. From our hypothesis, we have

$$\tilde{H}_i(X_{\preccurlyeq \mathbf{b}}; \Bbbk) = 0, \quad \tilde{H}_i(Y_{\preccurlyeq \mathbf{b}}; \Bbbk) = 0, \quad \text{and} \quad \tilde{H}_i\big((X \cap Y)_{\preccurlyeq \mathbf{b}}; \Bbbk\big) = 0$$

for all i and all $\mathbf{b} \in \mathbb{Z}^n$. Furthermore, it is clear from our labeling and the definition of Z that

$$Z_{\preccurlyeq \mathbf{b}} = (X \cup Y)_{\preccurlyeq \mathbf{b}} = X_{\preccurlyeq \mathbf{b}} \cup Y_{\preccurlyeq \mathbf{b}}$$

for all $\mathbf{b} \in \mathbb{Z}^n$. Then the Mayer–Vietoris Theorem gives us the following exact sequence

$$\tilde{H}_i(X_{\preccurlyeq \mathbf{b}}; \Bbbk) \oplus \tilde{H}_i(Y_{\preccurlyeq \mathbf{b}}; \Bbbk) \to \tilde{H}_i(Z_{\preccurlyeq \mathbf{b}}; \Bbbk) \to \tilde{H}_{i-1}\big((X \cap Y)_{\preccurlyeq \mathbf{b}}; \Bbbk\big).$$

Consequently, $\tilde{H}_i(Z_{\preccurlyeq \mathbf{b}}; \Bbbk) = 0$ and so $\mathbf{F}_Z$ is a cellular resolution of $I + J$ from Proposition 1. The minimality of $\mathbf{F}_Z$ follows immediately from the minimality of $\mathbf{F}_X$ and $\mathbf{F}_Y$ and Remark 3. $\square$

Remark 7. For any two monomial ideals I and J, we have

$$G(I+J) \subseteq G(I) \cup G(J).$$

A case where equality becomes true is when all elements of $G(I) \cup G(J)$ are of the same degree.

Lemma 8. *Let $I \subset \Bbbk[x_1, \ldots, x_k]$ and $J \subset \Bbbk[x_{k+1}, \ldots, x_n]$ be two monomial ideals. Suppose that X and Y are labeled regular cell complexes in some $\mathbb{R}^N$ of dimension $k-1$ and $n-k-1$, respectively, that support a minimal free resolution $\mathbf{F}_X$ and $\mathbf{F}_Y$ of I and J, respectively. Then the labeled regular cell complex $X \times Y$ supports a minimal free resolution $\mathbf{F}_{X\times Y}$ of IJ.*

Proof. Let $Z := X \times Y$ and label the vertices of Z with the product of the labels of the corresponding vertices of X and Y. This is a well defined labeling because from hypotheses it follows that $G(IJ) = G(I)G(J)$. Let $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2) \in \mathbb{Z}^n$, where $\mathbf{b}_1 \in \mathbb{Z}^k$ and $\mathbf{b}_2 \in \mathbb{Z}^{n-k}$. Then, it is easy to check that

$$Z_{\preccurlyeq \mathbf{b}} = (X \times Y)_{\preccurlyeq \mathbf{b}} = X_{\preccurlyeq \mathbf{b}_1} \times Y_{\preccurlyeq \mathbf{b}_2}.$$

From the Künneth Theorem for CW complexes, there is an isomorphism

$$\bigoplus_j \bigl(H_j(X_{\preccurlyeq \mathbf{b}_1}; \Bbbk) \otimes_{\Bbbk} H_{i-j}(Y_{\preccurlyeq \mathbf{b}_2}; \Bbbk)\bigr) \cong H_i(X_{\preccurlyeq \mathbf{b}_1} \times Y_{\preccurlyeq \mathbf{b}_2}; \Bbbk) = H_i(Z_{\preccurlyeq \mathbf{b}}; \Bbbk)$$

for all i. From our hypothesis, we have

$$\tilde{H}_i(X_{\preccurlyeq \mathbf{b}_1}; \Bbbk) = 0 \quad \text{and} \quad \tilde{H}_i(Y_{\preccurlyeq \mathbf{b}_2}; \Bbbk) = 0,$$

for all i. Therefore,

$$H_0(Z_{\preccurlyeq \mathbf{b}}; \Bbbk) = \Bbbk \otimes_{\Bbbk} \Bbbk = \Bbbk,$$

while

$$H_i(Z_{\preccurlyeq \mathbf{b}}; \Bbbk) = 0,$$

for $i > 0$. Therefore, $\mathbf{F}_Z$ is a cellular resolution of IJ from Proposition 1.

Now assume that the cellular resolutions $\mathbf{F}_X$ and $\mathbf{F}_Y$ are minimal and let $e_X \times e_Y$ and $\sigma_X \times \sigma_Y$ be two comparable faces of $X \times Y$ with the same label. That is,

$$e_X \subset \sigma_X \quad \text{and} \quad e_Y \subset \sigma_Y$$

and

$$\mathit{label}(e_X \times e_Y) = \mathit{label}(\sigma_X \times \sigma_Y) = (\mathbf{b}_1, \mathbf{b}_2)$$

where $\mathbf{b}_1 \in \mathbb{Z}^k$ and $\mathbf{b}_2 \in \mathbb{Z}^{n-k}$. Then

$$\mathit{label}(e_X) = \mathit{label}(\sigma_X) = (\mathbf{b}_1)$$

and

$$\mathit{label}(e_Y) = \mathit{label}(\sigma_Y) = (\mathbf{b}_2),$$

respectively. From Remark 3 and the minimality of $\mathbf{F}_X$ and $\mathbf{F}_Y$, it follows that $e_X = \sigma_X$ and $e_Y = \sigma_Y$, Therefore, $e_X \times e_Y = \sigma_X \times \sigma_Y$ and so $\mathbf{F}_Z$ is minimal. The proof is complete. $\square$

Remark 9. From our conclusion in Lemma 8, it follows that

$$\begin{aligned}\operatorname{proj}\dim(S/IJ) &= \dim(X \times Y) + 1\\ &= \dim(X) + \dim(Y) + 1\\ &= (k-1) + (n-k-1) + 1\\ &= n-1.\end{aligned}$$

Lemma 10. *Let $I \subset \Bbbk[x_1, \ldots, x_k]$ and $J \subset \Bbbk[x_k, \ldots, x_n]$ be two monomial ideals such that $G(IJ) = G(I)G(J)$. Suppose that there exists a regular cell complex X in $\mathbb{R}^N$ (for some N) of dimension $k-1$ and a regular cell complex Y in $\mathbb{R}^{n-k}$ of dimension $n-k$, which support a minimal free resolution $\mathbf{F}_X$ of I and $\mathbf{F}_Y$ J respectively. Then the regular cell complex $X \times Y$ supports a minimal free resolution $\mathbf{F}_{X\times Y}$ of IJ.*

