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# Hypergraphs and a functional equation of Bouwkamp and de Bruijn

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## Abstract

Let  $\Phi(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} u^m v^n$ . Bouwkamp and de Bruijn found that there exists a power series  $\Psi(u, v)$  satisfying the equation  $t\Psi(tz, z) = \log\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \exp(k\Phi(kz, z))\right)$ . We show that this result can be interpreted combinatorially using hypergraphs. We also explain some facts about  $\Phi(u, 0)$  and  $\Psi(u, 0)$ , shown by Bouwkamp and de Bruijn, by using hypertrees, and we use Lagrange inversion to count hypertrees by number of vertices and number of edges of a specified size.

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## 1. Introduction

In [3], Bouwkamp and de Bruijn use algebraic methods to prove some results concerning a power series expansion. Their original motivation arose from work by Harris and Park [7], who showed the asymptotic normality of the distribution of empty cells when some number of balls were placed in some number of equiprobable cells. To accomplish this, Harris and Park employed factorial cumulants; in particular, they showed that in the power

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series expansion of

$$\log\left(\sum_{k=0}^{\infty} \binom{N}{k} \left(1 - \frac{k}{N}\right)^N t^k\right),$$

the coefficient of  $t^n$  is  $O(N)$ . This work led Bouwkamp and de Bruijn (cf. [3, Section 4C]) to show that if  $\Phi(u, v)$  is a double power series of the form

$$\Phi(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} u^m v^n,$$

then there exists a power series  $\Psi(u, v)$  such that

$$\log\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \exp(k\Phi(kz, z))\right) = t\Psi(tz, z). \tag{1.1}$$

That is, the left-hand side can be written as  $t \sum_{n=0}^{\infty} t^n \theta_n(z) = \sum_{n=0}^{\infty} t^{n+1} \theta_n(z)$ , where  $\theta_n(z)$  is a power series which has no powers of  $z$  less than  $n$ . Bouwkamp and de Bruijn further demonstrate a result relating  $\psi(u) := \Psi(u, 0)$  and  $\phi(u) := \Phi(u, 0)$ . Note that  $\Psi(u, 0)$  yields the “leading terms” of (1.1), in the sense that  $t\Psi(tz, 0)$  is the series which contains all terms of  $t\Psi(tz, z)$  in which the power of  $z$  is one less than the power of  $t$ . Bouwkamp and de Bruijn show that if  $w$  is the power series in  $y$  satisfying

$$y = w \exp(-\phi(w) - w\phi'(w)), \tag{1.2}$$

then

$$\psi(y) = (w - w^2\phi'(w))/y. \tag{1.3}$$

We will show that these results from [3] are actually consequences of identities for hypergraphs and hypertrees. We will also give combinatorial interpretations of many other equations that were derived algebraically in [3].

A hypergraph is a generalization of a graph (the next section has exact definitions and basic facts; see [1] for further background). In general, edges can consist not only of a set of two vertices, but of a set of an arbitrary number of vertices. An edge consisting of  $i$  vertices will be called an  $i$ -edge. We will be concerned with hypergraphs without empty edges or loops (i.e., without 0-edges or 1-edges); therefore, when we use the term *hypergraph*, it will refer to hypergraphs whose edges have at least two vertices.

Bouwkamp and de Bruijn prove their results by analyzing the power series (see (1.2) in [3])

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \exp\left[\sum_{i=2}^{\infty} k^i x_{i-1}\right] \tag{1.4}$$

and then substituting power series for  $t$  and the  $x_i$ . We shall prove their results in a very similar way, by substituting power series for  $t$  and for the  $u_i$  in the generating function

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \exp \left[ \sum_{i=2}^{\infty} \binom{k}{i} u_i \right], \tag{1.5}$$

which we interpret as a generating function for hypergraphs.

If we wanted, we could prove the result by considering (1.4) to be the exponential generating function for a set of objects in which the “edges” are sequences of vertices (with repetitions allowed). However, since hypergraphs seem more natural, we choose to use them.

Section 2 provides definitions of hypergraphs and hypertrees. In Section 3, we prove (1.1) by showing it is a consequence of the hypergraph analogue of the fact that every connected graph with  $n$  vertices has at least  $n - 1$  edges. In Section 4, we interpret several equations obtained by Bouwkamp and de Bruijn in terms of hypergraphs. Section 5 provides interpretations of the leading terms,  $\psi(u)$  and  $\phi(u)$ , using hypertrees. We conclude in Section 6 by showing how this work and Lagrange inversion can be used to obtain previously known results on the enumeration of hypertrees.

## 2. Definitions and background

We define a hypergraph  $H$  on  $n$  vertices to be an ordered pair  $(V, E)$ , where  $V$  is the set of vertices, with  $|V| = n$ , and  $E$  is a multiset of subsets of  $V$ ; we also require the subsets in  $E$  contain at least two vertices. In particular, we allow multiple edges. For an arbitrary hypergraph  $H$ , we let  $v(H)$  denote the number of vertices of  $H$  and  $e(H)$  denote the number of edges of  $H$ . This definition differs from that in Berge [1] since we allow a hypergraph to have vertices which belong to no edge. Our definition of a hypergraph nearly agrees with that of Grieser [6]; the difference is that we do not allow loops.

