Finite Generation of Lifted $p$-Adic Homology with Compact Supports. Generalization of the Weil Conjectures to Singular, Non-complete Algebraic Varieties

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In Section 1, if $O$ is a c.d.v.r. with quotient field of characteristic zero and residue class field $k$, if $A$ is an $O$-algebra and if $A = A \otimes_O k$, then for algebraic families $X$ over $A$ that are polynomially properly embeddable over $A$, we define the lifted $p$-adic homology with compact supports $H^n_c(X, A \otimes \mathbb{Z}_p)$, which are functors with respect to proper maps. In Section 2, it is shown that, if $X$ is an algebraic variety over $k$ (i.e., if $A = k$), then the lifted $p$-adic homology of $X$ with compact supports with coefficients in $K$ is finite dimensional over $K = \text{quotient field of } O$. In Section 3, the results of Sections 1 and 2 are used to generalize both the statement and proof of the Weil “Lefschetz Theorem” Conjecture and the statement (but not the proof) of the Weil “Riemann Hypothesis” Conjecture, to non-complete, singular varieties over finite fields. In addition, the Weil zeta function of varieties over finite fields, is generalized by a device which we call the zeta matrices, $W^X(X)$, $0 < h < 2 \dim X$, of an algebraic variety $X$, to varieties over even infinite fields of non-zero characteristic. These are used to give formulas for the zeta functions of each variety in an algebraic family, by means of the zeta matrices of an algebraic family. Sketches only are given. In Section 4, some of the material is duplicated, to define a $q$-adic homology with compact supports, $q \neq \text{characteristic}$. The definition only makes sense for algebraic varieties; finite generation is proved. And the Weil “Lefschetz Theorem” Conjecture is established, even for singular, non-complete varieties, as well as a generalization of the Weil “Riemann Hypothesis” Conjecture. (However, zeta matrices do not make sense $q$-adically.) In Section 5, some special results are proved about $p$-adic homology with compact supports on affines. And the Weil “Riemann Hypothesis” conjecture is proved $p$-adically, $p = \text{characteristic}$, for projective, non-singular liftable varieties.

1. Lifted $p$-Adic Homology with Compact Supports

Let $O$ be a complete discrete valuation ring having a quotient field $K$ of characteristic zero, and with residue class field $k$. Let $A$ be an $O$-algebra and let $A = A \otimes_O k$. Let $C$ be a scheme of finite presentation over $A_{\text{red}}$ such that
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C is polynomially properly embeddable over $A$ (see below—e.g., it suffices that $C$ be quasiprojective over $A$). Then in this section we define the lifted $p$-adic homology groups with compact supports

$$H_{h^*}(C, (A^+) \otimes Z Q), \quad \text{all integers } h.$$  

(\star)

In the special case that $C$ is simple over $A_{\text{red}}$ with fibers of constant dimension $N$ and liftable over $A$, these are canonically isomorphic to $H^{2N-h}(C, (\Gamma^*(\mathcal{U}))^+ \otimes Z Q)$, all integers $h$, where $C$ is any simple and separated lifting of finite presentation over $A$. (In general the groups (\star) are defined to be $H^{2N-h}(X, X - C, (\Gamma^*(X))^+ \otimes Z Q)$, where $X$ is simple and separated of finite presentation over $A$, with fibers of constant dimension $N$ over $A$. $X = X \times_A k$ and $C$ is closed in $X$.) These groups are shown to be a functor with respect to maps of reduced schemes over $A_{\text{red}}$.

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We now begin.

Let $\mathcal{O}$ be a complete discrete valuation ring having quotient field $K$ and residue class field $k$. Let $A$ be an $\mathcal{O}$-algebra and let $A = A \otimes k$.

We first consider the following question. Let $C$ be a reduced prescheme over $\text{Spec}(A)$, and let $X$, resp. $D$, be preschemes simple of finite presentation over $\text{Spec}(A)$, such that $C$ is $A$-isomorphic to a closed subscheme of $X = X \times_A A$, resp. of $D = D \times_A A$, and such that $X - C$, resp. $D - C$, are quasicompact. Suppose that the dimensions of all the connected components of $X$, resp. $D$, over $A$ are all equal to the same integer $N$, respectively $M$. Then we find conditions under which we construct canonical isomorphisms:

$$H^{2N-h}(X, X - C, (\Gamma^*_d(X))^+ \otimes Z Q)$$

$$\cong H^{2M-h}(D, D - C, (\Gamma^*_d(D))^+ \otimes Z Q), \quad \text{all integers } h.$$  

(1)