Proof. Let $Z := X \times Y$ and label the vertices of Z with the product of the labels of the corresponding vertices of X and Y. This is a well defined labeling because $G(IJ) = G(I)G(J)$. Let $\mathbf{b} = (\mathbf{b}_1, \beta, \mathbf{b}_2) \in \mathbb{Z}^n$, where $\mathbf{b}_1 \in \mathbb{Z}^{k-1}$, $\beta \in \mathbb{Z}$ and $\mathbf{b}_2 \in \mathbb{Z}^{n-k}$. Then, for $1 \leqslant k \leqslant \beta - 1$ define iteratively $Z^{(k)}_{\preccurlyeq\mathbf{b}}$ as follows

$$Z^{(k+1)}_{\preccurlyeq\mathbf{b}} = Z^{(k)}_{\preccurlyeq\mathbf{b}} \cup (X_{\preccurlyeq(\mathbf{b}_1,k)} \times Y_{\preccurlyeq(\beta-k,\mathbf{b}_2)}),$$

where $Z^{(1)}_{\preccurlyeq\mathbf{b}} = X_{\preccurlyeq(\mathbf{b}_1,0)} \times Y_{\preccurlyeq(\beta,\mathbf{b}_2)}$, and note that

$$\begin{aligned}Z_{\preccurlyeq\mathbf{b}} = Z^{(\beta)}_{\preccurlyeq\mathbf{b}} = (X_{\preccurlyeq(\mathbf{b}_1,0)} \times Y_{\preccurlyeq(\beta,\mathbf{b}_2)}) \cup (X_{\preccurlyeq(\mathbf{b}_1,1)} \times Y_{\preccurlyeq(\beta-1,\mathbf{b}_2)}) \cup \cdots\\ \cup (X_{\preccurlyeq(\mathbf{b}_1,k)} \times Y_{\preccurlyeq(\beta-k,\mathbf{b}_2)}).\end{aligned}$$

Moreover,

$$(X_{\preccurlyeq(\mathbf{b}_1,0)} \times Y_{\preccurlyeq(\beta,\mathbf{b}_2)}) \cap (X_{\preccurlyeq(\mathbf{b}_1,1)} \times Y_{\preccurlyeq(\beta-1,\mathbf{b}_2)}) = X_{\preccurlyeq(\mathbf{b}_1,0)} \times Y_{\preccurlyeq(\beta-1,\mathbf{b}_2)},$$

or more generally,

$$Z^{(k)}_{\preccurlyeq\mathbf{b}} \cap (X_{\preccurlyeq(\mathbf{b}_1,k+1)} \times Y_{\preccurlyeq(\beta-k-1,\mathbf{b}_2)}) = X_{\preccurlyeq(\mathbf{b}_1,0)} \times Y_{\preccurlyeq(\beta-k-1,\mathbf{b}_2)}.$$

By combining the Mayer–Vietoris Theorem with the Künneth Theorem we get

$$\tilde{H}_i(Z_{\preccurlyeq\mathbf{b}}; \Bbbk) = 0.$$

Therefore, $\mathbf{F}_Z$ is a cellular resolution of IJ from Proposition 1. Now assume that the cellular resolutions $\mathbf{F}_X$ and $\mathbf{F}_Y$ are minimal and let $e_X \times e_Y$ and $\sigma_X \times \sigma_Y$ be two comparable faces of $X \times Y$ with the same label. That is,

$$e_X \subset \sigma_X \quad \text{and} \quad e_Y \subset \sigma_Y$$

and

$$label(e_X \times e_Y) = label(\sigma_X \times \sigma_Y) = (\mathbf{b}_1, \beta, \mathbf{b}_2)$$

where $\mathbf{b}_1 \in \mathbb{Z}^{k-1}$, $\beta \in \mathbb{Z}$ and $\mathbf{b}_2 \in \mathbb{Z}^{n-k}$. Then,

$$label(e_X) = (\mathbf{b}_1, \beta_1) \quad \text{and} \quad label(\sigma_X) = (\mathbf{b}_1, \beta_2),$$

which implies $\beta_1 \leqslant \beta_2$ and

$$label(e_Y) = (\beta - \beta_1, \mathbf{b}_2) \quad \text{and} \quad label(\sigma_Y) = (\beta - \beta_2, \mathbf{b}_2),$$

which implies $\beta - \beta_1 \leqslant \beta - \beta_2$, that is, $\beta_2 \leqslant \beta_1$. Thus we have $\beta_1 = \beta_2$ and then,

$$label(e_X) = label(\sigma_X) = (\mathbf{b}_1, \beta_1)$$

and

$$label(e_Y) = label(\sigma_Y) = (\beta - \beta_1, \mathbf{b}_2).$$

From Remark 3 and the minimality of $\mathbf{F}_X$ and $\mathbf{F}_Y$, it follows that $e_X = \sigma_X$ and $e_Y = \sigma_Y$. Therefore, $e_X \times e_Y = \sigma_X \times \sigma_Y$ and so $\mathbf{F}_Z$ is minimal. The proof is complete. □

Remark 11.

(1) For any two monomial ideals I and J, we have

$$G(IJ) \subseteq G(I)G(J).$$

Thus our assumption that $G(IJ) = G(I)G(J)$ is equivalent to $|G(IJ)| = |G(I)| \cdot |G(J)|$.

(2) As in Remark 9, from our conclusion in Lemma 10, it follows that

$$\begin{aligned} \operatorname{proj}\dim(S/IJ) &= \dim(X \times Y) + 1 \\ &= \dim(X) + \dim(Y) + 1 \\ &= (k-1) + (n-k) + 1 \\ &= n. \end{aligned}$$

(3) A lemma similar to Lemmas 8 and 10 for monomial ideals

$$I \subset \mathbb{k}[x_1, \ldots, x_{k-1}, x_k] \quad \text{and} \quad J \subset \mathbb{k}[x_{k-1}, x_k, \ldots, x_n]$$

and corresponding labeled regular cell complexes X and Y with

$$\dim(X) = k - 1 \quad \text{and} \quad \dim(Y) = n - k + 1$$

would fail because we would have

$$\begin{aligned}\dim(X \times Y) + 1 &= \dim(X) + \dim(Y) + 1 \\ &= (k-1) + (n-k+1) + 1 \\ &= n+1 \\ &> \operatorname{proj}\dim(S/IJ),\end{aligned}$$

which contradicts Hilbert's Syzygy Theorem (see, e.g. [8, p. 478]).

(4) Let X and Y be labeled regular cell complexes in some $\mathbb{R}^N$ that support a minimal free resolution $\mathbf{F}_X$ and $\mathbf{F}_Y$ of I and J, respectively. If $X \times Y$ is a labeled cell complex that supports a minimal free resolution of IJ, then

$$\mathbf{F}_{\mathbf{X}\times\mathbf{Y}} = \mathbf{F}_{\mathbf{X}} \otimes \mathbf{F}_{\mathbf{Y}}.$$

(see, e.g., pp. 280–282 in [15]).

Example 4. Let $R = \Bbbk[a, b, c]$. The resolution of $I = (a, b)$ is of the form

$$0 \to R(-2) \to R^2(-1) \to (a, b) \to 0$$

and the resolution of $J = (b, c)$ is of the form

$$0 \to R(-2) \to R^2(-1) \to (b, c) \to 0.$$

Therefore, the resolution of $IJ = (a, b)(b, c)$ is of the form

$$0 \to R(-4) \to R^4(-3) \to R^4(-2) \to IJ \to 0,$$

which is the tensor product of the first two resolutions.

3.2. Powers of the homogeneous maximal ideal

Now we may prove our first main result, which is about the powers of the homogeneous maximal ideal in S.