In general, we will consider hypergraphs labeled so that if the hypergraph has  $n$  vertices, they are labeled by the elements of  $[n] := \{1, 2, 3, \dots, n\}$ , and if the hypergraph has  $\lambda_i$   $i$ -edges, they are labeled by the elements of  $[\lambda_i]$ . For simplicity, we will call such objects *labeled hypergraphs*.

In what follows, we will always have  $\lambda_1 = 0$ , since our hypergraphs have no loops. Let  $u_2, u_3, u_4, \dots$  be indeterminates. We define the *weight* of  $H$  to be

$$u_2^{\lambda_2} u_3^{\lambda_3} \cdots u_n^{\lambda_n}$$

and we define the *edge magnitude* of  $H$  to be  $\sum_{i=2}^k (i - 1)\lambda_i$ .

An example of a labeled hypergraph is given in Fig. 1. The 2-edges are denoted by a segment connecting the two vertices; for edges with more than two vertices, the edge is represented by a closed curve which contains the vertices of the edge inside it. The vertices are labeled by numbers without subscripts; for clarity, the edges are labeled with subscripted numbers in which the subscript refers to the size of the edge being labeled. (The subscripts on the edge labels thus do not add structure to the hypergraph.) For the hypergraph in the

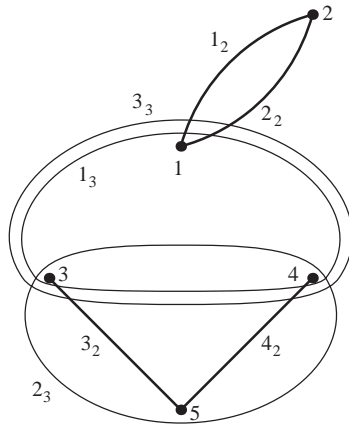


Fig. 1. A sample labeled hypergraph.

figure,  $V = [5]$ ;  $E = \{\{1, 2\}, \{1, 2\}, \{3, 5\}, \{4, 5\}, \{1, 3, 4\}, \{1, 3, 4\}, \{3, 4, 5\}\}$ ;  $\lambda_2 = 4, \lambda_3 = 3$ ; the weight is  $u_2^4 u_3^3$ ; and the edge magnitude is 10.

We define a *walk* in a hypergraph to be a sequence

$$v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n,$$

where for all  $i, v_i \in V, e_i \in E$ , and for each  $e_i, \{v_{i-1}, v_i\} \subseteq e_i$ . We define a *path* in a hypergraph to be a walk in which all  $v_i$  are distinct and all  $e_i$  are distinct. A walk is a *cycle* if the walk contains at least two edges, all  $e_i$  are distinct, and all  $v_i$  are distinct except  $v_0 = v_n$ .

A hypergraph is *connected* if for every pair of vertices  $v, v'$  in the hypergraph, there is a path starting at  $v$  and ending at  $v'$ . The hypergraph in Fig. 1 is connected. For example, a path between vertices 2 and 5 is

$$2, 1_2 = \{1, 2\}, 1, 3_3 = \{1, 3, 4\}, 3, 3_2 = \{3, 5\}, 5.$$

We define a *hypertree* to be a connected hypergraph with no cycles.

The *degree* of a vertex  $v \in V$ , denoted  $\text{deg}(v)$ , is defined as

$$\text{deg}(v) := |\{e \in E \mid v \in e\}|;$$

i.e., the degree of  $v$  is the number of edges to which  $v$  belongs. Two vertices in a hypergraph are *adjacent* if there is an edge containing both.

We now note some basic facts about hypertrees. First, two edges in a hypertree have at most one vertex in common; for if edges  $e_1, e_2$  have two vertices  $v_1, v_2$  in common, then the hypergraph has a cycle  $v_1, e_1, v_2, e_2, v_1$ . Next, we prove the following lemma. It is known; for another proof, see [6].

**Lemma.** *A connected hypergraph on  $n$  vertices is a hypertree if and only if it has edge magnitude  $n - 1$ . Furthermore, the minimum edge magnitude of a connected hypergraph is  $n - 1$ .*

**Proof.** First we prove by induction on  $n$  that a hypertree on  $n$  vertices has edge magnitude  $n - 1$ . This is clearly true for  $n = 1$ . Now suppose that  $H$  is a hypertree with  $n > 1$  vertices and that every hypertree with  $n - 1$  vertices has edge magnitude  $n - 2$ .

Let  $v_0, e_1, \dots, e_m, v_m$  be a longest path in  $H$ . Suppose that  $v_m$  is contained in some edge  $e$  other than  $e_m$ . We will show that this assumption leads to a contradiction. If  $e \in \{e_1, \dots, e_{m-1}\}$  then  $H$  contains a cycle, so  $e \notin \{e_1, \dots, e_{m-1}\}$ . Let  $v$  be a vertex of  $e$  other than  $v_m$ . Then  $v_0, e_1, \dots, e_m, v_m, e, v$  is either a longer path than  $v_0, e_1, \dots, e_m, v_m$  (if  $v \notin \{v_0, \dots, v_{m-1}\}$ ) or contains a cycle (if  $v \in \{v_0, \dots, v_{m-1}\}$ ). Since both are impossible,  $v_m$  cannot be contained in any edge of  $H$  other than  $e_m$ .