**Lemma 1.** $C$, $X$ and $D$ as above, suppose that there exists an $A$-map $f: X_{\text{red}} \rightarrow D_{\text{red}}$ such that the restriction of $f$ induces the identity isomorphism from $C \subset X_{\text{red}}$ onto $C \subset D_{\text{red}}$. Let $\nabla$ denote the image of $C$ under the mapping: $C \rightarrow X \times_A D$ such that the composites with the first and second projections are the inclusions: $C \rightarrow X$ and $C \rightarrow D$ respectively. Then we have isomorphisms:

$$H^{2M+2N-h}(X \times_A D, X \times_A D - \nabla, (\Gamma^*_d(X \times_A D))^+ \otimes Z Q)$$

$$\cong H^{2N-h}(X, X - C, \Gamma^*_d(X))^+ \otimes Z Q).$$  

(2)
Sketch of proof. Consider the composite mapping:

$$H^{2N-h}(X, X - C) \xrightarrow{(\text{map induced by projection})} H^{2N-h}(X \times_A D, X \times_A D - C \times_A D) \xrightarrow{\text{cupping with } u_{X \times_A D, \Gamma'_f}} H^{2N+2M-h}(X \times_A D, X \times_A D - \mathbb{V}),$$  \hspace{1cm} (3)

where $u_{X \times_A D, \Gamma'_f} \in H^{2M}(X \times_A D, X \times_A D - \Gamma'_f)$ is the canonical class of $\Gamma'_f$ on $X \times_A D$ [2]. This gives an $(\mathcal{A} \uparrow \otimes \mathbb{Z} \mathbb{Q})$-homomorphism between the groups that we wish to prove isomorphic.

By the second Leray spectral sequence of relative hypercohomology [1] we have the first quadrant spectral sequence

$$E_2^{0,0} = H^0(C, H_X^0(X, X - C, (\Gamma'_d(X)\uparrow) \otimes \mathbb{Z} \mathbb{Q})) \Rightarrow H^n(X, X - C, (\Gamma'_d(X)\uparrow) \otimes \mathbb{Z} \mathbb{Q}),$$  \hspace{1cm} (4)

and a similar second Leray spectral sequence of relative hypercohomology, call it (5), abutting at the first groups in Eq. (2). The construction of the composite mapping (3) defines a mapping from the spectral sequence (4) into the spectral sequence (5). Therefore the proof that the composite mapping (3) is an isomorphism becomes a local problem—i.e., it suffices to prove the analogous assertion for any collection of open subsets of $C$ that are an open base for the topology of $C$. But an open base for the topology of $C$ consists of those open subsets $C'$ of $C$, such that there exists an open neighborhood $X'$ of $C'$ in $X$, such that, if $f' = f| X'$, then $\Gamma'_f$ is globally regularly embedded [2] as $\mathcal{O}$-space in $X \times_A D$, and Eq. (1) has been proved in that case in [2].

Remark. The proof of Lemma 1 shows more generally, “Let $C, \bar{X}$ and $\mathcal{D}$ be as above, and suppose that we have an $A$-map $\pi: D_{\text{red}} \rightarrow X_{\text{red}}$ that induces the identity from $C$ onto $C$, and also that there exists a reduced closed subscheme of finite presentation $E$ of $D$ such that, all points of $E$ that are generic in their fiber over $\text{Spec}(A)$ are simple points of $E$ over $\text{Spec}(A_{\text{red}})$, and of codimension $M - N$ on $D_{\text{red}}$, and such that $\pi^{-1}(C) \cap E$ is the closed subset $C$ of $D_{\text{red}}$, the intersection being in general position and transverse regular (as defined in [3, Proposition II.5.2, p. 231]). Then Eq. (1) holds.”

Lemma 2. If there exist $A$-maps: $X_{\text{red}} \rightarrow D_{\text{red}}$ and $D_{\text{red}} \rightarrow X_{\text{red}}$ that both induce the identity mapping of $C$, then one can establish isomorphisms as in Eq. (1).

Proof. By Lemma 1 we have the isomorphisms (2). Lemma 1 with “$D$” and “$\mathcal{X}$” interchanged completes the proof. Q.E.D.