Theorem 12. *There exists a shellable polytopal cell complex $P_d(x_1, \dots, x_n)$ that supports a minimal free resolution of $(x_1, \dots, x_n)^d$. Moreover, $P_d(x_1, \dots, x_n)$ is a polytopal subdivision of the $(n-1)$-simplex $\Delta_{n-1}(x_1^d, x_2^d, \dots, x_n^d)$.*

Proof. The proof will be by induction on d. It is clear that if $d = 1$, then the standard $(n-1)$-simplex denoted by $\Delta_{n-1}(x_1, x_2, \dots, x_n)$, supports a minimal free resolution of $(x_1, \dots, x_n)$ for all $n \geqslant 1$. Thus

$$P_1(x_1, \dots, x_n) = \Delta_{n-1}(x_1, x_2, \dots, x_n)$$

for all $n \geqslant 1$. Also, $P_1(x_{k+1}, \dots, x_n)$ is a subcomplex of $P_1(x_k, \dots, x_n)$ for all $k < n$. Assume that for some $d \geqslant 1$ we have constructed $P_d(x_1, \dots, x_n)$ for all $n \geqslant 1$ and that $P_d(x_{k+1}, \dots, x_n)$ is a subcomplex of $P_d(x_k, \dots, x_n)$ for all $k < n$. Define the ideals

$$I_k = (x_1, x_2, \dots, x_k)(x_k, x_{k+1}, \dots, x_n)^d$$

and note that an easy (finite) induction on k gives us

$$I_1 + \cdots + I_k = (x_1, \dots, x_k)(x_1, x_2, \dots, x_n)^d$$

for all $1 \leqslant k \leqslant n$. Indeed, assuming that we have proved it for $k-1$, for some $k > 1$, then we have

$$\begin{aligned} I_1 + \cdots + I_{k-1} + I_k &= (x_1, \dots, x_{k-1})(x_1, x_2, \dots, x_n)^d + (x_1, \dots, x_k)(x_k, \dots, x_n)^d \\ &= (x_1, \dots, x_{k-1})(x_1, x_2, \dots, x_n)^d + x_k(x_k, \dots, x_n)^d \\ &= (x_1, \dots, x_{k-1})(x_1, x_2, \dots, x_n)^d + x_k(x_1, \dots, x_n)^d \\ &= (x_1, \dots, x_k)(x_1, \dots, x_n)^d. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} (I_1 + \cdots + I_k) \cap I_{k+1} &= (x_1, \dots, x_k)(x_1, \dots, x_n)^d \cap (x_1, \dots, x_{k+1})(x_{k+1}, \dots, x_n)^d \\ &= (x_1, \dots, x_k)(x_{k+1}, \dots, x_n)^d. \end{aligned}$$

From Lemmas 8 and 10, we conclude that the polytopal cell complexes C_k and D_k ($k = 1, 2, \dots, n$) defined by

$$C_k := \Delta_{k-1}(x_1, x_2, \dots, x_k) \times P_d(x_k, \dots, x_n)$$

and

$$D_k := C_k \cap C_{k+1} = \Delta_{k-1}(x_1, x_2, \dots, x_k) \times P_d(x_{k+1}, \dots, x_n)$$

support a minimal free resolution for I_k and $(I_1 + \cdots + I_k) \cap I_{k+1}$, respectively.

Thus from this and Lemma 6, the polytopal cell complex C'_k, which is defined recursively by

$$C'_1 = C_1, \quad \text{and} \quad C'_{k+1} = C'_k \cup C_{k+1}$$

for $k \geqslant 1$, supports a (minimal) free resolution for $(x_1, \dots, x_k)(x_1, x_2, \dots, x_n)^d$. Accordingly, set

$$P_{d+1}(x_1, x_2, \dots, x_n) := C'_n = C_1 \cup C_2 \cup \cdots \cup C_n$$

and the construction of our polytopal cell complex is done by induction. The intersection of any two polytopes of $P_d(x_1, \dots, x_n)$ is empty or another polytope of $P_d(x_1, \dots, x_n)$ smaller dimension. The $(n-1)$-simplex $\Delta_{n-1}(x_1^d, x_2^d, \dots, x_n^d)$ is the union of the polytopes in $P_d(x_1, \dots, x_n)$, and so $P_d(x_1, \dots, x_n)$ is a polytopal subdivision of $\Delta_{n-1}(x_1^d, x_2^d, \dots, x_n^d)$. From our construction and Example 3, it follows that $P_d(x_1, \dots, x_n)$ is a regular subdivision of $\Delta_{n-1}(x_1^d, x_2^d, \dots, x_n^d)$. Therefore, $P_d(x_1, \dots, x_n)$ is shellable. □

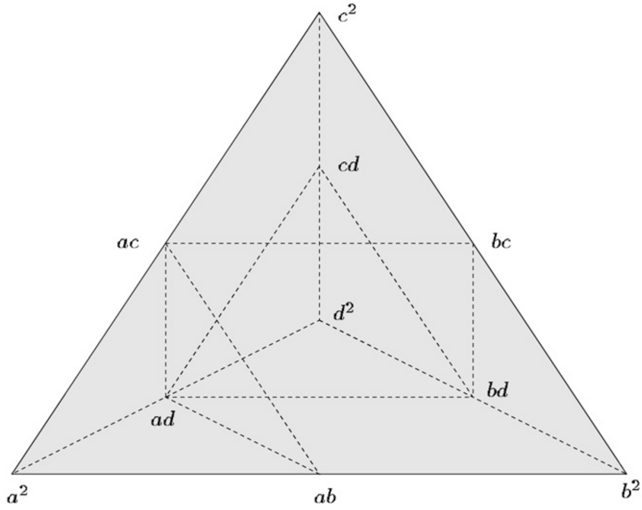


Fig. 1.

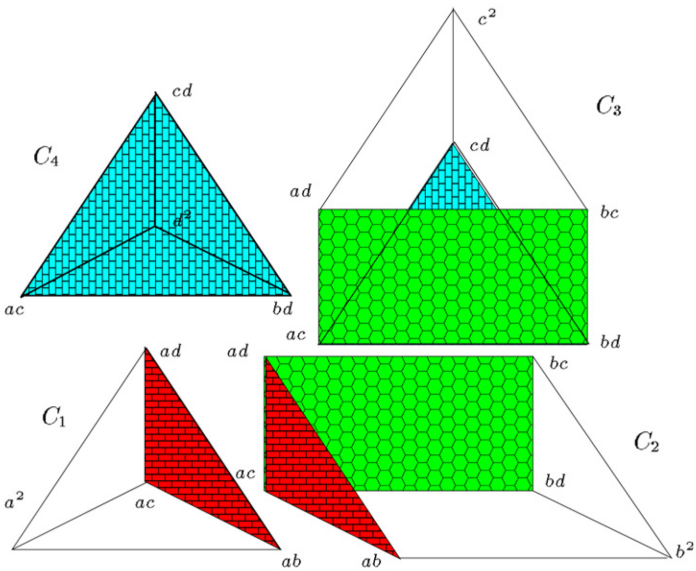


Fig. 2.

Example 5. Let $I = (a, b, c, d)^2$. Using the software package MACAULAY 2 [12], we see that the polytopal cell complex that supports the minimal free resolution of I is as shown in Fig. 1.

This can be decomposed as shown in Fig. 2.

Another cell complex that supports a minimal free resolution of I is shown in Fig. 3 (Morse complex), which supports the Eliahou–Kervaire resolution of I. In particular, note that it is not polytopal.

