JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **151**, 275–290 (1990)

Fractal Surfaces

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Submitted by Bruce C. Berndt

Received April 28, 1989

I. INTRODUCTION

We construct surfaces in $\mathbf{R}^3$ which are attractors of iterated function systems. These surfaces are graphs of continuous functions from the oriented standard simplex $\sigma^2 \subset \mathbf{R}^2$ into $\mathbf{R}$, and have in general a non-integral dimension. For this reason we refer to them as fractal surfaces. This construction of fractal surfaces can be extended to any topological space via singular simplices.

Fractal surfaces are currently being considered in areas such as chemistry, metallurgy, and surface physics. We give a general construction for a fractal surface which we believe can be used to describe or better understand the surfaces in the above-mentioned areas.

The structure of this paper is as follows: In Section II we present a general introduction to iterated function systems (i.f.s.) and state some relevant results. In Section III we construct fractal surfaces using i.f.s.'s. Projecting these fractal surfaces onto appropriate coordinate planes yields the graph of a fractal function. Section IV is devoted to the study of the code space and the dynamical system associated with the fractal surface. The Lyapunov dimension Λ is introduced and for a special class of fractal surfaces we present a formula for Λ. In Section V we look at $\mathbf{p}$-balanced measures and the moment theory. The main result is then presented in Section VI; a formula for the fractal dimension d of a fractal surface is derived. This is done for a special case of affine generating maps. In Section VII we show Hölder continuity and prove that the fractal surface has finite d-dimensional Hausdorff measure.

II. PRELIMINARIES

Let $X = (X, d)$ be a compact metric space or a closed subset of $\mathbf{R}^n$, $n \in \mathbf{N}$, with metric d. Denote by $H(X)$ the set of all non-empty closed subsets of X. With the Hausdorff metric $h: H(X) \times H(X) \to \mathbf{R}_0^+$,

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0022-247X/90 $3.00

$$h(A, B) = \max\{\sup\{d(x, B) : x \in A\}, \sup\{d(A, y) : y \in B\}\}$$

$(H(X), h)$ is a complete metric space.

Let $\mathbf{F} = \{f_i : X \to X : i = 1, ..., N\}$ be a collection of continuous functions on X. The pair $\mathbf{F}_X = (X, \mathbf{F})$ is called an iterated function system on X. If the maps f_i, $i \in \mathbf{I} = \{1, ..., N\}$, are contractive then $\mathbf{F}_X$ is called a hyperbolic i.f.s. If we define a set-valued map $\hat{f}: H(X) \to H(X)$, by

$$\hat{f}(A) = \bigcup f_i(A), \qquad \forall A \in H(X)$$

then $\hat{f}$ is a contraction on $H(X)$, thus possessing a unique fixed point, called the attractor of the i.f.s. $\mathbf{F}_X$ and denoted by $A(\mathbf{F}_X)$ (we usually omit the dependence of A on $\mathbf{F}_X$ if it is clear from the context which i.f.s. is used). Every i.f.s. $\mathbf{F}_X$ admits a stationary measure μ, called the **p**-balanced measure, given by

$$\int_X g \, d\mu = \sum_i p_i \int_X g f \, d\mu$$

$\forall g \in C^0(X)$ and a set of non-zero probabilities $\mathbf{p} = \{p_i : i \in \mathbf{I}\}$.

For a more detailed and elaborate presentation of i.f.s.'s we refer the reader to [1, 2].

Now let $X = [0, 1] \times \mathbf{R}$ and consider a set of $N + 1$ interpolation points $\mathfrak{I} = \{(x_j, y_j) : \ 0 = x_0 < \cdots < x_N = 1, \ y_j \in \mathbf{R}, \ j \in \mathbf{J} = \{0\} \cup \mathbf{I}\}$, define maps $f_i : X \to X$ by

$$f_i(x, y) = \begin{bmatrix} \Delta x_i & 0 \\ b_i & e_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_{i-1} \\ k_{i-1} \end{bmatrix}$$

with $\Delta x_i = x_i - x_{i-1}$, $b_i = y_i - y_{i-1} - e_i(y_N - y_0)$, $k_{i-1} = y_{i-1} - e_i y_0$, $|e_i| < 1$, $i \in \mathbf{I}$. Then $(X, \mathscr{D})$ with $\mathscr{D}((x, y), (\tilde{x}, \tilde{y})) = |x - \tilde{x}| + (1 - \max\{\Delta x_i\})(2\beta)^{-1} |y - \tilde{y}|$, where $\beta > \max\{|b_i|\}$, is a complete metric space and the f_i, $i \in \mathbf{I}$, are contractive on $(X, \mathscr{D})$.

The unique fixed point of this i.f.s. is the graph of a continuous function $f: [0, 1] \to \mathbf{R}$ with $f(x_j) = y_j$, $j \in \mathbf{J}$. Since the graph of f is in general a fractal, f is called a fractal function. At this point the reader may want to consult the following publications which deal with fractal functions and their properties: [1, 3–10].

Figure 1 shows the construction of a fractal function. The first two iterates of a base set and the final attractor are shown.

Recall that the fractal dimension (also called capacity or similarity dimension) of a bounded set $S \subset \mathbf{R}^n$, dim S, is defined as

$$\limsup_{\varepsilon \to 0} \frac{\log \Sigma(\varepsilon)}{\log(\varepsilon^{-1})},$$

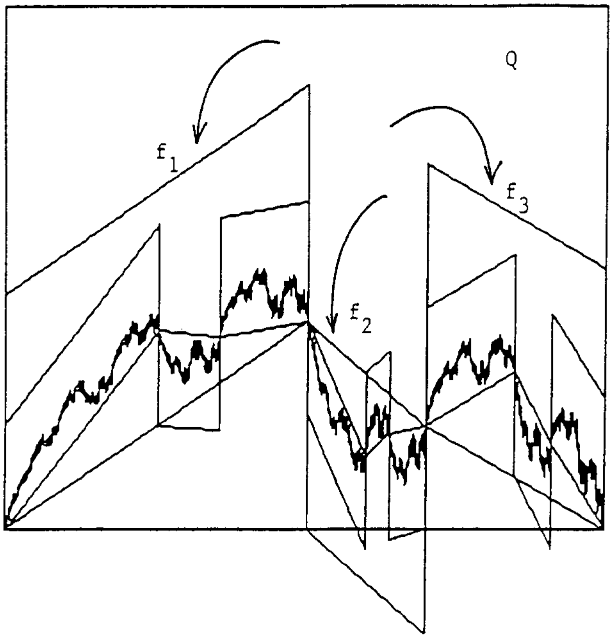

FIGURE 1

where $\Sigma(\varepsilon)=\min\{|\mathscr{C}(\varepsilon)|:\mathscr{C}(\varepsilon)$ cover of S by balls of radius $\varepsilon>0\}$. We remark that $\dim S$ remains unchanged if the continuous variable ε is replaced by any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ and

$$\frac{\log \varepsilon_{n+1}}{\log \varepsilon_n} \to 0.$$

At this point we quote a result from [5] or [7]:

THEOREM 1. *Let* $G=\operatorname{graph}(f)$, *where* f *is as defined above. Suppose that* $\Sigma|e_i|>1$ *and* $\mathfrak{I}$ *is not collinear. Then* $\dim G$ *is the unique positive solution of*

$$\sum_i |e_i|(\Delta x_i)^{d-1}=1;$$

otherwise $\dim G=1$.