For the moment, Lemma 2 above will suffice for the next set of applications.
Next we define a category which we denote \( \mathcal{C}'_{\mathcal{E},\mathcal{A}} \). The objects of \( \mathcal{C}'_{\mathcal{E},\mathcal{A}} \) are the pairs \((C, \overline{X})\) where \( C \) is a reduced scheme over \( A \) and \( \overline{X} \) is proper and of finite presentation over \( \text{Spec}(A) \), and such that \( C \) is a locally closed sub \( A \)-scheme of \( \overline{X} = \overline{X} \times_{\mathcal{E}} k \), and such that \( C \) is contained in some open subscheme \( X \) of \( \overline{X} \) such that \( X \) is simple of finite presentation over \( A \), such that \( \overline{X} - C \) is quasicompact and such that all the connected components of all the fibers of \( X \) over points of \( \text{Spec}(A) \) are of the same constant dimension \( N \).

**Example.** If \( C \) is simple over \( \text{Spec}(A_{\text{red}}) \), such that there exists \( \mathcal{C}' \) simple of finite presentation over \( \text{Spec}(A) \) such that \( C = (\mathcal{C} \times_{\mathcal{E}} k)_{\text{red}} \), and such that there exists \( \overline{X} \) proper of finite presentation, all the connected components of the fibres of which over \( \text{Spec}(A) \) have the same dimension, such that \( \mathcal{C} \) is an open subscheme of \( \overline{X} \), then \((C, \overline{X}) \in \mathcal{C}'_{\mathcal{E},\mathcal{A}} \). (Therefore the reader will verify that lifted \( p \)-adic homology with compact supports, as defined below, generalizes the hypercohomology of a flat lifting with coefficients in the sheaves of differential forms, when such a lifting exists, of a simple scheme).

Given two such objects \((C, \overline{X})\) and \((D, \overline{Y})\) in \( \mathcal{C}'_{\mathcal{E},\mathcal{A}} \), a map from \((C, \overline{X})\) into \((D, \overline{Y})\) is a pair \((\iota, f)\) where

\[
\iota: C \to D \text{ is a proper map over } A_{\text{red}},
\]

\[
f: X_{\text{red}} \to Y_{\text{red}} \text{ is an } A\text{-map extending } \iota,
\]

where \( X \) is a “sufficiently small” open neighborhood of \( C \) in \( \overline{X} = \overline{X} \times_{\mathcal{E}} k \) and where \( Y = \overline{Y} \times_{\mathcal{E}} k \) (two such “\( f \)”s, defined on different open neighborhoods “\( X \)” of \( C \) in \( \overline{X} \), are considered to be the same if they agree on some smaller neighborhood of \( C \) in \( \overline{X} \)).

**Proposition 3.** For each object \((C, \overline{X}) \in \mathcal{C}'_{\mathcal{E},\mathcal{A}} \), define

\[
H^h_\mathcal{C}'(C, \overline{X}, A_\mathcal{A}^{\dagger} \otimes_\mathbb{Z} \mathbb{Q}) = H^{2N-h}(X, X - C, (\Gamma_\mathcal{A}^*(\overline{X})^{\dagger}) \otimes_\mathbb{Z} \mathbb{Q}),
\]

where \( X \) is some open neighborhood of \( C \) in \( \overline{X} \) containing \( C \) as a closed subset such that \( X \) is simple over \( \text{Spec}(A) \) and such that the dimensions of all the connected components of all fibers of \( X \) over \( \text{Spec}(A) \) are equal to some constant integer \( N \). Assume for simplicity that the ring \( A \) is normal (i.e., is isomorphic to the direct product of finitely many integral domains, each of which is integrally closed in its field of quotients). (We will show later to remove this hypothesis.) Then the assignment:

\[
(C, \overline{X}) \mapsto H^h_\mathcal{C}'(C, \overline{X}, (A_\mathcal{A}^{\dagger}) \otimes_\mathbb{Z} \mathbb{Q})
\]

is in a natural way a covariant functor from the category \( \mathcal{C}'_{\mathcal{E},\mathcal{A}} \) into the category of \((A_\mathcal{A}^{\dagger}) \otimes_\mathbb{Z} \mathbb{Q}\)-modules, all integers \( h \).
**Sketch of proof.** The proof in many ways resembles that of III.1.7, p. 249, of [3]. (That theorem is basically the special case of this proposition in which \( \mathcal{A} = \emptyset \), and we restrict attention to those \((C, \mathcal{X}) \in \mathcal{C}_v \), such that \( C = \mathcal{X} \times k \).)