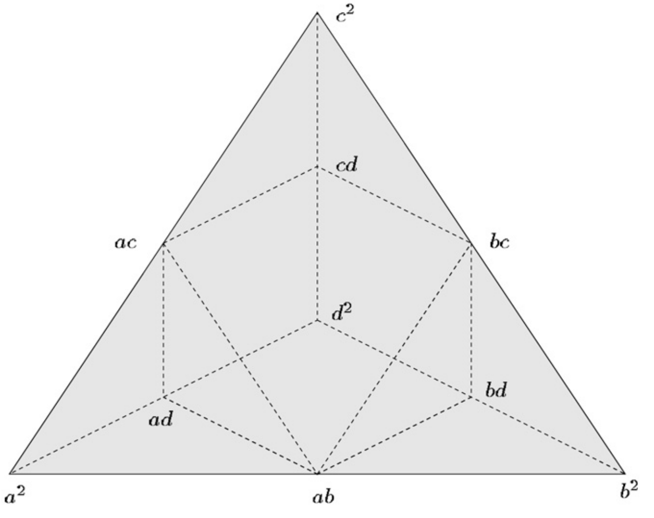


Fig. 3.

Remark 13. From Theorem 12 and Corollary 2, we may get a minimal *cellular* resolution for all ideals of the form $I_{\preccurlyeq\mathbf{b}}$ ($\mathbf{b} \in \mathbb{Z}^n$) (see also [17]).

3.3. Borel principal ideals

Our next goal is to prove a more general result for Borel principal ideals. Note that the following theorem includes Theorem 12 as a special case, since

$$(x_1, \dots, x_n)^d = \langle x_n^d \rangle.$$

Theorem 14. *There exists a shellable polytopal cell complex $Q(\mathbf{m})$ that supports a minimal free resolution of the Borel principal ideal*

$$I = \langle \mathbf{m} \rangle = \prod_{j=1}^{s} I_{\lambda_j}^{d_j},$$

where $\mathbf{m} = x_{\lambda_1}^{d_1} x_{\lambda_2}^{d_2} \cdots x_{\lambda_s}^{d_s}$, $I_i = (x_1, x_2, \dots, x_i)$ and $1 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_s \leqslant n$. Moreover, $Q(\mathbf{m})$ is a subcomplex of $P_d(x_1, \dots, x_n)$, where $d = \deg(\mathbf{m})$. In particular, $Q(\mathbf{m})$ is the union of all the convex polytopes (i.e. the faces) of the polytopal cell complex $P_d(x_1, \dots, x_n)$, with vertices in $\langle \mathbf{m} \rangle$.

Remark 15. If $s = 1$, then $\mathbf{m} = x_{\lambda_1}^{d_1}$, and so $Q(\mathbf{m}) = P_{d_1}(x_1, x_2, \dots, x_{\lambda_1})$. If $\lambda_{s-1} = 1$, then $s = 2$ and $\mathbf{m} = x_1^{d_1} x_{\lambda_2}^{d_2}$, so $Q(\mathbf{m})$ is obtained by multiplying all the labels of the vertices of $P_{d_2}(x_1, x_2, \dots, x_{\lambda_2})$ by $x_1^{d_1}$.

Before we prove the above theorem we need a lemma. Because of the above remark, we may assume that $s > 1$ and $\lambda_{s-1} > 1$.

Lemma 16. *Let I be a Borel principal ideal as above. Define the ideals*

$$N_k = \langle x_{\lambda_1}^{d_1} x_{\lambda_2}^{d_2} \cdots x_{\lambda_j}^{d_j} x_k^{d_{j+1}+\cdots+d_{s-1}} \rangle \quad \text{and} \quad J_k = N_k \cdot (x_k, \ldots, x_\mu)^{d_s}$$

for $\lambda_j < k \leqslant \lambda_{j+1}$ ($j < s-1$ and $\lambda_0 = 0$). Then for $\lambda_{s-1} < \mu \leqslant \lambda_s$

(a) $N_i \cdot (x_1, \ldots, x_\mu)^{d_s} = J_1 + J_2 + \cdots + J_i$ *for $i = 1, 2, \ldots, \lambda_{s-1}$, and*
(b) $N_j \cdot (x_1, \ldots, x_\mu)^{d_s} \cap N_{j+1} \cdot (x_{j+1}, \ldots, x_\mu)^{d_s} = N_j \cdot (x_{j+1}, \ldots, x_\mu)^{d_s}$

for $j = 1, 2, \ldots, \lambda_{s-1} - 1$.

Proof. (a) First it is clear that

$$N_i \cdot (x_1, \ldots, x_\mu)^{d_s} \supseteq J_1 + J_2 + \cdots + J_i.$$

Note that the ideal $N_i \cdot (x_1, \ldots, x_\mu)^{d_s}$ is Borel principal, Borel generated by the monomial

$$x_{\lambda_1}^{d_1} x_{\lambda_2}^{d_2} \cdots x_{\lambda_j}^{d_j} x_i^{d_{j+1}+\cdots+d_{s-1}} x_\mu^{d_s} \in N_i \cdot (x_i, \ldots, x_\mu)^{d_s} = J_i$$

($\lambda_j < i \leqslant \lambda_{j+1} < \mu$, since $j < s-1$). To complete the proof of part (a), it suffices to show that the ideal $J_1 + J_2 + \cdots + J_i$ is Borel fixed. Since J_1 is Borel fixed, assume by induction that for some $1 \leqslant k < i$, the ideal $J_1 + \cdots + J_k$ is Borel fixed and let $\mathbf{n} \in J_1 + \cdots + J_k + J_{k+1}$. If $\mathbf{n} \in J_1 + \cdots + J_k$, we are done, so assume that $\mathbf{n} \in G(N_{k+1}) \setminus (J_1 + \cdots + J_k)$. Then write

$$\mathbf{n} = \mathbf{n}'\mathbf{n}''$$

for some $\mathbf{n}' \in G(N_{k+1})$ and $\mathbf{n}'' \in G((x_{k+1}, \ldots, x_\mu)^{d_s})$. Now observe that x_{k+1} must divide $\mathbf{n}'$ because $\mathbf{n} \notin J_1 + \cdots + J_k$. If $r < t$ and x_t divides $\mathbf{n}$, then we see that $\mathbf{n}_{t \to r}$ is in J_{k+1}. Indeed, it is easy to verify this when x_t divides $\mathbf{n}'$, because N_{k+1} is Borel fixed, so assume that x_t does not divide $\mathbf{n}'$. Then x_t divides $\mathbf{n}''$, so $t \geqslant k+1$. If $k+1 \leqslant r$, then we have

$$\frac{\mathbf{n}x_r}{x_t} = \mathbf{n}' \cdot \frac{\mathbf{n}''x_r}{x_t} \in N_{k+1} \cdot (x_{k+1}, \ldots, x_\mu)^{d_s} = J_{k+1},$$

while if $r < k+1$,

$$\frac{\mathbf{n}x_r}{x_t} = \frac{\mathbf{n}'x_r}{x_{k+1}} \cdot \frac{\mathbf{n}''x_{k+1}}{x_t} \in N_{k+1} \cdot (x_{k+1}, \ldots, x_\mu)^{d_s} = J_{k+1},$$

because N_{k+1} is Borel fixed. Thus

$$\frac{\mathbf{n}x_r}{x_t} \in J_1 + \cdots + J_k + J_{k+1}$$

in all cases and the proof of part (a) is complete.