Let us note that every compact one-dimensional C^0-submanifold of $\mathbf{R}^2$ containing $\mathfrak{I}$ will converge in the Hausdorff metric to G.

III. Construction of Fractal Surfaces

Before we commence with the construction of fractal surfaces, let us briefly recall some basic notions from algebraic topology (see for instance [11] or [12]. In [12] n-dimensional parallelepipeds are used instead of n-dimensional simplices).

Definition 1. Let $\{v_0, ..., v_k\}$ be a set of geometrically independent points in $\mathbf{R}^n$, $n \geqslant k$. The k-dimensional geometric simplex or k-simplex, denoted by σ^k, is defined as

$$\sigma^k = \left\{ x \in \mathbf{R}^n : \exists \lambda_0, ..., \lambda_k \in \mathbf{R}^+, x = \sum_{j=0}^{k} \lambda_j v_j, \sum_{j=0}^{k} \lambda_j = 1 \right\}.$$

The points $\{v_0\}, ..., \{v_k\}$ are called the vertices of the simplex σ^k.

A simplex σ^m is called a face of a simplex σ^k, $k \geqslant m$, if each vertex of σ^m is also a vertex of σ^k. We denote a k-simplex σ^k also by $\sigma^k = \langle v_0, ..., v_k \rangle$, where $\{v_0\}, ..., \{v_k\}$ are the vertices of σ^k. The k-simplex σ^k endowed with the Euclidean subspace topology of $\mathbf{R}^n$ is denoted by $|\sigma^k|$, and is called the geometric carrier of σ^k.

Definition 2. A k-dimensional singular simplex is a C^0-map $\Xi: |\sigma^k| \to Y$, Y a topological space. The compact set $\Xi(|\sigma^k|)$ is called the trace of σ^k in Y; it is also denoted by $|\Xi|$. The free abelian group $C_k = C_k(Y)$ generated by the k-dimensional singular simplices is called the group of k-chains of Y and its elements are called k-chains.

A k-chain C_k on Y is thus a finite linear combination of k-dimensional singular simplices with coefficients in $\mathbf{Z}$: $K = \Sigma m_\nu \Xi_\nu$. The zero chain, i.e., the zero element in C_k, is denoted by 0. The trace of 0 is defined as the empty set $\varnothing$; the trace of an arbitrary chain K as $|K| = \bigcup |\Xi_\nu|$.

The boundary of a k-simplex σ^k is defined the usual way and extended to k-dimensional singular simplices via Ξ: For $\sigma^k = \langle v_0, ..., v_k \rangle$, and $\partial_q \sigma^k = \langle v_0, ..., \hat{v}_q, ..., v_k \rangle$ $0 \leqslant q \leqslant k$ (where $\hat{v}_q$ indicates that the vertex v_q is to be omitted) we have

$$\partial \sigma^k = \sum_{q=0}^{k} (-1)^q \, \partial_q \sigma^k.$$

Since $\sigma^k = \mathrm{id}(\sigma^k)$, $\partial_q \Xi = \Xi \partial_q$, $0 \leqslant q \leqslant k$, from which follows $\partial \Xi = \Sigma(-1)^k \, \Xi \partial_k$.

∂ induces a derivation from C_k into C_{k-1} by linearly extending it to a homomorphism $\partial: C_k \to C_{k-1}$ (we use the same symbol, namely ∂, to denote the boundary operator and the derivation).

DEFINITION 3. A compact set $Y_0 \subset Y$ is called a singular complex if

(a) $Y_0 = |K|$, K n-chain on Y,

(b) $K = \Sigma \varepsilon_\nu \Xi_\nu$ with $\varepsilon_\nu = \pm 1$, Ξ_ν injective on $\mathring{\sigma}^k$,

(c) for $\nu \neq \mu$, $|\mathring{\Xi}_\nu| \cap |\mathring{\Xi}_\mu| = \varnothing$,

(d) if F is any face of Ξ_ν, then $|F| \cap |\mathring{\Xi}_\nu| = \varnothing$.

If we set $Y = \mathbf{R}^n$ and $\Xi = \mathrm{id}$, then Y_0 is called a geometric complex. $|K|$ is called the polyhedron associated with K.

DEFINITION 4. A topological space Y is called triangulable if there exists a geometric complex K such that $Y = |K|$. K is then called a triangulation of Y.

We now commence with the construction of fractal surfaces in $\mathbf{R}^3$. Let $v_0 = (0, 0)$, $v_1 = (1, 0)$ and $v_2 = (0, 1)$, and let $\sigma^2 = \langle v_0, v_1, v_2 \rangle$ with orientation induced by $0 < 1 < 2$. Let $X = Q \times \mathbf{R}$, where $Q = [0, 1] \times [0, 1]$.

Define $f_i \colon X \to X$ by

$$f_i(x, y, z) = \begin{bmatrix} \Phi_i(x, y) \\ \Psi_i(x, y, z) \end{bmatrix}$$

with $\Phi_i \colon Q \to Q$, $\Psi_i \colon X \to \mathbf{R}$, $i \in \mathbf{I} = \{1, ..., N\}$, $N \in \mathbf{N}$, $N \geqslant 2$, and such that f_i maps σ^2 onto subsimplices $\sigma_i^2 = f_i(\sigma^2)$, and $\partial\sigma^2 = \Sigma \partial \sigma_i^2$, i.e., if $\sigma_i^2 \cap \sigma_j^2 = F_{ij} \subset \sigma^2$, $i \neq j$, then F_{ij} has opposite orientations in σ_i^2 and σ_j^2, $i, j \in \mathbf{I}$.

Furthermore we require that Φ_i maps σ^2 homeomorphically onto σ_i^2, and that Ψ_i is Lipschitz in the first and second variable and contractive in the third, $i \in \mathbf{I}$. We also assume that $\Phi_1(0, 0) = (0, 0)$, $\Psi_1(0, 0, 0) = 0$, and $\exists i_0, i_1 \in \mathbf{I}$, such that $\Phi_{i_0}(1, 0) = (1, 0)$, $\Phi_{i_1}(0, 1) = (0, 1)$, $\Psi_{i_0}(1, 0, 0) = \Psi_{i_1}(0, 1, 0) = 0$ (see Fig. 2).