Let  $H'$  be obtained from  $H$  by removing vertex  $v_m$  and then either replacing edge  $e_m$  with  $e_m - \{v_m\}$ , if  $|e_m| > 2$ ; or deleting  $e_m$ , if  $|e_m| = 2$ . It is clear that  $H'$  is a hypertree and that its edge magnitude is one less than that of  $H$ . By the inductive hypothesis,  $H'$  has edge magnitude  $n - 2$ , so  $H$  has edge magnitude  $n - 1$ .

Now suppose that  $H$  is a connected hypergraph on  $n$  vertices that is not a hypertree. Then  $H$  has a cycle  $v_0, e_1, v_1, \dots, e_n, v_0$ . Replacing  $e_1$  with  $e_1 - \{v_0\}$  if  $|e_1| > 2$ , or deleting  $e_1$  if  $|e_1| = 2$ , leaves a connected hypergraph on  $n$  vertices with edge magnitude one less than that of  $H$ . Repeating this reduction on cycles eventually yields a hypertree. Thus  $H$  has edge magnitude greater than  $n - 1$ .  $\square$

### 3. Proof of (1.1)

We turn our attention to proving the main result of [3] using the exponential generating function for labeled hypergraphs. We adopt the convention that if  $\lambda = (\lambda_2, \lambda_3, \dots)$  is a sequence of integers with finitely many non-zero parts, then

$$u^\lambda = u_2^{\lambda_2} u_3^{\lambda_3} u_4^{\lambda_4} \dots$$

and

$$\frac{u^\lambda}{\lambda!} = \frac{u^\lambda}{\lambda_2! \lambda_3! \lambda_4! \dots}$$

We do not use  $u_1$  since we will not consider hypergraphs with loops.

Using exponential generating functions, we now count labeled hypergraphs with vertices and edges labeled as in Fig. 1. (For background on the combinatorics of exponential generating functions, see [4, Chapter 3, Section 2], [11, Chapter 5], or, for an approach using species [2, Chapter 1].) Consider the exponential generating function

$$(e^{u_i})^{\binom{k}{i}} = \left( 1 + u_i + \frac{u_i^2}{2!} + \dots \right)^{\binom{k}{i}}.$$

We view the term  $\frac{u_i^j}{j!}$  in the expansion of  $e^{u_i}$  as representing  $j$  multiple copies of a particular  $i$ -edge. Since there are  $\binom{k}{i}$   $i$ -subsets of vertices in a hypergraph with  $k$  vertices, the previous expression counts labeled hypergraphs on  $k$  vertices whose edges are all of size  $i$ .

Therefore,

$$\prod_{i=2}^{\infty} (e^{u_i})^{\binom{k}{i}} = \exp\left[\binom{k}{2}u_2 + \binom{k}{3}u_3 + \dots\right]$$

is the exponential generating function for labeled hypergraphs on  $k$  vertices, where the coefficient of  $\frac{t^k}{k!}$  is the number of labeled hypergraphs on  $k$  vertices with  $\lambda_i$   $i$ -edges for each  $i \geq 2$ . From this, we see that the exponential generating function for labeled hypergraphs with vertices weighted  $t$  and  $i$ -edges weighted by  $u_i$  is

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \exp\left[\sum_{i=2}^{\infty} \binom{k}{i} u_i\right].$$

Since the edge magnitude of a hypergraph counted by the coefficient of  $\frac{t^k}{k!}$  is  $\sum_{i \geq 2} \lambda_i(i-1)$ , we define the *magnitude* of  $u^\lambda$  to be the same expression.

We now consider connected labeled hypergraphs. Since a labeled hypergraph is a set of connected labeled hypergraphs, if  $C := C(t; u_2, u_3, \dots)$  is the exponential generating function for connected labeled hypergraphs, we have

$$e^C = \sum_{k=0}^{\infty} \frac{t^k}{k!} \exp\left[\sum_{i=2}^{\infty} \binom{k}{i} u_i\right].$$

Hence,

$$C = \log\left[\sum_{k=0}^{\infty} \frac{t^k}{k!} \exp\left(\binom{k}{2}u_2 + \binom{k}{3}u_3 + \binom{k}{4}u_4 + \dots\right)\right]. \tag{3.1}$$

We know from Section 2 that the edge magnitude of a connected hypergraph on  $k$  vertices must be at least  $k - 1$ . So if we write

$$C = \sum_{k=1}^{\infty} \frac{t^k}{k!} f_k(u_2, u_3, u_4, \dots), \tag{3.2}$$

the minimum magnitude of terms of  $f_k$  is  $k - 1$ .