For part (b), let $\mathbf{m} \in G(N_j \cdot (x_1, \ldots, x_\mu)^{d_s})$ and $\mathbf{n} \in G(N_{j+1} \cdot (x_{j+1}, \ldots, x_\mu)^{d_s})$, and write $\mathbf{m} = \mathbf{m}_1\mathbf{m}_2$ and $\mathbf{n} = \mathbf{n}_1\mathbf{n}_2$, where $\mathbf{m}_1 \in G(N_j)$, $\mathbf{m}_2 \in G((x_1, \ldots, x_\mu)^{d_s})$, $\mathbf{n}_1 \in G(N_{j+1})$ and

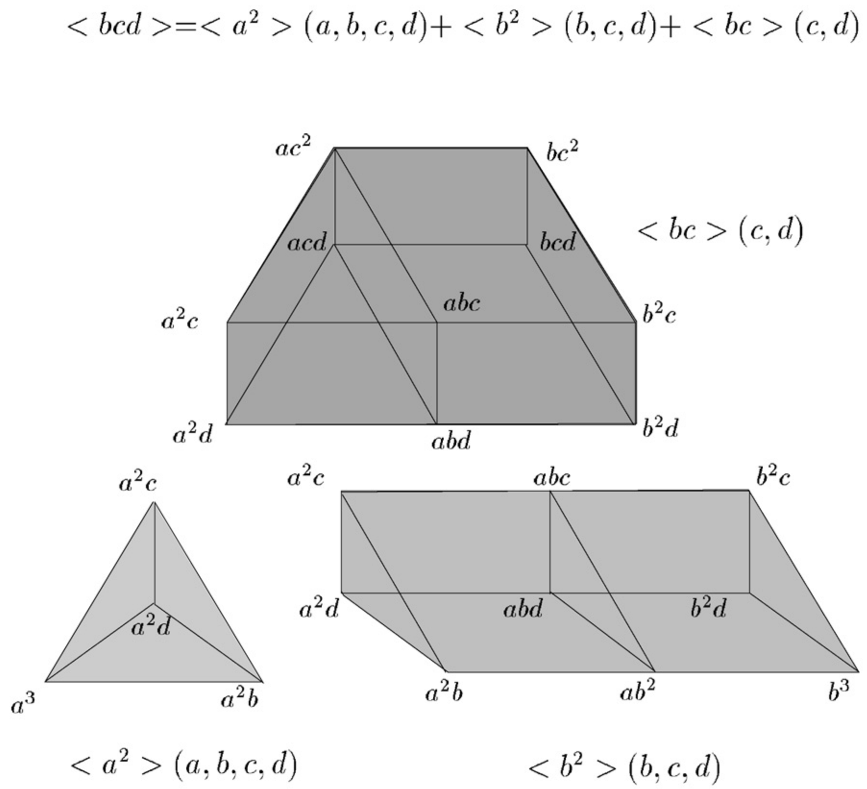


Fig. 4.

$\mathbf{n}_2 \in G((x_{j+1}, \ldots, x_\mu)^{d_s})$. Then, note that $\mathbf{m}_1\mathbf{n}_2$ divides $\operatorname{lcm}(\mathbf{m}, \mathbf{n})$. This implies that $\operatorname{lcm}(\mathbf{m}, \mathbf{n})$ is in $N_j \cdot (x_{j+1}, \ldots, x_\mu)^{d_s}$, and so

$$N_j \cdot (x_1, \ldots, x_\mu)^{d_s} \cap N_{j+1} \cdot (x_{j+1}, \ldots, x_\mu)^{d_s} \subseteq N_j \cdot (x_{j+1}, \ldots, x_\mu)^{d_s}.$$

The opposite containment is obvious, so the proof of part (b) is complete. □

Remark 17. Part (a) with $i = \lambda_{s-1}$ and $\mu = \lambda_s$ yields

$$I = N_1 \cdot (x_1, \ldots, x_{\lambda_s})^{d_s} + N_2 \cdot (x_2, \ldots, x_{\lambda_s})^{d_s} + \cdots + N_{\lambda_{s-1}} \cdot (x_{\lambda_{s-1}}, \ldots, x_{\lambda_s})^{d_s}.$$

Example 6.

(i) For the ideal $I = \langle bd^2 \rangle$ in $\Bbbk[a, b, c, d]$, we have $s = 2$, $\lambda_1 = 2$, $d_1 = 1$, $\lambda_2 = 4$ and $d_2 = 2$. Moreover, $N_1 = \langle a \rangle$, $N_2 = \langle b \rangle = (a, b)$. Therefore,

$$I = \langle a \rangle (a, b, c, d)^2 + \langle b \rangle (b, c, d)^2.$$

(ii) For the ideal $I = \langle bcd \rangle$ in $\Bbbk[a,b,c,d]$, we have $s = 3$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$ and $d_1 = d_2 = d_3 = 1$. Moreover, $N_1 = \langle a^2 \rangle$, $N_2 = \langle b^2 \rangle$ and $N_3 = \langle bc \rangle$. Therefore, as is also shown in Fig. 4,

$$I = \langle a^2 \rangle (a,b,c,d) + \langle b^2 \rangle (b,c,d) + \langle bc \rangle (c,d).$$

Proof of Theorem 14. By induction on s. For $s = 1$, we are done. Assume that $s > 1$ and that we have obtained $Q(\prod_{j=1}^{k} I_{\lambda_j}^{d_j})$ for all $k < s$. By the inductive hypothesis, there is a polytopal cell complex $Q(N_i)$ that supports a minimal free resolution for the ideals N_i, for all $1 \leqslant i \leqslant \lambda_{s-1}$. In particular, $Q(N_i)$ is the union of all the convex polytopes of $P_{d-d_s}(x_1, \ldots, x_n)$ with vertices in N_i. Thus $Q(N_i)$ is a subcomplex of $Q(N_{i+1})$ for all $1 \leqslant i < \lambda_{s-1}$. Set $J_i = N_i \cdot (x_i, \ldots, x_{\lambda_s})^{d_s}$ ($i = 1, 2, \ldots, \lambda_{s-1} - 1$). From Lemmas 8, 10 and 16 it follows that the polytopal cell complexes C_i and D_i ($i = 1, 2, \ldots, \lambda_s$) defined by

$$C_i := Q(N_i) \times P_{d_s}(x_i, \ldots, x_{\lambda_s})$$

and

$$D_i := C_i \cap C_{i+1} = Q(N_i) \times P_{d_s}(x_{i+1}, \ldots, x_{\lambda_s})$$

support a minimal free resolution of J_i and $(J_1 + \cdots + J_i) \cap J_{i+1}$, respectively, for all $1 \leqslant i < \lambda_{s-1}$. Thus Lemma 6 implies that the polytopal cell complex C'_k, which is defined recursively by

$$C'_1 = C_1, \quad \text{and} \quad C'_{i+1} = C'_i \cup C_{i+1}$$

for $1 \leqslant i < \lambda_{s-1}$, supports a minimal free resolution of $J_1 + J_2 + \cdots + J_i$. Accordingly, set

$$Q(\mathbf{m}) := C'_{\lambda_{s-1}} = C_1 \cup C_2 \cup \cdots \cup C_{\lambda_{s-1}}$$

and the construction of our polytopal cell complex is done by induction. Also, from our construction it follows inductively that $Q(\mathbf{m})$ is the union of all the convex polytopes of $P_d(x_1, \ldots, x_n)$ with vertices in $\langle \mathbf{m} \rangle$. Therefore, $Q(\mathbf{m})$ is subcomplex of $P_d(x_1, \ldots, x_n)$, where $d = \deg(\mathbf{m})$, as desired. As in Theorem 12, $Q(\mathbf{m})$ is a regular subdivision of $\Delta_{n-1}(x_1^d, x_2^d, \ldots, x_n^d)$ and hence shellable. The proof is complete. □

3.4. d-generated Borel fixed ideals

Next we generalize Theorem 14 to the case of any Borel fixed ideal generated in one degree. We need some preliminary lemmas first.