The pair $\mathbf{F}_X = (X, \mathbf{F})$ with $\mathbf{F} = \{f_i \colon X \to X : i \in \mathbf{I}\}$ is an i.f.s. on X. To show that $\mathbf{F}_X$, with an appropriate metric, is a hyperbolic i.f.s. we need the following lemma whose proof is elementary.

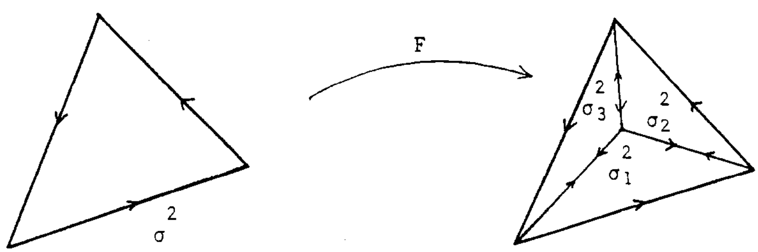

FIGURE 2

LEMMA 1. *Let* $d: \mathbf{R}^2 \to \mathbf{R}_0^+$ *be a metric on* $\mathbf{R}$. *Then* $\mathfrak{D}_{\alpha,\beta,\gamma}((x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})) = \alpha d(x, \tilde{x}) + \beta d(y, \tilde{y}) + \gamma d(z, \tilde{z})$ *is a metric on* $\mathbf{R}^3$, $\forall \alpha, \beta, \gamma > 0$.

PROPOSITION 1. $\mathbf{F}_X$ *is a hyperbolic i.f.s. on the complete metric space* $(X, \mathfrak{D}_\kappa)$, *for fixed* $\kappa > 1$, *where*

$$\mathfrak{D}_\kappa(x, y) = |x_1 - y_1| + |x_2 - y_2| + \frac{1 - \max\{\operatorname{contr} \Phi_i(x, \cdot), \operatorname{contr} \Phi_i(\cdot, y)\}}{\kappa \max\{\operatorname{Lip} \Psi_i(x, \cdot, \cdot), \operatorname{Lip} \Psi_i(\cdot, y, \cdot)\}} |x_3 - y_3|$$

and $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$.

Proof. The proof is straightforward and is omitted. ∎

$\mathbf{F}_X$ as a hyperbolic i.f.s. has a unique attractor $A \in H(X)$. The following theorem characterizes this attractor.

THEOREM 2. $A = A(\mathbf{F}_X)$, *where* $\mathbf{F}_X$ *is the i.f.s. constructed above, is the graph of a* C^0*-function* $\chi: \sigma^2 \to \mathbf{R}$, *passing through the vertices of* σ_i^2, $i \in \mathbf{I}$.

Proof. Let $\Theta: \mathbf{R}^2 \to \mathbf{R}$ be a function with domain $\operatorname{dom}(\Theta) \supset \sigma^2$, $\Theta \in BC^0(\mathbf{R}^2, \mathbf{R})$ with $\Theta(0, 0) = \Theta(1, 0) = \Theta(0, 1) = 0$. Denote this collection of functions by $C^*(\mathbf{R}^2, \mathbf{R})$. Let $\mathscr{S} = \{$compact two-dimensional C^0-submanifolds of $\mathbf{R}^3$ given by $z = \Theta(x, y)$, $\Theta \in C^*(\mathbf{R}^2, \mathbf{R})\}$. Clearly $\mathscr{S} \neq \varnothing$.

Choose a metric $D: \mathscr{S} \times \mathscr{S} \to \mathbf{R}_0^+$, $D(\Theta_1, \Theta_2) = \sup\{|\Theta_1(x, y) - \Theta_2(x, y)| : (x, y) \in \mathbf{R}^2\}$. Then $(\mathscr{S}, D)$ is a complete metric space. Define an operator $T: \mathscr{S} \to \mathscr{S}$ by

$$(T\Theta)(x, y) = \Psi_i(g_i(x, y), \Theta g_i(x, y)), \qquad \forall (x, y) \in \sigma_i^2$$

with $g_i \in \mathscr{L}(\sigma_i^2, \sigma^2)$, where $\mathscr{L}(\sigma_i^2, \sigma^2)$ denotes the set of all nonsingular affine maps from σ_i^2 onto σ^2 and $i \in \mathbf{I}$.

The operator T is well-defined and contractive: $D(T\Theta_1, T\Theta_2) \leqslant \alpha D(\Theta_1 g_i, \Phi_2 g_i) \leqslant \alpha D(\Theta_1, \Theta_2)$, where $\alpha = \operatorname{contr}\{\Psi_i(x, y, \cdot) : i \in \mathbf{I}\} < 1$. Thus by the Banach Fixed Point Theorem T has a unique fixed point $\chi \in \mathscr{S}$. That $\operatorname{graph}(\chi) \subset \mathbf{R}^3$ contains $\{v_0\}$, $\{v_1\}$, and $\{v_2\}$ is clear. ∎

The graph of χ is in general a fractal set, and $\operatorname{graph}(\chi) = \bigcup f_i(\operatorname{graph}(\chi))$; i.e., $\operatorname{graph}(\chi)$ is self-similar. We therefore refer to $\operatorname{graph}(\chi)$ as a *fractal surface*. Note that χ is a fractal function from $\mathbf{R}^2$ into $\mathbf{R}$.

In what follows we shall mostly deal with a special class of maps f_i, $i \in \mathbf{I}$, namely affine ones. Let

$$\Phi_i(x, y) = \begin{bmatrix} a_i & d_i \\ b_i & c_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \qquad i \in \mathbf{I}$$

$$\Psi_i(x, y, z) = k_i x + l_i y + s_i z + \gamma_i,$$

where the constants a_i, b_i, c_i, d_i, k_i, l_i, α_i, β_i, and γ_i are determined by the conditions on f_i, and the constants s_i, $|s_i| < 1$, are arbitrary parameters.