We finish the proof of (1.1) with an argument from Bouwkamp and de Bruijn [3, Section 1]. For  $m \geq 1$ , let  $P_m(z) = \sum_{i \geq 0} p_{m,i} z^i$  be power series in  $z$ . If we make the substitutions

$$t \exp(P_1(z)) \mapsto t, \quad z^{m-1} P_m(z) \mapsto u_m,$$

in (3.2), then the coefficient of  $t^k$  is a power series in  $z$  with no term of degree less than  $k - 1$ . Thus, the resulting power series is of form  $t\Psi(tz, z)$ . If we make the same substitutions in (3.1), we obtain

$$C = \log\left[\sum_{k=0}^{\infty} \frac{t^k}{k!} \exp\left(\sum_{m \geq 1} \binom{k}{m} z^{m-1} \sum_{i \geq 0} p_{m,i} z^i\right)\right].$$

Here the argument of the exponential function may be written

$$\sum_{j=0}^{\infty} z^j \sum_{m=1}^{j+1} \binom{k}{m} P_{m, j-m+1}. \tag{3.3}$$

Now let  $\Phi(u, v) = \sum_{m,n=0}^{\infty} c_{mn} u^m v^n$  be any double power series, so

$$k\Phi(kz, z) = \sum_{m,n \geq 0} k^{m+1} c_{mn} z^{m+n} = \sum_{j=0}^{\infty} z^j \sum_{l=1}^{j+1} k^l c_{l-1, j-l+1}. \tag{3.4}$$

Since the polynomials  $\binom{k}{m}$ , for  $m = 1, \dots, j + 1$ , form a basis for the polynomials of degree at most  $j + 1$  in  $k$  that vanish at 0, it is possible to choose power series  $P_m(z)$  that make (3.3) equal to (3.4) and thus make (3.1) equal to the left side of (1.1). This proves (1.1).

#### 4. Further combinatorial interpretations

We remark here that some calculations done by Bouwkamp and de Bruijn [3, Sections 2–3] correspond to simple manipulations involving  $C$  which yield various ways of decomposing hypergraphs into other hypergraphs. Differentiation of  $e^C$  with respect to  $u_j$  yields (cf. [3, Section 2])

$$\begin{aligned} \frac{\partial e^C}{\partial u_j} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \binom{k}{j} \exp\left(\binom{k}{2} u_2 + \binom{k}{3} u_3 + \binom{k}{4} u_4 + \dots\right) \\ &= \frac{t^j}{j!} \frac{\partial^j e^C}{\partial t^j}. \end{aligned} \tag{4.1}$$

By properties of exponential generating functions [4, pp. 167–168],  $\frac{\partial e^C}{\partial u_j}$  counts hypergraphs rooted at an unlabeled  $j$ -edge. (The generating function  $u_j \frac{\partial e^C}{\partial u_j}$  would count those rooted at a labeled  $j$ -edge.) Also, operating on  $e^C$  by  $t^j \frac{\partial^j}{\partial t^j}$  counts hypergraphs which are equipped with an ordered  $j$ -tuple of  $j$  distinct vertices. By dividing by  $j!$  as in (4.1), we count hypergraphs rooted at  $j$  vertices. Therefore (4.1) represents a bijection between hypergraphs rooted at an unlabeled  $j$ -edge and hypergraphs with  $j$  rooted vertices.

From (4.1) in the case  $j = 2$ , we obtain (cf. [3, (2.1)])

$$\frac{\partial C}{\partial u_2} = \frac{t^2}{2!} \left[ \left( \frac{\partial C}{\partial t} \right)^2 + \frac{\partial^2 C}{\partial t^2} \right]. \tag{4.2}$$

This represents a way to decompose connected hypergraphs rooted at an unlabeled 2-edge. By removing the rooted 2-edge, we either obtain two vertex-rooted connected hypergraphs or else we obtain a doubly vertex-rooted connected hypergraph. See Fig. 2 for an example;

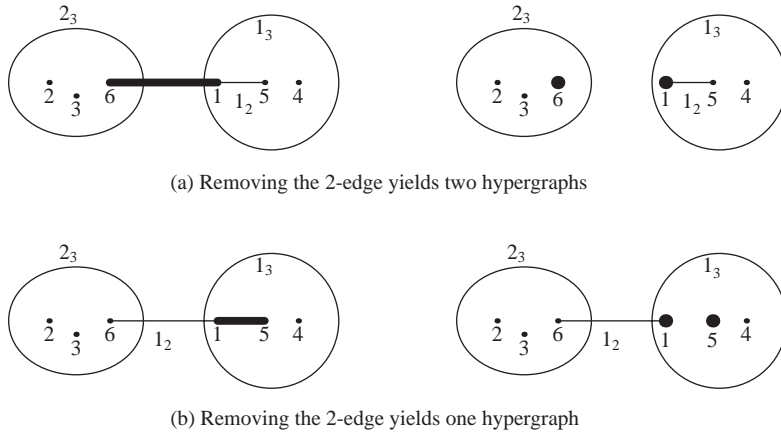


Fig. 2. Decomposition of (4.2).

in the figure, rooted objects are marked heavily (thick lines or larger dots). Since  $(t \frac{\partial C}{\partial t})^2$  counts ordered pairs of rooted hypergraphs,  $\frac{1}{2!} (t \frac{\partial C}{\partial t})^2$  counts sets containing two rooted hypergraphs, as in Fig. 2(a). Also, since  $t^2 \frac{\partial^2 C}{\partial t^2}$  counts hypergraphs rooted at an ordered pair of vertices,  $\frac{t^2}{2!} \frac{\partial^2 C}{\partial t^2}$  counts doubly vertex-rooted hypergraphs, as in Fig. 2(b).