Lemma 18. *Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two monomials of the same degree d. Then*

$$\langle \mathbf{m}_1 \rangle \cap \langle \mathbf{m}_2 \rangle$$

is a Borel principal ideal, Borel generated by a monomial $\mathbf{m}$ of degree d. Moreover,

$$Q(\mathbf{m}_1) \cap Q(\mathbf{m}_2) = Q(\mathbf{m}).$$

Proof. First assume that $\mathbf{m}_1 = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\mathbf{m}_2 = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$. Then define $\mu_n = \min\{a_n, b_n\}$ and the natural numbers μ_i for $i = n-1, \dots, 1$ recursively, by setting

$$\mu_i = \min\{a_i + \cdots + a_n, b_i + \cdots + b_n\} - (\mu_{i+1} + \cdots + \mu_n).$$

Define the following monomial of degree d

$$\mathbf{MIN}(\mathbf{m}_1, \mathbf{m}_2) := x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}.$$

From our choice of the μ_i's, we have

$$\mu_{n-i} + \mu_{n-i+1} + \cdots + \mu_n \leqslant a_{n-i} + a_{n-i+1} + \cdots + a_n,$$

and

$$\mu_{n-i} + \mu_{n-i+1} + \cdots + \mu_n \leqslant b_{n-i} + b_{n-i+1} + \cdots + b_n,$$

for all $i = 0, 1, \dots, n-1$. Therefore,

$$\big\langle \mathbf{MIN}(\mathbf{m}_1, \mathbf{m}_2) \big\rangle \subseteq \langle \mathbf{m}_1 \rangle \cap \langle \mathbf{m}_2 \rangle.$$

Now let $\mathbf{n}_1 = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}$ be in $G(\langle \mathbf{m}_1 \rangle)$ and let $\mathbf{n}_2 = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ be in $G(\langle \mathbf{m}_2 \rangle)$. We want to show that $\operatorname{lcm}(\mathbf{n}_1, \mathbf{n}_2)$ is in $\langle \mathbf{MIN}(\mathbf{m}_1, \mathbf{m}_2) \rangle$, so we may assume that $c_1 < d$ and $d_1 < d$. Let k be the largest positive integer such that

$$\max\{c_1, d_1\} + \cdots + \max\{c_k, d_k\} < d.$$

Then set $\nu_i = \max\{c_i, d_i\}$ for $i = 1, 2, \dots, k$ and $\nu_{k+1} = d - (\nu_1 + \cdots + \nu_k)$. Since

$$\begin{aligned} \nu_1 + \cdots + \nu_i &\geqslant \max\{c_1 + \cdots + c_i, d_1 + \cdots + d_i\} \\ &\geqslant \max\{a_1 + \cdots + a_i, b_1 + \cdots + b_i\}, \end{aligned}$$

for all $1 \leqslant i \leqslant k$, we see that

$$\begin{aligned} \nu_{i+1} + \cdots + \nu_{k+1} &\leqslant d - \max\{a_1 + \cdots + a_i, b_1 + \cdots + b_i\} \\ &= \min\{a_{i+1} + \cdots + a_n, b_{i+1} + \cdots + b_n\} \\ &= \mu_{i+1} + \cdots + \mu_n. \end{aligned}$$

Therefore, the monomial $x_1^{\nu_1} x_2^{\nu_2} \cdots x_{k+1}^{\nu_{k+1}}$ is a minimal generator of $\langle \mathbf{MIN}(\mathbf{m}_1, \mathbf{m}_2) \rangle$ and divides $\operatorname{lcm}(\mathbf{n}_1, \mathbf{n}_2) = x_1^{\max\{c_1, d_1\}} x_2^{\max\{c_2, d_2\}} \cdots x_n^{\max\{c_n, d_n\}}$. Thus $\operatorname{lcm}(\mathbf{n}_1, \mathbf{n}_2)$ is in $\langle \mathbf{MIN}(\mathbf{m}_1, \mathbf{m}_2) \rangle$, and so

$$\langle \mathbf{m}_1 \rangle \cap \langle \mathbf{m}_2 \rangle \subseteq \big\langle \mathbf{MIN}(\mathbf{m}_1, \mathbf{m}_2) \big\rangle.$$

The proof of our first claim is complete with $\mathbf{m} := \mathbf{MIN}(\mathbf{m}_1, \mathbf{m}_2)$. Now note that $Q(\mathbf{m}_1) \cap Q(\mathbf{m}_2)$ is the union of all the convex polytopes of the polytopal cell complex $P_d(x_1, \dots, x_n)$ with vertices

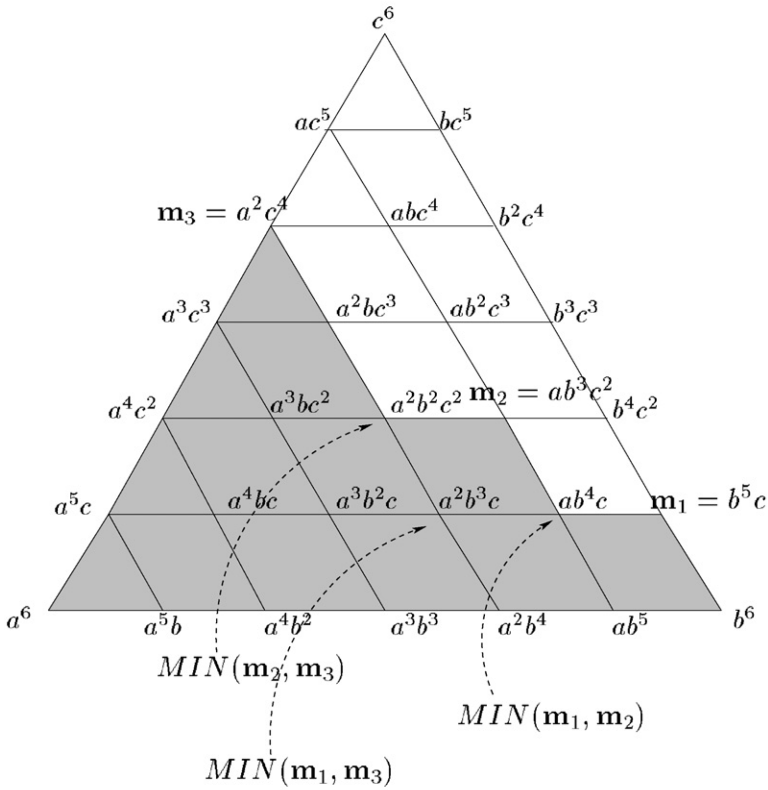


Fig. 5.

in $\langle \mathbf{m}_1 \rangle \cap \langle \mathbf{m}_2 \rangle = \langle \mathbf{m} \rangle$. Since $Q(\mathbf{m})$ is the union of all the convex polytopes of the polytopal cell complex $P_d(x_1, \dots, x_n)$ with vertices in $\langle \mathbf{m} \rangle$, we must have

$$Q(\mathbf{m}_1) \cap Q(\mathbf{m}_2) = Q(\mathbf{m}). \qquad \square$$

Example 7. Let $\mathbf{m}_1 = b^5c$, $\mathbf{m}_2 = ab^3c^2$ and $\mathbf{m}_3 = a^2c^4$ in $\Bbbk[a,b,c]$. Then, as we may easily check from Fig. 5,

$$\langle \mathbf{m}_1 \rangle \cap \langle \mathbf{m}_2 \rangle = \langle b^5c \rangle \cap \langle ab^3c^2 \rangle = \langle ab^4c \rangle,$$
$$\langle \mathbf{m}_1 \rangle \cap \langle \mathbf{m}_3 \rangle = \langle b^5c \rangle \cap \langle a^2c^4 \rangle = \langle a^2b^3c \rangle.$$

Also,

$$\langle \mathbf{m}_1 \rangle \cap \langle \mathbf{m}_2, \mathbf{m}_3 \rangle = \langle ab^4c, a^2b^3c \rangle = \langle ab^4c \rangle.$$

In general, the intersection of a Borel principal ideal with a Borel nonprincipal ideal is not principal. For example, in $\Bbbk[a,b,c,d]$ we have

$$\langle ab^4c^3d \rangle \cap \langle a^2b^4cd^2, a^3bc^2d^3 \rangle = \langle a^3b^2c^3d, a^2b^4c^2d \rangle.$$

The following lemma is sufficient for our purposes.