An even more special class of affine maps is obtained by choosing $N+1$ equally spaced vertices on $\partial\sigma^2$, from which we then obtain N^2 subsimplices σ_i and thus N^2 maps f_i. Denote these maps by $\mathfrak{A}_i$, $i \in I = \{1, ..., N^2\}$. Then

$$\mathfrak{A}_i(x, y, z) = \begin{bmatrix} \pm\frac{1}{N} & 0 & 0 \\ 0 & \pm\frac{1}{N} & 0 \\ k_i & l_i & s_i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{bmatrix} \tag{$*$}$$

Associated with the i.f.s. $\mathbf{F}_X$ is the i.f.s. $\hat{\mathbf{F}}_\sigma = (\sigma^2, \hat{\mathbf{F}})$, where $\hat{\mathbf{F}} = \{\Phi_i : i \in \mathbf{I}\}$. The attractor $\hat{A} = \hat{A}(\mathbf{F}_\sigma) = \sigma^2$. We may call $\mathbf{F}_\sigma$ the projection of $\mathbf{F}_\mathbf{X}$ onto $\mathbf{R}^2$ via $\pi_z : \mathbf{R}^3 \to \mathbf{R}^2$, $\pi_z(x, y, z) = (x, y)$, $\forall x, y, z \in \mathbf{R}$. The notation $\mathbf{F}_\sigma = \pi_z^* \mathbf{F}_\mathbf{X}$ is then self-evident. The i.f.s.'s $\hat{\mathbf{F}}_{\mathbf{xz}} = \pi_y^* \mathbf{F}_\mathbf{X}$ and $\hat{\mathbf{F}}_{\mathbf{yz}} = \pi_x^* \mathbf{F}_\mathbf{X}$ generate fractal functions from $\mathbf{R} \times \{0\} \times \mathbf{R}$ and $\{0\} \times \mathbf{R} \times \mathbf{R}$ onto $\mathbf{R}$, respectively. In the case of affine maps we obtain in particular:

$\hat{\mathbf{F}}_{\mathbf{xz}}$: $\quad g_i : [0, 1] \times \{0\} \times \mathbf{R}$ ⟲

$$g_i(x, z) = \begin{bmatrix} a_i & 0 \\ k_i & s_i \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \alpha_i \\ \gamma_i \end{bmatrix}$$

$\hat{\mathbf{F}}_{\mathbf{yz}}$: $\quad h_i : \{0\} \times [0, 1] \times \mathbf{R}$ ⟲

$$h_i(y, z) = \begin{bmatrix} b_i & 0 \\ l_i & s_i \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} \beta_i \\ \gamma_i \end{bmatrix}$$

If we have a geometric complex K consisting of a finite number of two-dimensional simplices, each having its own i.f.s. $\mathbf{F}_\sigma$ with attractor $A(\mathbf{F}_\sigma)$, we obtain a fractal surface from this complex by setting $A(K) = \bigcup A(\mathbf{F}_\sigma)$. Hence every triangulable topological space can be used to generate fractal surfaces.

The construction of fractal surfaces from singular simplices and complexes is done via the maps Ξ, as defined above; the i.f.s. consists of maps of the form $\Xi \circ f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_m}$, where $\omega_1, ..., \omega_m \in \mathbf{I}$, $m \in \mathbf{N}$.

Fractal surfaces on compact two-dimensional Riemannian manifolds are a special case of the above-mentioned construction: the Ξ's are the charts in the differentiable atlas of the manifold.

By restricting the i.f.s. $\mathbf{F}_\mathbf{X}$ to $\pi_y^* \mathbf{F}_\mathbf{X}$ or $\pi_x^* \mathbf{F}_\mathbf{X}$ we are able to construct fractal functions from topological spaces into $\mathbf{R}$, including compact two-dimensional Riemannian manifolds.

IV. The Code Space and the Dynamical System Associated with an I.F.S.

Let $\mathbf{F_X}$ be an i.f.s. consisting of N maps $f_i \in \mathbf{F}$. Let A be the attractor of $\mathbf{F_X}$. Define $\Omega = \mathbf{I}^{\mathbf{N}}$. Ω endowed with the Fréchet metric $|\cdot, \cdot|: \Omega \times \Omega \to \mathbf{R}_0^+$,

$$|\omega, \tilde{\omega}| = \sum_{m=1}^{\infty} \frac{|\omega_m - \tilde{\omega}_m|}{(N+1)^m}, \quad \omega = (\omega_m)_{m \in \mathbf{N}}, \quad \tilde{\omega} = (\tilde{\omega}_m)_{m \in \mathbf{N}}$$

is a compact metric space, homeomorphic to the classical Cantor set on N symbols.

It can be shown (see for instance [2]) that there exists a surjection $S \in C^0(\Omega, A)$,

$$S(\omega) = \lim_{m \to \infty} f_{\omega_1} f_{\omega_2} \cdots f_{\omega_m}(x_0)$$

with $\omega = (\omega_m) \in \Omega$, $m \in \mathbf{N}$, $x_0 \in X$, and this limit is uniformly independent of x_0.

Convention. From now on we use the notation $f_{\omega_1 \cdots \omega_m}$ to denote $f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_m}$.

Now let $\mathbf{F_X}$ be the i.f.s. defining a fractal function in $\mathbf{R}^3$ as defined in III. Every point $(x, y) \in \sigma^2 = A(\hat{\mathbf{F}}_\sigma)$ can then be written as $(x, y) = \hat{S}(\omega) = \lim \Phi_{\omega_1 \dots \omega_m}(x_0, y_0)$, some $(x_0, y_0) \in \sigma^2$, where $\hat{S}$ is a continuous surjection from Ω onto σ^2 (the code spaces for $\mathbf{F_X}$ and $\mathbf{F}_\sigma$ are clearly identical). Since $A(\mathbf{F_X}) = \text{graph}(\chi)$ with $\chi \in C^0(\sigma^2, \mathbf{R})$ we have

$$\chi(\hat{S}\omega) = \lim_{m \to +\infty} \Psi_{\omega_1}(\hat{S}\sigma_+ \omega, \Psi_{\omega_2}(\hat{S}\sigma_+^2 \omega, \dots, \Psi_{\omega_m}(\hat{S}\sigma_+^m \omega, z_0))\dots)$$

some $z_0 \in \mathbf{R}$.

As we can associate a code space with every i.f.s. $\mathbf{F_X}$, we can also associate a dynamical system $\mathbf{D} = \mathbf{D}(\tilde{X}, \tilde{\xi}, \tilde{B}, \tilde{\mu})$ with $\mathbf{F_X}$. This can be done as follows (see also [3] or [13]):

Let μ be the $\mathbf{p}$-balanced measure of $\mathbf{F_X}$ and let m be the uniform Lebesgue measure on $[0, 1]$. Let $\tilde{X} = X \times [0, 1]$ and let $\tilde{\mu} = \mu \times m$ be a measure on $\tilde{X}$. Denote by $\tilde{B} = \tilde{B}(\tilde{X})$ the σ-algebra of all $\tilde{\mu}$-measurable subsets of $\tilde{X}$. Let $\mathbf{p} = \{p_i : i \in \mathbf{I}\}$ be a set of non-zero probabilities. Define a $\tilde{\mu}$-measurable map $\tilde{\xi}: \tilde{X} \to \tilde{X}$ by

$$\tilde{\xi}(x, t) = \left(f_i(x), \frac{t - \Sigma_{i-1}}{p_i} \right),$$

for $(x, t) \in [\Sigma_{i-1}, \Sigma_i]$ with $\Sigma_i = p_1 + \cdots + p_i$, $\Sigma_0 = 0$, $i \in \mathbf{I}$.