Next, note that (4.1) implies

$$\frac{\partial e^C}{\partial u_j} = \frac{1}{j} \left( t \frac{\partial}{\partial t} - (j - 1) \right) \frac{\partial e^C}{\partial u_{j-1}}.$$

From this we obtain (cf. [3, (2.2)])

$$\frac{\partial C}{\partial u_j} = \frac{1}{j} \left[ t \frac{\partial C}{\partial t} \frac{\partial C}{\partial u_{j-1}} + t \frac{\partial^2 C}{\partial t \partial u_{j-1}} - (j - 1) \frac{\partial C}{\partial u_{j-1}} \right]. \tag{4.3}$$

Combinatorially, the term  $t \frac{\partial C}{\partial t} \frac{\partial C}{\partial u_{j-1}}$  on the right-hand side of (4.3) counts pairs of connected hypergraphs, in which one of the pair is rooted at a vertex and one is rooted at an unlabeled  $(j - 1)$ -edge. The next term,  $t \frac{\partial^2 C}{\partial t \partial u_{j-1}}$ , counts connected hypergraphs rooted at both an unlabeled  $(j - 1)$ -edge and a vertex. Finally,  $(j - 1) \frac{\partial C}{\partial u_{j-1}}$  counts connected hypergraphs rooted at an unlabeled  $(j - 1)$ -edge and at a vertex *in that rooted edge*. Thus, (4.3) says there are  $j$  ways to decompose a connected hypergraph rooted at an unlabeled  $j$ -edge either into a pair of connected hypergraphs, one rooted at a vertex and the other rooted at an unlabeled  $(j - 1)$ -edge (for example, see Fig. 3(b)(i)); or into a single connected hypergraph, rooted at an unlabeled  $(j - 1)$ -edge and at a vertex *not in the rooted edge* (see Fig. 3(b)(ii), (iii)). To perform the decomposition given a hypergraph rooted at an unlabeled  $j$ -edge, simply choose a vertex  $v$  in the rooted edge  $e$ . Then remove the edge  $e$ , add the edge  $e - \{v\}$ , and root the new object at  $v$  and at the added edge.



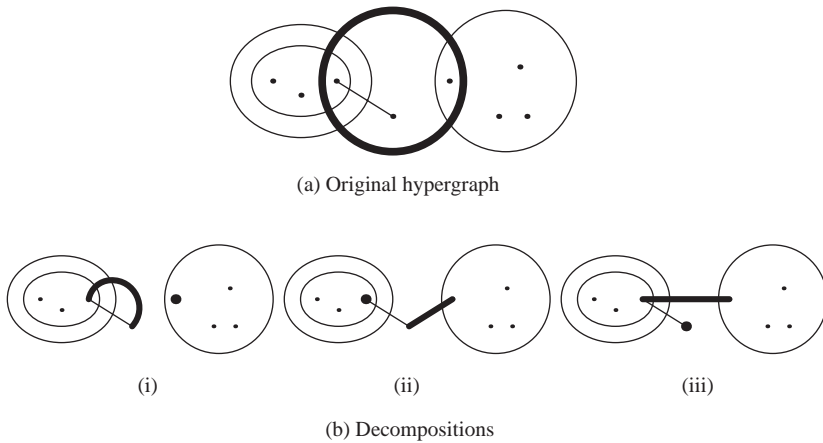


Fig. 3. Decompositions according to (4.3).

### 5. Interpretations of the leading terms

We now consider the combinatorial interpretation of the results in [3] about the leading terms of (1.1). It turns out that much of the work leading to the results in [3] involves differential equations related to decompositions of hypertrees.

We define for  $n \geq 1$ ,

$$C_n = C_n(u_2, u_3, \dots) := \left[ \frac{t^n}{n!} \right] C;$$

that is,  $C_n$  is the coefficient of  $\frac{t^n}{n!}$  in the power series  $C$ . Thus, the coefficient of  $\frac{u_j^\lambda}{\lambda!}$  in  $C_n$  is the number of connected labeled hypergraphs on  $[n]$  with  $\lambda_j$   $j$ -edges for  $j \geq 2$ . From (4.2) we obtain (cf. [3, (2.3)]),

$$\frac{\partial C_n}{\partial u_2} = \frac{1}{2!} \left[ \sum_{i=1}^{n-1} \binom{n}{i} i(n-i) C_i C_{n-i} + n(n-1) C_n \right] \tag{5.1}$$

and from (4.3) (cf. [3, (2.4)]),

$$\frac{\partial C_n}{\partial u_j} = \frac{1}{j} \left[ \sum_{i=1}^{n-1} \binom{n}{i} (n-i) \left( \frac{\partial C_i}{\partial u_{j-1}} \right) C_{n-i} + (n-(j-1)) \frac{\partial C_n}{\partial u_{j-1}} \right]. \tag{5.2}$$

Now, we define

$$T_n = T_n(u_2, u_3, \dots) := \text{all terms of magnitude } n-1 \text{ in } C_n,$$

and let

$$T(t; u_2, u_3, \dots) = \sum_{n=1}^{\infty} \frac{t^n}{n!} T_n(u_2, u_3, \dots). \tag{5.3}$$

The generating function  $T$  contains the “leading terms” of  $C$ , in the sense  $T_n$  is the sum of the terms of minimal magnitude in  $C_n$ . Hence, if  $\tau_{n,\lambda}$  is the number of labeled hypertrees on  $[n]$  with  $\lambda_i$   $i$ -edges, we have

$$T_n = \sum_{\lambda=(\lambda_2,\lambda_3,\dots)} \tau_{n,\lambda} \frac{u^\lambda}{\lambda!},$$

where  $\tau_{n,\lambda} = 0$  unless  $\sum_{i \geq 2} (i - 1)\lambda_i = n - 1$ . We note that  $T_n$  corresponds to  $\eta_n^*$  in [3, Section 3].