Lemma 19. *Let* $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_s$ *be* s *monomials of the same degree* d. *Then for all* $j < s$,

$$\langle \mathbf{m}_j \rangle \cap \langle \mathbf{m}_{j+1}, \ldots, \mathbf{m}_s \rangle$$

is a Borel fixed ideal, which is Borel generated by at most $s - j$ *monomials* $\mathbf{n}_{j+1}, \ldots, \mathbf{n}_s$ *of degree* d.

Proof. First note that the case where $j = s - 1$ is Lemma 18. Now for all $j < s$, we have

$$\begin{aligned} \langle \mathbf{m}_j \rangle \cap \langle \mathbf{m}_{j+1}, \ldots, \mathbf{m}_s \rangle &= \big(\langle \mathbf{m}_j \rangle \cap \langle \mathbf{m}_{j+1} \rangle \big) + \cdots + \big(\langle \mathbf{m}_j \rangle \cap \langle \mathbf{m}_s \rangle \big) \\ &= \big\langle \mathbf{MIN}(\mathbf{m}_j, \mathbf{m}_{j+1}) \big\rangle + \cdots + \big\langle \mathbf{MIN}(\mathbf{m}_j, \mathbf{m}_s) \big\rangle \\ &= \langle \mathbf{n}_{j+1}, \ldots, \mathbf{n}_s \rangle \end{aligned}$$

where

$$\mathbf{n}_k := \mathbf{MIN}(\mathbf{m}_j, \mathbf{m}_k)$$

for $k = j + 1, \ldots, s$. Some of the $\mathbf{n}_k$'s might be redundant, so the above intersection is Borel generated by *at most* $s - j$ monomials $\mathbf{n}_{j+1}, \ldots, \mathbf{n}_s$ of degree d. □

Now we are ready to prove our most general result.

Theorem 20. *Let* $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_s$ *be* s *monomials of the same degree* d *and let*

$$I = \langle \mathbf{m}_1, \ldots, \mathbf{m}_s \rangle.$$

Then there exists a polytopal cell complex $Q(\mathbf{m}_1, \ldots, \mathbf{m}_s)$ *that supports a minimal free resolution of* I. *Moreover,* $Q(\mathbf{m}_1, \ldots, \mathbf{m}_s)$ *is the union of all the convex polytopes of the polytopal cell complex* $P_d(x_1, \ldots, x_n)$ *with vertices in* I.

Proof. For $s = 2$ both of our claims were proved in Lemma 18. So assume that $s > 2$, and for all $j < s$ set

$$I_j = \langle \mathbf{m}_j, \ldots, \mathbf{m}_s \rangle.$$

Next suppose that for some $j < s$ we have constructed a polytopal cell complex $Q(K)$ that supports a minimal free resolution of any Borel fixed ideal K, which is Borel generated by at most $s - j$ monomials of the same degree d. Assume also that $Q(K)$ is the union of all the convex polytopes of the polytopal cell complex $P_d(x_1, \ldots, x_n)$ with vertices in K. From Lemma 19, we see that

$$\langle \mathbf{m}_j \rangle \cap \langle \mathbf{m}_{j+1}, \ldots, \mathbf{m}_s \rangle = \langle \mathbf{n}_{j+1}, \ldots, \mathbf{n}_s \rangle$$

is a Borel fixed ideal, which is Borel generated by *at most* $s - j$ monomials $\mathbf{n}_{j+1}, \ldots, \mathbf{n}_s$ of degree d. So far we have constructed the polytopal cell complex $Q(\mathbf{m}_j)$ in Theorem 14, and the polytopal cell complexes $Q(\mathbf{m}_{j+1},\dots,\mathbf{m}_s)$ and $Q(\mathbf{n}_{j+1},\dots,\mathbf{n}_s)$, by the inductive hypothesis. Also by the inductive hypothesis, $Q(\mathbf{m}_j)\cap Q(\mathbf{m}_{j+1},\dots,\mathbf{m}_s)$ is the union of all the convex polytopes of $P_d(x_1,\dots,x_n)$ with vertices in $\langle\mathbf{m}_j\rangle\cap\langle\mathbf{m}_{j+1},\dots,\mathbf{m}_s\rangle$. Since $Q(\mathbf{n}_{j+1},\dots,\mathbf{n}_s)$ is the union of all the convex polytopes of $P_d(x_1,\dots,x_n)$ with vertices in $\langle\mathbf{n}_{j+1},\dots,\mathbf{n}_s\rangle$, we must have

$$Q(\mathbf{m}_j)\cap Q(\mathbf{m}_{j+1},\dots,\mathbf{m}_s)=Q(\mathbf{n}_{j+1},\dots,\mathbf{n}_s).$$

Since the rest of the hypotheses of Lemma 6 are easily verified, we conclude that the complex

$$X_j:=Q(\mathbf{m}_j)\cup Q(I_{j+1})=Q(\mathbf{m}_j)\cup Q(\mathbf{m}_{j+1})\cup\cdots\cup Q(\mathbf{m}_s)$$

supports a minimal free resolution of the ideal I_j. Thus

$$X:=X_1=Q(\mathbf{m}_1)\cup Q(\mathbf{m}_2)\cup\cdots\cup Q(\mathbf{m}_s)$$

supports a minimal free resolution of $I_1=I$. □

4. The lcm-lattice

The lcm-lattice of an arbitrary monomial ideal I was introduced in [11], where the authors show how its structure relates to the Betti numbers and the maps in the minimal free resolution of I. The lcm-lattice of I, with $G(I)=\{\mathbf{m}_1,\mathbf{m}_2,\dots,\mathbf{m}_r\}$, is denoted by L_I. This is the lattice with elements labeled by the least common multiple of $\mathbf{m}_1,\mathbf{m}_2,\dots,\mathbf{m}_r$ ordered by divisibility; that is, if $\mathbf{n}$ and $\mathbf{m}$ are distinct elements of L_I, then $\mathbf{m}\prec\mathbf{n}$ if and only if $\mathbf{m}$ divides $\mathbf{n}$. Moreover, we include $\hat{0}:=1$ as the bottom element and $\hat{1}=\operatorname{lcm}(\mathbf{m}_1,\mathbf{m}_2,\dots,\mathbf{m}_r)$ as the top one. We say that $\mathbf{n}$ *covers* $\mathbf{m}$ and we write $\mathbf{m}\to\mathbf{n}$, if $\mathbf{m}\prec\mathbf{n}$ and if there is no element $\mathbf{k}\neq\mathbf{n},\mathbf{m}$ of L_I such that $\mathbf{m}\prec\mathbf{k}\prec\mathbf{n}$. A *chain* of L_I is a set of elements of L_I in which every two monomials are comparable. A chain is *maximal* if it is not a proper subset of another chain. If every maximal chain has the same length, L_I is called *ranked*.