It is easy to see that the attractor of $\mathbf{D}$ is $A(\tilde{X}) = A(X) \times [0, 1]$, and thus $\dim A(\tilde{X}) = 1 + \dim A(X)$. Let us now introduce the Lyapunov dimension of a dynamical system:

DEFINITION 5. Let $\mathbf{D} = \mathbf{D}(\tilde{X}, \tilde{\xi}, \tilde{B}, \tilde{\mu})$ be the dynamical system associated with the i.f.s. $\mathbf{F_X}$, where $\tilde{\mathbf{X}}$ is a compact k-dimensional Riemannian manifold. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ be the Lyapunov exponents of $\tilde{\xi}$. Let $m_0 = \sup\{m \in \{1, ..., k\}: \lambda_1 + \cdots + \lambda_m > 0\}$. If $1 \leqslant m_0 < m$, then the Lyapunov dimension $\Lambda(\tilde{\mu})$ of $\tilde{\mu}$ is defined as

$$\Lambda(\tilde{\mu}) = m_0 + \frac{\lambda_1 + \cdots + \lambda_{m_0}}{|\lambda_{m_0+1}|}.$$

If $m_0 = m$, then $\Lambda(\tilde{\mu}) = k$ and if no such m_0 exists, $\Lambda(\tilde{\mu}) = 0$.

Remark. The Lyapunov exponents are calculated via Oseledec's Multiplicative Ergodic Theorem (see [14]).

In the case of affine maps $\mathfrak{A}_i$, $i \in \mathbf{I}$, it is straightforward to calculate the Lyapunov exponents and thus the Lyapunov dimension of the associated dynamical system $\mathbf{D}$. Indeed, $\tilde{\xi}: \tilde{X} \to \tilde{X}$ is given by

$$\tilde{\xi}(x, y, z, t) = \begin{bmatrix} \dfrac{x}{N} \\ \dfrac{y}{N} \\ k_i x + l_i y + s_i z \\ \dfrac{t - \Sigma_{i-1}}{p_i} \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{bmatrix}$$

for $(x, y, z, t) \in X \times [\Sigma_{i-1}, \Sigma_i]$, $i = 1, ..., N^2$ (α, β, and γ are determined by the constraints on $\mathfrak{A}_i$), from which follows that (using the law of large numbers to determine the resulting limit of random products)

$$\lambda_1 = \sum_{i=1}^{N^2} p_i \log\left(\frac{1}{p_i}\right)$$

$$\lambda_2 = \sum_{i=1}^{N^2} p_i \log(|s_i|)$$

$$\lambda_3 = \lambda_4 = \log\left(\frac{1}{N}\right).$$

If $\Sigma |s_i| > N$ (which ensures that $\dim \Lambda(\tilde{X}) > 3$), then $\lambda_1 \geqslant \lambda_2 > \lambda_3 = \lambda_4$. Thus

$$\Lambda(\tilde{\mu}, \mathbf{p}) = 3 + \frac{\sum p_i \log(|s_i|/Np_i)}{\log N}.$$

If $\Sigma |s_i| \leqslant N$, $\Lambda(\tilde{\mu}, p) = 3$. Using methods from advanced calculus we find a set of probabilities $\mathbf{p}^*$ that maximizes $\Lambda(\tilde{\mu}, \mathbf{p})$. This maximized Lyapunov dimension Λ^* is then given by

$$\Lambda^* = 2 + \frac{\log(\Sigma |s_i|)}{\log N}.$$

V. p-Balanced Measures and Moment Theory

The i.f.s. $\mathbf{F_X}$, as defined in III, admits a stationary measure μ whose support is graph(χ). The "projected" i.f.s. $\mathbf{F}_\sigma$ also admits a stationary measure $\hat{\mu}$ with $\operatorname{supp} \hat{\mu} = |\sigma^2|$. Denote by $\mathscr{M}(\sigma^2)$ and $\mathscr{M}(G)$, $G = \text{graph}(\chi)$, the measure spaces of σ^2 and G, respectively. We can define a homeomorphism $H: \sigma^2 \to G$, $H(x, y) = (x, y, \chi(x, y))$, and obtain the following relation between measures on $\mathbf{F_X}$ and measures on $\mathbf{F}_\sigma$:

Theorem 3. *$H: \sigma^2 \to G$, as defined above, induces a contravariant homeomorphism $\mathscr{M}(H): \mathscr{M}(G) \to \mathscr{M}(\sigma^2)$. Furthermore, $\hat{\mu}(\hat{E}) = \mu(H(\hat{E}))$, $\forall \hat{E} \in \hat{B}(\sigma^2)$. If $g \in L^1(X, \mu)$ then*

$$\int_G g \, d\mu = \int_{\sigma^2} gH \, d\hat{\mu} = \int_{H^{-1}\sigma^2} g \, d\hat{\mu}.$$

Proof. Define $\mathscr{M}(H)$: $\mathscr{M}(G) \to \mathscr{M}(\sigma^2)$ by $\mathscr{M}(H)(\mu)(\hat{E}) = \mu(H(\hat{E}))$, $\forall \hat{E} \in \hat{B}(\sigma^2)$, $\forall \mu \in \mathscr{M}(G)$. Now let μ be any $\mathbf{p}$-balanced measure on $\mathscr{M}(G)$; then $\mu(E) = \Sigma p_i \mu(f_i^{-1}(E))$, $\forall E \in B(G)$. Also $\mathscr{M}(H)(\mu)(\hat{E}) = \mu(H(\hat{E})) = \Sigma p_i \mu(f_i^{-1}(H(\hat{E}))) = \Sigma p_i \mu(H\Phi_i^{-1}(\hat{E})) = \Sigma p_i \hat{\mu}(\Phi_i^{-1}(\hat{E}))$. But for the $\mathbf{p}$-balanced measure $\hat{\mu}$ we have $\hat{\mu}(\hat{E}) = \Sigma p_i \hat{\mu}(\Phi_i^{-1}(\hat{E}))$, $\forall \hat{E} \in \mathscr{M}(\sigma^2)$. Hence $\mathscr{M}(H)(\mu) = \hat{\mu}$. The integral identity follows now easily. ∎

Theorem 3 implies the following integral relation:

Corollary 1. *If the probabilities p_i are chosen according to $p_i = \text{area}(\sigma_i^2)$, $i \in \mathbf{I}$, and if dA denotes two-dimensional Lebesgue measure on $\mathbf{R}^2$, then*

$$\int_G g \, d\mu = \int_{\sigma^2} g \circ H \, dA = \int_0^1 \int_0^{1-x} g \circ H \, dy \, dx.$$

Proof. With the above choice of probabilities we have $dA(\Phi_i^{-1}(\hat{B})) = dA(\hat{B})$, $\forall \hat{B} \in B(\sigma^2)$. ∎

Corollary 1 together with the stationarity of the **p**-balanced measure μ can be used to calculate moments. Let $g \in L^1(X, \mu)$ and let $p_i' = \text{area}(\sigma_i^2)$, $i \in \mathbf{I}$. Then

$$\begin{aligned}
\int_{\text{graph}(\chi)} & g(x, y, z)\, d\mu(x, y, z) \\
&= \sum_i p_i' \int g \circ f_i(x, y, z)\, d\mu(x, y, z) \\
&= \sum_i p_i' \int_{\text{graph}(\chi)} g(\Phi_i(x, y), \Psi_i(x, y, z))\, d\mu(x, y, z) \\
&= \int_{\sigma^2} g(x, y, \chi(x, y))\, dA.
\end{aligned}$$

Also

$$\int_{\text{graph}(\chi)} g(x, y, z)\, d\mu(x, y, z) = \sum_i p_i' \int_{\sigma^2} g(\Phi_i(x, y), \Psi_i(x, y, \chi(x, y)))\, dA.$$

Hence

$$\int_{\sigma^2} g(x, y, \chi(x, y))\, dA = \sum_i p_i' \int_{\sigma^2} g(\Phi_i(x, y), \Psi_i(x, y, \chi(x, y)))\, dA.$$

DEFINITION 6. Let μ be the **p**-balanced measure for the i.f.s. $\mathbf{F_X}$. Then the nth moment of χ with respect to μ is defined as

$$\mathbf{M}(\rho; \mu) = \int_X x^\rho \chi(x, y)\, d\mu, \qquad \rho \in \mathbf{N}_0$$

and the generalized moment of χ with respect to μ as

$$\mathbf{M}(\rho, \sigma, \tau; \mu) = \int_X x^\rho y^\sigma \chi^\tau\, d\mu, \qquad \rho, \sigma, \tau \in \mathbf{N}_0.$$

Convention. Since we only consider moments with respect to μ, we simply write $\mathbf{M}_\rho$ for $\mathbf{M}(\rho; \mu)$ and $\mathbf{M}_{\rho,\sigma,\tau}$ for $\mathbf{M}(\rho, \sigma, \tau; \mu)$.

THEOREM 4. *Let graph*(χ) *be the fractal surface generated by* $\mathbf{F_X}$ *with* $f_i = \mathfrak{A}_i$, $i \in \mathbf{I}$. *Then the moments* $\mathbf{M}_\rho$ *and* $\mathbf{M}_{\rho,\sigma,\tau}$ *are determined uniquely and recursively by the lower order moments.*

Proof.

$$\mathbf{M}_\rho = \int_{\sigma^2} x^\rho \chi \, dA = \sum_i (2N^2)^{-1} \int_{\sigma^2} \left(\frac{x}{N} + \alpha_i\right)^\rho (k_i x + l_i y + s_i \chi + \gamma_i) \, dA.$$

Now the right-hand side equals

$$\sum_i (2N^2)^{-1} \int_{\sigma^2} \left\{ \sum_{r=0}^{\rho} \binom{\rho}{r} \frac{x^r}{N^r} \alpha_i^{\rho - r} \right\} \{k_i x + l_i y + s_i \chi + \gamma_i\} \, dA$$

$$= \sum_i (2N^2)^{-1} \int_{\sigma^2} \left[\sum_{r=0}^{\rho-1} \binom{\rho}{r} \frac{x^r}{N^r} \alpha_i^{\rho-r} s_i \chi + \left(\frac{x}{N} + \alpha_i\right)^\rho (k_i x + l_i y + \gamma_i) \right] dA$$

$$= \sum_i \sum_{r=0}^{\rho-1} \binom{\rho}{r} \frac{\alpha_i^\rho s_i}{2N^{r+3}} \int_{\sigma^2} x^r \chi \, dA + P_\rho,$$

where

$$P_\rho = \sum_i (2N^2)^{-1} \int_{\sigma^2} \left(\frac{x}{N} + \alpha_i\right)^\rho (k_i x + l_i y + \gamma_i) \, dA.$$

Hence

$$\mathbf{M}_\rho = \sum_i \frac{s_i}{2N^{r+3}} \mathbf{M}_\rho + \sum_i \sum_{r=0}^{\rho-1} \binom{\rho}{r} \frac{\alpha_i^{\rho-r} s_i}{2N^{r+3}} \mathbf{M}_r + P_\rho$$

$$\Leftrightarrow \mathbf{M}_\rho = \frac{\sum_i \sum_{r=0}^{\rho-1} \binom{\rho}{r} (\alpha_i^{\rho-r} s_i / 2N^{r+3}) \, \mathbf{M}_r + P_\rho}{1 - \sum_i s_i / 2N^{r+3}}$$

(note that $\Sigma \, |s_i| > N$).

For $\mathbf{M}_{\rho,\sigma,\tau}$ we obtain

$$\mathbf{M}_{\rho,\sigma,\tau} = \int_{\sigma^2} x^\rho y^\sigma \chi^\tau \, dA$$

$$= \sum_i (2N^2)^{-1} \int_{\sigma^2} \left(\frac{x}{N} + \alpha_i\right)^\rho \left(\frac{y}{N} + \beta_i\right)^\sigma (k_i x + l_i y + s_i \chi + \gamma_i)^\tau \, dA$$

$$= \sum_i (2N^2)^{-1} \int_{\sigma^2} \sum_{r,s,t} P_{r,s,t}(N, \alpha_i, \beta_i, \gamma_i, k_i, l_i, s_i) \, x^r y^s \chi^t \, dA$$

$$+ \sum_i (2N^2)^{-1} \int_{\sigma^2} \frac{x^\rho}{N^\rho} \frac{y^\sigma}{N^\sigma} s_i \chi^\tau \, dA,$$

where $P_{r,s,t}$ is an appropriate polynomial. Thus

$$\mathbf{M}_{\rho,\sigma,\tau} = \frac{\sum_i \sum_{r,s,t} (2N^2)^{-1} P_{r,s,t} \mathbf{M}_{r,s,t}}{1 - \sum_i (s_i^\tau / 2N^{\rho+\sigma+2})}. \quad ∎$$

VI. The Fractal Dimension of Graph(χ)

Consider the i.f.s. $\mathbf{F_X}$ with $f_i = \mathfrak{A}_i$, $i \in \mathbf{I} = \{1, ..., N^2\}$. We derive a formula for the fractal dimension of graph(χ), where $\chi: \sigma^2 \to \mathbf{R}$ is the fractal function representing the fractal surface generated by $\mathbf{F_X}$.