We can get differential equations for  $T$  using the differential equations for  $C_n$ . By taking terms with the minimal magnitude  $n - 2$  on both sides of (5.1) we get (cf. [3, (3.1)])

$$\frac{\partial T_n}{\partial u_2} = \frac{1}{2!} \left( \sum_{i=1}^{n-1} \binom{n}{i} i(n-i) T_i T_{n-i} \right).$$

There is no contribution to this equation from the term  $\frac{1}{2}n(n-1)C_n$  in the summation on the right-hand side of (5.1). That term corresponds to the case in which the removal of a 2-edge from a connected hypergraph yields a single connected hypergraph. For hypertrees, removing a 2-edge must yield two hypertrees.

From the last equation, we obtain

$$\frac{\partial T}{\partial u_2} = \frac{1}{2!} \left( t \frac{\partial T}{\partial t} \right)^2. \tag{5.4}$$

If we take terms with the minimal magnitude  $n - 1 - (j - 1) = n - j$  on both sides of (5.2), we get (cf. [3, (3.2)])

$$\frac{\partial T_n}{\partial u_j} = \frac{1}{j} \left( \sum_{i=1}^{n-1} \binom{n}{i} (n-i) \frac{\partial T_i}{\partial u_{j-1}} T_{n-i} \right),$$

implying

$$\frac{\partial T}{\partial u_j} = \frac{1}{j} \left( t \frac{\partial T}{\partial t} \right) \left( \frac{\partial T}{\partial u_{j-1}} \right). \tag{5.5}$$

We can then conclude from (5.4) and (5.5) that (cf. [3, (3.4)])

$$\frac{\partial T}{\partial u_j} = \frac{1}{j!} \left( t \frac{\partial T}{\partial t} \right)^j. \tag{5.6}$$

This equation describes a correspondence between hypertrees rooted at an unlabeled  $j$ -edge and sets of  $j$  hypertrees each rooted at a vertex. Husimi [8] was the first to obtain (5.6).

We now return to (5.3), the definition of  $T$ . We apply the operator  $u_j \frac{\partial}{\partial u_j}$  to both sides; note that this will count hypertrees rooted at a labeled  $j$ -edge. Writing  $\sum_{\lambda} \tau_{n,\lambda} \frac{u^\lambda}{\lambda!}$  for  $T_n$ ,

we get

$$\begin{aligned}
 u_j \frac{\partial T}{\partial u_j} &= \sum_{n=1}^{\infty} \frac{t^n}{n!} u_j \frac{\partial T_n}{\partial u_j} \\
 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} u_j \frac{\partial}{\partial u_j} \left( \sum_{\lambda} \tau_{n,\lambda} \frac{u^\lambda}{\lambda!} \right) \\
 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\lambda} \tau_{n,\lambda} \lambda_j \frac{u^\lambda}{\lambda!}.
 \end{aligned}$$

Multiplying both sides by  $j - 1$  and then summing on  $j$  yields

$$\begin{aligned}
 \sum_{j=2}^{\infty} (j - 1) u_j \frac{\partial T}{\partial u_j} &= \sum_{j=2}^{\infty} (j - 1) \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\lambda} \tau_{n,\lambda} \lambda_j \frac{u^\lambda}{\lambda!} \\
 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\lambda} \tau_{n,\lambda} \frac{u^\lambda}{\lambda!} \sum_{j=2}^n (j - 1) \lambda_j \\
 &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\lambda} \tau_{n,\lambda} \frac{u^\lambda}{\lambda!} (n - 1) \\
 &= \sum_{n=1}^{\infty} (n - 1) \frac{t^n}{n!} T_n.
 \end{aligned}$$

In the above, the third equality follows from the second because  $\lambda_j$  is the number of  $j$ -edges in a hypertree, and the edge magnitude of a hypertree is  $n - 1$ . We conclude that (cf. [3, (3.5)])

$$\sum_{j=2}^{\infty} (j - 1) u_j \frac{\partial T}{\partial u_j} = t \frac{\partial T}{\partial t} - T. \tag{5.7}$$

This equation describes two ways to count each hypertree on  $n$  vertices with multiplicity  $n - 1$ . It is clear that the right-hand side does this. The terms on the left-hand side count every hypertree rooted at a labeled  $j$ -edge  $j - 1$  times. But since the edge magnitude of a hypertree on  $[n]$  is  $n - 1$ , the left-hand side counts every hypertree  $n - 1$  times.

We now define  $R$  to be

$$R = t \frac{\partial T}{\partial t}.$$

In [3, (3.9)], the expression  $w$  corresponds to  $R$ , which is the exponential generating function for hypertrees rooted at a labeled vertex, counting hypertrees by weight and number of edges. We shall refer to the objects counted by  $R$  as *rooted hypertrees*. Using this definition, and

using (5.6) and (5.7), we obtain

$$T = R - \sum_{j=2}^{\infty} (j-1)u_j \frac{1}{j!} R^j, \tag{5.8}$$

which was also first derived by Husimi [8].