The key step is, if \((\iota, f): (C, \mathcal{X}) \rightarrow (D, \mathcal{Y})\) is a map in \( \mathcal{C}_v \), then we must construct

\[
H^h_\eta(\iota, f, \mathcal{A} \uparrow \otimes \mathbb{Q}): H^h_\eta(C, \mathcal{X}, (\mathcal{A} \uparrow) \otimes \mathbb{Q}) \rightarrow H^h_\eta(D, \mathcal{Y}, (\mathcal{A} \uparrow) \otimes \mathbb{Q}),
\]

a homomorphism of \( \mathcal{A} \uparrow \otimes \mathbb{Q} \)-modules, all integers \( h \). First, we can easily reduce to the case \( \mathcal{A} \) an integral domain, which we assume. Also, it is easy to reduce to the case in which \( \mathcal{X} \) and \( \mathcal{Y} \) are connected, which we assume. Let \( \mathcal{X} \), respectively: \( \mathcal{Y} \), be an open neighborhood of \( C \) in \( \mathcal{X} \), respectively \( D \) in \( \mathcal{Y} \), such that \( \mathcal{X} \) and \( \mathcal{Y} \) are simple over \( \text{Spec}(\mathcal{A}) \), such that \( C \), respectively \( D \), is closed in \( \mathcal{X} \), respectively \( \mathcal{Y} \), such that there exists an integer \( N \), respectively: \( M \), such that all of the connected components of fibers of \( \mathcal{X} \), respectively: \( \mathcal{Y} \), over \( \text{Spec}(\mathcal{A}) \) are of dimension \( N \), respectively: \( M \). Let \( X = \mathcal{X} \times k \), \( Y = \mathcal{Y} \times k \), \( \mathcal{X} = \mathcal{X} \times k \), \( \mathcal{Y} = \mathcal{Y} \times k \). Replacing \( \mathcal{X} \) by an open subset if necessary, we can assume that the map \( f \) in (6) is defined on \( X \) and maps into \( Y \). Then by Lemma 2 with \( D = \mathcal{X} \times \mathcal{A} \), we have isomorphisms

\[
H^{2N-h}(X, X - C, (\mathcal{G}^*(\mathcal{X})) \uparrow) \otimes \mathbb{Q}) \\
\approx H^{2N+2M-h}(X \times_A Y, X \times_A Y - \Gamma_x, (\mathcal{G}^*(\mathcal{X})) \uparrow) \otimes \mathbb{Q}),
\]

all integers \( h \), (7)

where \( \Gamma_x \) is the graph of the map (5). (The maps, the projection: \( X_{\text{red}} \times_A Y_{\text{red}} \rightarrow X_{\text{red}} \) and \( x \rightarrow (x, f(x)): X_{\text{red}} \rightarrow X_{\text{red}} \times_A Y_{\text{red}} \) fulfill the requirements of Lemma 2.)

Since the map \( \iota: C \rightarrow D \) is a proper map over \( A \) (see (5)), since \( \Gamma_x \) is \( D \)-isomorphic to \( C \) and since \( D \) is closed in \( Y \) it follows that both \( \Gamma_x \) and \( X \times_A Y \) are proper over \( Y \). Therefore \( \Gamma_x \) is closed in \( X \times_A Y \). Therefore we have the excision isomorphism [3, I.6.4, pp. 146–147],

\[
H^{2N+2M-h}(X \times_A Y, X \times_A Y - \Gamma_x, (\mathcal{G}^*(\mathcal{X} \times_A Y)) \uparrow) \otimes \mathbb{Q}) \\
\approx H^{2N+2M-h}(\mathcal{X} \times_A Y, \mathcal{X} \times_A Y - \Gamma_x, (\mathcal{G}^*(\mathcal{X} \times_A Y)) \uparrow) \otimes \mathbb{Q}) \ (8)
\]

all integers \( h \), where

\[
\Gamma_x(\mathcal{X} \times_A Y) = \Gamma_x(\mathcal{X} \times_A Y), \quad j \leq N + M, \\
= 0, \quad j \geq N + M + 1
\]

all integers \( j \geq 0 \). The restriction maps the right side of Eq. (8) into
\[ H^{2N+2M-h}(X \times_A Y, X \times_A Y - X \times_A D, ('T^*_d(X \times_A Y) \pm) \otimes \mathbb{Z} \mathbb{Q}), \]  which we prefer writing as
\[ H^{2N+2M-h}(X \times_A (Y, Y - D), ('T^*_d(X \times_A Y) \pm) \otimes \mathbb{Z} \mathbb{Q}). \]