Theorem 5. *Assume that* $Z = \Sigma |s_i| > N$ *and* $\bigcup |\sigma_i^2|$ *is not coplanar. Then the fractal dimension* d *of* $G = \mathrm{graph}(\chi)$ *is given by* $d = 1 + \log_N(\Sigma |s_i|)$; *otherwise* $d = 2$.

Before we proceed with the proof of this theorem, let us state a lemma which we use in the proof.

Lemma 2. *Let* B_ε *denote a ball of radius* $\varepsilon > 0$ *in* $\mathbf{R}^3$. *Let* Δ_ε *be the* ε*-triangular set* $\varepsilon\sigma^2 \times \varepsilon I$, $I = [0, 1]$ (εA *of a bounded set* $A \subset \mathbf{R}^n$ *is the image of* A *under* $E: \mathbf{R}^n \to \mathbf{R}^n$, $E(x) = \varepsilon x$, $\varepsilon > 0$, *usually* $\varepsilon \ll 1$). *Let* S *be a bounded set in* $\mathbf{R}^3$ *and let* $\mathcal{N}(B_\varepsilon)$ *be the minimum number of balls* B_ε *needed to cover* S. *Then* $\mathcal{N}(B_\varepsilon) \leqslant \mathcal{N}(\Delta_\varepsilon) \leqslant 8\mathcal{N}(B_\varepsilon)$, *where* $\mathcal{N}(\Delta_\varepsilon)$ *is the minimum number of* ε*-triangular sets necessary to cover* S.

Proof. Every B_ε can be covered by at most $8\Delta_\varepsilon$, and every Δ_ε can be covered by one B_ε. ∎

Proof of Theorem 5. By the remark following the definition of fractal dimension it suffices to choose $\varepsilon = \varepsilon_m = (1/N)^m$, $m \in \mathbf{N}$.

Let $\mathscr{C}_m$ be a collection of covers of G by ε_m-trianular sets $\Delta = \Delta(m, p, q)$ of the form $\pi_z \Delta = |\sigma_p^2|$, $p = 1, ..., N^{2m}$, $\sigma_p^2 = f_{\omega_p}(\sigma^2)$, $\omega_p \in \Omega$, and $|\omega_p| = m$, $q = 1, ..., \mathcal{N}(m, p)$, where $\mathcal{N}(m, p) \in \mathbf{N}$ denotes the number of such ε_m-triangular sets above $|\sigma_p^2|$. Furthermore require that $\Delta(m, p, q)$ and $\Delta(m, p, q+1)$ intersect along their respective faces. Let $C_m \in \mathscr{C}_m$ be a cover of minimal cardinality $\mathcal{N}(m) \in \mathbf{N}$ (note that the existence of such a minimal cover is guaranteed by the compactness of G). Observe that $\mathcal{N}(m) = \Sigma \mathcal{N}(m, p)$. Let $G_p = \bigcup \Delta(m, p, q)$, $p = 1, ..., N^{2m}$. Then $H^k(G_p) = 0$, $k = 0, 1, 2$, where H^k denotes the kth homology functor.

Now apply map f_i, $i \in I$, to G_p. $G_{pi} = f_i(G_p)$ is then a compact set in $\sigma_p^2 \times \mathbf{R}^2$, above the ith subsimplex σ_{pi}^2 of σ_p^2. Since $G = \bigcup f_i(G)$ we have $G \subset \bigcup f_i(\bigcup G_p)$. G_{pi} is contained in the triangular set $(N^{m+1})^{-1} |\sigma^2| \times (\mathcal{N}(m, p) |s_i| + |k_i| + |l_i|)(N^{-m}) I$. Hence if we denote by $\mathcal{N}(m+1, p, i)$ the number of ε_{m+1}-triangular sets above σ_{pi}^2, we obtain $\mathcal{N}(m+1, p, i) \leqslant N(\mathcal{N}(m, p) |s_i| + |k_i| + |l_i|) + 1$. Observe that $\mathcal{N}(m+1) = \Sigma\Sigma\mathcal{N}(m+1, p, i)$. Summing the above inequality over p and i yields $\mathcal{N}(m+1) \leqslant (NZ) \mathcal{N}(m)$

$+c_1 N^{2m+1}$, with $c_1 = \Sigma(|k_i| + |l_i| + N) > 0$. If $Z \leqslant N$ induction on m yields $\mathcal{N}(m) \leqslant \mathcal{N}(1) + c_1 m N^{-1}) N^{2m}$, then it follows that (see Fig. 3)

$$\limsup_{m \to \infty} \frac{\log \mathcal{N}(m)}{\log(N^m)} \leqslant 2.$$

Hence $\dim(G) = 2$. If $\bigcup |\sigma_i^2|$ is coplanar, i.e., if $\bigcup |\sigma_i^2| \subset \Pi$, Π any hyperplane of $\mathbf{R}^3$, then $G = \Pi$ and thus $\dim(G) = 2$.

If $Z > N$, we again obtain by induction on m

$$\begin{aligned}\mathcal{N}(m) &\leqslant (NZ)^m \left[\mathcal{N}(1) + \frac{c_1}{Z}\left(1 + \frac{N}{Z} + \cdots + \left(\frac{N}{Z}\right)^{m-1}\right)\right] \\ &\leqslant (NZ)^m \left[\mathcal{N}(1) + \frac{c_1}{Z-N}\right],\end{aligned}$$

which implies $\dim(G) = \limsup(\log \mathcal{N}(m))/(\log(N^m)) \leqslant 1 + \log_N(Z)$.