We now obtain a functional equation for  $R$ , using a slightly different path from that in [8]. Differentiating both sides of (5.8) with respect to  $t$ , we get

$$\frac{R}{t} = \frac{\partial T}{\partial t} = \frac{\partial R}{\partial t} - \sum_{j=2}^{\infty} (j-1)u_j \frac{1}{j!} j R^{j-1} \frac{\partial R}{\partial t},$$

so

$$\frac{dt}{t} = \frac{\partial R}{R} - \sum_{j=2}^{\infty} u_j \frac{1}{(j-2)!} R^{j-2} \partial R. \tag{5.9}$$

Integrating this yields

$$\frac{R}{t} = \exp\left(\sum_{j=1}^{\infty} u_{j+1} \frac{R^j}{j!}\right), \tag{5.10}$$

where we obtain the constant of integration by noting that  $\frac{R}{t}|_{t=0} = 1$  (since the number of rooted hypertrees on a single vertex is 1). We can rewrite this as (cf. [8])

$$R = t \exp\left(\sum_{j=1}^{\infty} u_{j+1} \frac{R^j}{j!}\right). \tag{5.11}$$

Eq. (5.11) describes a way of decomposing rooted hypertrees into a set of other rooted hypertrees. Note that in a rooted hypertree, if  $v_1$  and  $v_2$  are both adjacent to the root of the original hypertree, then  $v_1$  and  $v_2$  cannot both be in an edge which does not contain the root. Thus, if the root of a hypertree is contained in  $i$  edges containing  $j_1 + 1, j_2 + 1, \dots, j_i + 1$  vertices, then when we remove the root and those edges from the original hypertree, we are left with  $i$  sets of hypertrees, containing  $j_1, j_2, j_3, \dots, j_i$  hypertrees. In addition, each hypertree in each set is rooted at the vertex formerly in an edge with the root.

This is exactly what (5.11) is describing. For a given  $j$ , the term  $u_{j+1} \frac{R^j}{j!}$  corresponds to a set of  $j$  rooted hypertrees and another edge of  $j + 1$  vertices; this extra edge consists of the roots of the  $j$  hypertrees and a new vertex (counted by the leading  $t$  in (5.11)) which becomes the root of the new hypertree.

Fig. 4 depicts a hypertree rooted at the vertex labeled 1 and, below, the decomposition resulting from removing the root and all edges containing it. The roots of the hypertrees resulting from the decomposition are denoted by larger dots. The original hypertree that is shown is decomposed into three sets of hypertrees, indicated in the figure.

We note here that (5.11) and (5.8) can be obtained from (1.2) (which is [3, (1.6)]) and (1.3) (which is [3, (1.7)]), respectively. In (1.2) and (1.3), we substitute  $t$  for  $y$ ;  $R$  for  $w$ ;  $T$  for  $y\phi(y)$ ; and set  $\phi(x)$  equal to the power series  $\sum_{i=1}^{\infty} \frac{u_{i+1}x^i}{(i+1)!}$ .

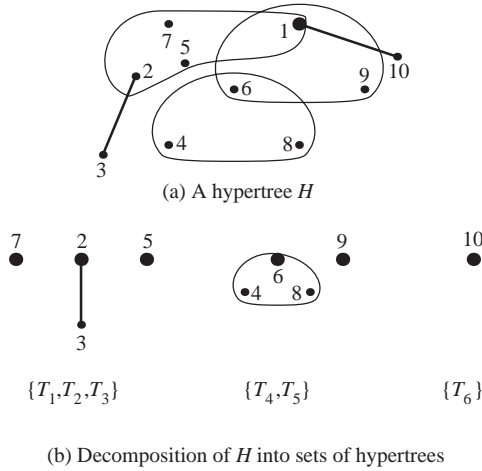


Fig. 4. Decomposition of (5.11).

To count hypertrees only according to the total number of vertices, we can set  $u_j = 1$  in (5.8) for all  $j$ . Let  $\tilde{T}$  be the expression obtained from  $T$  by this substitution, and let  $\tilde{R}$  be the analogous expression for  $R$ . From (5.8), we get a simple expression relating  $\tilde{T}$  to  $\tilde{R}$ , where each exponential generating function now counts hypertrees only by number of vertices:

$$\begin{aligned}
 \tilde{T} &= \tilde{R} - \tilde{R} \sum_{j=1}^{\infty} \frac{\tilde{R}^j}{j!} + \sum_{j=2}^{\infty} \frac{\tilde{R}^j}{j!} \\
 &= \tilde{R} - \tilde{R}(e^{\tilde{R}} - 1) + (e^{\tilde{R}} - 1 - \tilde{R}) \\
 &= (e^{\tilde{R}} - 1) - \tilde{R}(e^{\tilde{R}} - 1) \\
 &= (e^{\tilde{R}} - 1)(1 - \tilde{R}).
 \end{aligned}
 \tag{5.12}$$