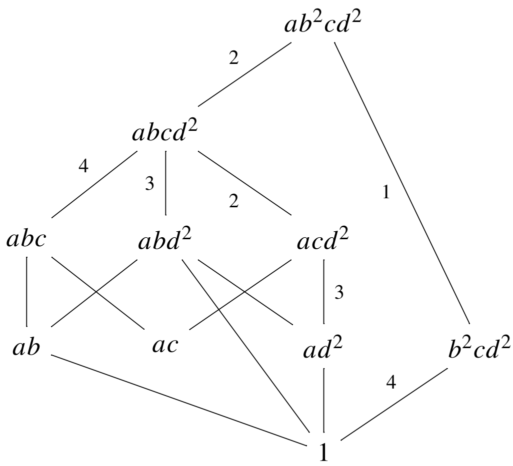
We would like to find a labeling of the edges of L_I with the following property: for all elements $\mathbf{m}$ and $\mathbf{n}$ in L_I with $\mathbf{m}\prec\mathbf{n}$, there exists a unique *increasing* maximal chain from $\mathbf{m}$ to $\mathbf{n}$, which is also lexicographically first among all other maximal chains from $\mathbf{m}$ to $\mathbf{n}$. This would prove that L_I is shellable (see [5,6]) in a way different from the one given in [1]. Finding such a labeling is still an open problem.

Remark 21. The natural labeling which assigns to each edge $\mathbf{m}\to\mathbf{n}$ the integer $\max(\frac{\mathbf{n}}{\mathbf{m}}):=\max\{i\,|\,x_i \text{ divides } \frac{\mathbf{n}}{\mathbf{m}}\}$ does not work. For this labeling, one can prove that the desired property holds only provided that $\mathbf{m}\neq\hat{0}=1$.

Example 8. Let

$$\begin{aligned}I&=\langle ab,ac,ad^2,b^2cd^2\rangle\\&=(a^2,ab,b^5,ac,b^4c,b^3c^2,b^2c^3,b^4d,b^3cd,b^2c^2d,ad^2,b^3d^2,b^2cd^2).\end{aligned}$$

The interval $[1, ab^2cd^2]$ of L_I is



Hence there is no decreasing sequence of labels from ab^2cd^2 to 1 (or even to abc).

The above example shows also that the lcm lattice of a Borel fixed ideal need not be ranked in general. However, if I is generated in the same degree then we prove the following proposition, which was also proved independently (and in greater generality) in [18]. The advantage of our proof is that it is very elementary; it only requires a minimum knowledge of lattices and the definition of a Borel fixed ideal.

Proposition 22. *The* lcm-*lattice L_I of a d-generated Borel fixed ideal I is ranked.*

Proof. Let $I = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r)$ be minimally generated by $\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_r}$ in the same degree d and let $\mathbf{m} \neq \hat{1} = \operatorname{lcm}(\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r)$ be in the lcm-lattice L_I of I. Assume that $\mathbf{m} = \operatorname{lcm}(\mathbf{m}_\alpha, \mathbf{m}_\beta, \ldots, \mathbf{m}_\gamma)$, with

$$e(\mathbf{m}_\alpha) = (a_1, a_2, \ldots, a_n), \qquad e(\mathbf{m}_\beta) = (b_1, b_2, \ldots, b_n), \qquad \ldots, \qquad e(\mathbf{m}_\gamma) = (c_1, c_2, \ldots, c_n).$$

In order to show that the lattice is ranked it suffices to prove that $\deg(\mathbf{n}) = 1 + \deg(\mathbf{m})$ for all $\mathbf{n}$ that cover $\mathbf{m}$. There exists an $\mathbf{m}_\delta$ in I, with $e(\mathbf{m}_\delta) = (d_1, d_2, \ldots, d_n)$ such that $\mathbf{n} = \operatorname{lcm}(\mathbf{m}_\alpha, \mathbf{m}_\beta, \ldots, \mathbf{m}_\gamma, \mathbf{m}_\delta) = \operatorname{lcm}(\mathbf{m}, \mathbf{m}_\delta)$. Also, there is at least one j such that $d_j > \max\{a_j, b_j, \ldots, c_j\}$. Without loss of generality assume that for that j, $\max\{a_j, b_j, \ldots, c_j\} = a_j$. If there is some k with $j < k \leqslant n$ such that $a_k \neq 0$, then

$$\ell := \operatorname{lcm}\big((\mathbf{m}_\alpha)_{k \to j}, \mathbf{m}_\alpha, \mathbf{m}_\beta, \ldots, \mathbf{m}_\gamma\big)$$

has degree $\deg(\ell) = 1 + \deg(\mathbf{m})$, divides $\mathbf{n}$ and is divisible by $\mathbf{m}$. The minimality of $\mathbf{n}$ forces $\ell = \mathbf{n}$ and so $\deg(\mathbf{n}) = 1 + \deg(\mathbf{m})$. Now assume that $a_k = 0$ for all $j < k \leqslant n$. Then, there is an $i < j$

such that $\max\{a_i, b_i, \dots, c_i\} > d_i$. [Indeed, suppose to the contrary that $d_i \geqslant \max\{a_i, b_i, \dots, c_i\}$ for all $i < j$. Then,

$$d \geqslant \sum_{i=1}^{j} d_i > \sum_{i=1}^{j} \max\{a_i, b_i, \dots, c_i\} \geqslant \sum_{i=1}^{j} a_i = \sum_{i=1}^{n} a_i = d,$$

a contradiction.] Then

$$\ell := \operatorname{lcm}\big((\mathbf{m}_\delta)_{j\to i}, \mathbf{m}_\alpha, \mathbf{m}_\beta, \dots, \mathbf{m}_\gamma\big)$$

has degree $\deg(\mathbf{n}) - 1$, divides $\mathbf{n}$ and is divisible by $\mathbf{m}$. Hence, $\ell = \mathbf{m}$ and so $\deg(\mathbf{n}) = 1 + \deg(\mathbf{m})$, as desired. The proof is complete. □

Remark 23.

(1) The above proof applies with minor modifications to the case of a *strongly stable square-free* ideal generated in the same degree. A monomial ideal I is called *strongly stable square-free* if all monomials in $G(I)$ are square-free and for every $\mathbf{m}$ in $G(I)$, if x_t divides m and x_s does not divide $\mathbf{m}$ ($1 \leqslant s < t$), then $\mathbf{m}_{t\to s}$ is in I.

(2) There exists a d-generated Borel fixed ideal $I = (\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_r)$ minimally generated by $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_r$ and an element $\mathbf{m}$ of L_I of degree $d + 1$, such that for some $1 \leqslant s < t \leqslant n$,
  (i) x_t divides $\mathbf{m}$,
  (ii) $x_s^{d_s}$ does not divide $\mathbf{m}$, where d_i is the largest positive integer such that $x_i^{d_i}$ divides $\operatorname{lcm}(\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_r)$, and
  (iii) $\mathbf{m}_{t\to s}$ is not in L_I.

Example 9. Let

$$I = \langle x_1 x_3^3, x_2^2 x_3 x_4 \rangle.$$

Then $d_3 = 3$ and $x_2^2 x_3^2 x_4 = \operatorname{lcm}(x_2^2 x_3 x_4, x_2^2 x_3^2)$ is in L_I, but $x_2^2 x_3^3 = (x_2^2 x_3^2 x_4)_{4\to 3}$ is *not* in L_I.

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