To obtain a lower bound for $\dim(G)$ if $Z > N$ and $\bigcup |\sigma_i^2|$ not coplanar, we proceed as follows. Apply f_i^{-1} to G_p, assuming $s_i \neq 0$, $i \in \mathbf{I}$. $f_i^{-1}(G_p)$ is then contained in a triangular set $(N^{-m+1}) |\sigma_i^2| \times N^{-m}(\mathcal{N}(m, p) |s_i^{-1}| + |k_i s_i^{-1}| N) I$. Hence with the same notation as above: $\mathcal{N}(m-1, p, i) \leqslant |s_i^{-1}| \mathcal{N}(m, p, i) + |k_i s_i^{-1}| N + |l_i s_i^{-1}| N + 1$. Summing over p and i yields $\mathcal{N}(m) \geqslant (NZ) \mathcal{N}(m-1) - c_2 N^{2m-1}$, $c_2 = N\Sigma(|k_i s_i^{-1}| + |l_i s_i^{-1}| + N) > 0$ (let us remark that the above inequality holds trivially for $s_i = 0$), and induction on m for all $m_0 \in \{1, ..., m\}$: $\mathcal{N}(m) \geqslant (NZ)^{m-m_0} (\mathcal{N}(m_0) - c_2 N^{2m_0}(Z-N)^{-1})$. Lemma 3 below shows that one can choose m_0 large enough to guarantee $\mathcal{N}(m_0) > c_2 N^{2m_0}(Z-N)^{-1}$. If we set $c_3 = (NZ)^{-m_0} (\mathcal{N}(m_0) - c_2 N^{2m_0}(Z-N)^{-1}) > 0$, we get $\mathcal{N}(m) \geqslant c_3 (NZ)^m$. Hence $\limsup (\log \mathcal{N}(m)/\log(N^m)) \geqslant 1 + \log_N(Z)$. ∎

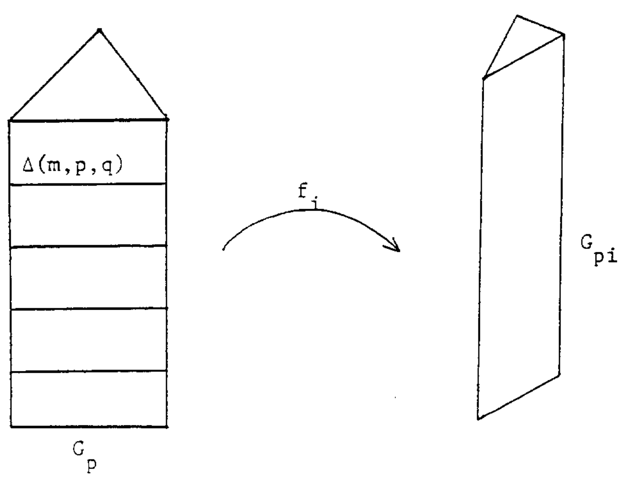

FIGURE 3

LEMMA 3. *If $Z > N$ and $\bigcup |\sigma_i^2|$ is not coplanar, then*

$$\lim_{m \to \infty} \frac{N^{2m}}{\mathcal{N}(m)} = 0.$$

Proof. Since $\bigcup |\sigma_i^2|$ is not coplanar $\exists i_0 \in \mathbf{I}$ such that $V = |z_{i_0}| > 0$. Notice that $V \leqslant \max\{|z_i - z_{i_0}| : i \in \mathbf{I} \setminus \{i_0\}\}$. We then have $\mathcal{N}(m) \geqslant VN^{2m}$, and consequently $\mathcal{N}(m) \geqslant \Sigma |s_i| \, VN^{2m}$, $m \in \mathbf{N}$. By induction

$$\mathcal{N}(m) \geqslant \sum_{i_1} \sum_{i_2} \cdots \sum_{i_p} |s_{i_1} \cdots s_{i_p}| \, VN^{2m}, \qquad \forall p \leqslant m,$$

and thus $\mathcal{N}(m) \geqslant [V(\Sigma |s_i|)^p - 1] \, N^{2m}$. ▮

Let us remark that $\dim(G) = \Lambda^*(\mu) - 1$.

VII. HÖLDER CONTINUITY

We show that χ is Hölder continuous with Hölder exponent $\delta = 3 - \dim(G)$. Throughout this section we assume that $G = \operatorname{graph}(\chi)$ is generated by $\mathfrak{A}_i \in \mathbf{F}$, $i \in \mathbf{I} = \{1, ..., N^2\}$.

THEOREM 6. *Let χ be the fractal function representing the fractal surface $G = \operatorname{graph}(\chi)$ generated by $\mathbf{F_X}$ with $\mathfrak{A}_i \in \mathbf{F}$, $i \in \mathbf{I} = \{1, ..., N^2\}$. Then if $0 \leqslant h < 1$,*

$$|\chi(x+h, y+h) - \chi(x, y)| \leqslant ch^\delta,$$

$\forall (x, y) \in \sigma^2$, with $c > 0$ and $\delta = 3 - \dim(G) = 2 - \log_N(Z)$.

Proof. Let $h \in [0, 1)$ be given. Let m be the least integer such that there exists a code $\omega \in \Omega$ with $|\omega| = m$, so that (x, y) and $(x+h, y+h) \in f_\omega(\sigma^2)$. Then $|\chi(x+h, y+h) - \chi(x, y)| \leqslant N^{-m} \max\{\mathcal{N}(m, p) : p = 1, ..., N^{2m}\}$ (we use here the same terminology and notation as in the proof of Theorem 5). Thus since $\max \mathcal{N}(m, p) \geqslant N^{-2m} \mathcal{N}(m) \geqslant cN^{-2m} (N^{-m})^{-d}$, where $d = \dim(G)$, it follows that

$$\begin{aligned}
\frac{\log|\chi(x+h, y+h) - \chi(x, y)|}{\log h} &\geqslant \frac{\log(cN^{-3m}(N^{-m})^{-d})}{\log h} \\
&\geqslant \frac{\log c}{\log h} + \frac{\log(N^{-3m} N^{md})}{\log(N^{-m})} \\
&= \frac{\log c}{\log h} + 3 - d,
\end{aligned}$$

since $h \leqslant N^{-m}$. ▮

The fact that χ is Hölder continuous with exponent $\delta = 3 - \dim(G)$ has the following consequence.

THEOREM 7. *Suppose that χ is Hölder continuous with exponent $\delta = 3 - \dim(G)$. Then $\mathscr{H}^{\dim(G)}(G) < +\infty$.*

Proof. Let $h \in (0, 1)$ and let m be the least integer such that $h \leqslant N^{-m}$. Let σ_i^2 be any of the N^{2m} subsimplices of σ^2. Then $G|_{\sigma_i}$ may be covered by at most $N^{2m}(ch^\delta + 1)\, N^{-m}$-triangular sets. Thus

$$\begin{aligned}\mathscr{H}^{\dim(G)}_{N^{-2m}}(G) &\leqslant N^{2m}(\mathrm{ch}^{3-\dim(G)} + 1)(N^{-m})^{\dim(G)}\\ &\leqslant N^{2m}(cN^{(-m)(3-\dim(G))} + 1)(N^{-m})^{\dim(G)}\\ &= cN^{-m} + (N^{-m})^{\dim(G)-2} \leqslant \frac{c}{N} + \left(\frac{1}{N}\right)^{\dim(G)-2} < +\infty.\end{aligned}$$

Hence $\mathscr{H}^{\dim(G)}(G) < +\infty$. ∎

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