We can understand (5.12) by considering the penultimate form of the equation. The expression

$$u_2 \frac{\partial T}{\partial u_2} + u_3 \frac{\partial T}{\partial u_3} + u_4 \frac{\partial T}{\partial u_4} + \dots
 \tag{5.13}$$

counts hypertrees rooted at an edge. Thus, each unrooted hypertree  $H$  is counted  $e(H)$  times in (5.13). From (5.6), we see that if we replace each  $u_j$  by 1 in (5.13), the resulting expression is equal to  $e^{\tilde{R}} - \tilde{R} - 1$ . But each unrooted hypertree  $H$  is counted  $v(H)$  times by  $\tilde{R}$ , so that each hypertree  $H$  is counted by  $e^{\tilde{R}} - 1$  with multiplicity  $e(H) + v(H)$ . On the other hand, we can decompose a hypertree rooted at an edge and a vertex in the rooted edge by removing the rooted edge (but no vertices). We are left with a hypertree rooted at the previously rooted vertex and a set of hypertrees each rooted at a vertex which used to be in the rooted edge. These objects are exactly counted by  $\tilde{R}(e^{\tilde{R}} - 1)$ , which therefore counts (with multiplicity one) each hypertree rooted at an edge and a vertex in that edge. If

as before we denote the number of  $i$ -edges of  $H$  by  $\lambda_i$ , then the number of ways to root it at an edge and a vertex in that edge is  $\sum_i i \lambda_i$ . But

$$\sum_i i \lambda_i = \sum_i (i - 1) \lambda_i + \sum_i \lambda_i = (v(H) - 1) + e(H).$$

Therefore, in  $\tilde{R}(e^{\tilde{R}} - 1)$ , each hypertree  $H$  is counted  $v(H) - 1 + e(H)$  times, and so subtracting that expression from  $e^{\tilde{R}} - 1$  produces an expression in which every hypertree is counted exactly once. This explains (5.12).

### 6. Application to enumeration of hypertrees

By Lagrange inversion, we can find an explicit formula for rooted hypertrees by weight and number of edges. The numbers are well-known; cf. Husimi [8], Greene and Iba [5], and Kreweras [9] (in which hypertrees are called “dendroids”).

Since we can write, from (5.11),

$$R = t \prod_{j=1}^{\infty} e^{u_{j+1} \frac{R^j}{j!}}, \tag{6.1}$$

we get, using Lagrange inversion ([4, Theorem 1.2.4, p. 17]),

$$[t^n]R = \frac{1}{n} [t^{n-1}] \prod_{j=1}^{\infty} \left( 1 + \binom{n}{j} u_{j+1} \frac{t^j}{j!} + \frac{\binom{n}{j} u_{j+1} \frac{t^j}{j!}^2}{2!} + \frac{\binom{n}{j} u_{j+1} \frac{t^j}{j!}^3}{3!} + \dots \right).$$

Letting  $\lambda \vdash n - 1$  denote that  $\lambda$  is a partition of  $n - 1$  and  $a_i$  denote the number of parts of size  $i$  in  $\lambda$ , we calculate

$$\left[ \frac{t^n}{n!} \right] R = \sum_{\lambda \vdash n-1} \binom{n-1}{\lambda_1, \lambda_2, \dots} \prod_i \frac{(nu_{i+1})^{a_i}}{a_i!}. \tag{6.2}$$

Since there are  $n$  ways to root a hypertree on  $n$  vertices,  $\left[ \frac{t^n}{n!} \right] T = \frac{1}{n} \left[ \frac{t^n}{n!} \right] R$ , so for hypertrees on no more than 6 vertices,

$$\begin{aligned} \left[ \frac{t}{1!} \right] T &= 1, \\ \left[ \frac{t^2}{2!} \right] T &= u_2, \\ \left[ \frac{t^3}{3!} \right] T &= u_3 + 3u_2^2, \\ \left[ \frac{t^4}{4!} \right] T &= u_4 + 12u_2u_3 + 16u_2^3, \\ \left[ \frac{t^5}{5!} \right] T &= u_5 + 20u_2u_4 + 15u_3^2 + 150u_3u_2^2 + 125u_2^4, \end{aligned}$$

$$\left[ \frac{t^6}{6!} \right] T = u_6 + 30u_2u_5 + 60u_4u_3 + 360u_4u_2^2 + 540u_2u_3^2 + 2160u_3u_1^3 + 1296u_2^5.$$

If we set  $u_j = u$  for all  $j$  in (6.1), we can obtain the enumerator for rooted hypertrees on  $[n]$  by number of edges. If we let  $\bar{R}$  be the generating function resulting from setting  $u_j = u$  in  $R$ , then from (6.1),  $\bar{R} = te^{u(e^{\bar{R}}-1)}$ . However,  $e^{u(e^t-1)} = \sum_{n \geq 0} \frac{t^n}{n!} \left( \sum_{k=0}^n S(n, k) u^k \right)$  is the generating function for Stirling numbers of the second kind (cf. [10, p. 34]). Therefore, Lagrange inversion yields

$$\begin{aligned} \left[ \frac{t^n}{n!} \right] \bar{R} &= \frac{1}{n} n! [t^{n-1}] e^{nu(e^t-1)} \\ &= \sum_{k=1}^{n-1} (nu)^k S(n-1, k), \end{aligned}$$

so the number of rooted hypertrees on  $[n]$  with  $k$  edges is  $n^k S(n-1, k)$ . In particular, the total number of rooted hypertrees on  $[n]$  is  $\sum_k n^k S(n-1, k)$ .

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