Since \( X \) is proper of finite presentation over \( A \), there exists a blowing up \( X' \) of \( X \) that is projective over \( \text{Spec}(A) \). Then we have the natural mapping, induced by the morphism: \( X' \rightarrow X \),
\[ H^{2N+2M-h}(X \times_A (Y, Y - D)) \rightarrow H^{2N+2M-h}(X' \times_A (Y, Y - D)), \]
where \( X' = X' \times \mathbb{C} \mathbb{K} \) (and where the coefficients are in \( ('T^*_d(X \times_A Y) \pm) \otimes \mathbb{Z} \mathbb{Q} \) and in \( ('T^*_d(X' \times_A Y) \pm) \otimes \mathbb{Z} \mathbb{Q} \) respectively). Since \( X' \) is projective over \( A \), there exists a finite map (meaning a map of module finite presentation)
\[ \overline{X}' \rightarrow \mathbb{P}^N(A). \]

Since \( \overline{X}' \) is connected and finite over the normal scheme \( \mathbb{P}^N(A) \), there exists \( \overline{X}'' \) finite over \( \overline{X}' \) such that \( \overline{X}'' \) is normal, and such that \( \overline{X}'' \) is Galois over \( \mathbb{P}^N(A) \). Let \( d \) be the degree of the finite covering \( \overline{X}'' \) of \( \overline{X}' \) (thus \( d = [K(\overline{X}''): K(\overline{X}')] \)), and let \( G \) be the Galois group of \( \overline{X}'' \) over \( \mathbb{P}^N(A) \). Then we have the natural mapping from the group on the right of Eq. (10) into
\[ H^{2N+2M-h}(\overline{X}'' \times_A (Y, Y - D)) \]
where \( \overline{X}'' = \overline{X}'' \times \mathbb{C} \mathbb{K} \), and the coefficients are in \( ('T^*_d(\overline{X}'' \times_A Y) \pm) \otimes \mathbb{Z} \mathbb{Q} \).

Let
\[ H^{2N+2M-h}(\overline{X}' \times_A (Y, Y - D)) \]
\[
\frac{1}{d} \text{times natural mapping} \rightarrow H^{2N+2M-h}(\overline{X}'' \times_A (Y, Y - D)),
\]
denote \( 1/d \) times the natural mapping, all integers \( h \). Since \( \overline{X}'' \) is finite, and therefore affine, over \( \mathbb{P}^N(A) \), by Corollary II.3.1.2, p. 191, of [3] we have the isomorphisms:
\[ H^{2N+2M-h}(\overline{X}'' \times_A (Y, Y - D), ('T^*_d(\overline{X}'' \times_A Y) \pm) \otimes \mathbb{Z} \mathbb{Q}) \]
\[ \approx H^{2N+2M-h}(\mathbb{P}^N(A) \times_A (Y, Y - D), \]
\[ (\pi \times_A \text{id}_Y)_* (('T^*_d(\overline{X}'' \times_A Y) \pm) \otimes \mathbb{Z} \mathbb{Q})), \]
where \( \pi: \overline{X}'' \rightarrow \mathbb{P}^N(A) \) is the morphism. Since \( \mathbb{P}^N(A) \) is normal and since \( \overline{X}'' \) is a Galois finite covering of \( \mathbb{P}^N(A) \) with Galois group \( G \), it is easy to see, by arguments similar to those in [3, proof of Theorem II.4.5, pp. 209–210], that
\[ \pi_*(\mathcal{O}_{\overline{X}''})^G = \mathcal{O}_{\mathbb{P}^N(A)}. \]
It follows (by methods not unlike those in [3, proof of II.4.5]) that

\[ ((\pi \times_d \text{id}_Y)_* (\Gamma^*_d(\mathbb{X}^* \times_d Y)\uparrow) \otimes \mathbb{Z} \mathbb{Q})^G \approx (\Gamma^*_d(\mathbb{P}^N(A))\uparrow) \otimes \mathbb{Z} \mathbb{Q}. \]  

(13)

The assignment: \( \alpha \mapsto \sum_{g\in G} \alpha \cdot g \) in each stalk defines a mapping of cochain complexes of sheaves of \((\mathbb{A} \uparrow) \otimes \mathbb{Z} \mathbb{Q}\) modules over \(\mathbb{P}^N(A)\) from

\[ (\pi \times_d \text{id}_Y)_* ((\Gamma^*_d(\mathbb{X}^* \times_d Y)\uparrow) \otimes \mathbb{Z} \mathbb{Q}) \]

into

\[ ((\pi \times_d \text{id}_Y)_* ((\Gamma^*_d(\mathbb{X}^* \times_d Y)\uparrow) \otimes \mathbb{Z} \mathbb{Q}))^G. \]

Combining this observation with Eq. (13) we obtain the transfer mapping from the right side of Eq. (12) into

\[ H^{N+2h-h}(\mathbb{P}^N(A) \times_A (Y, Y - D), \Gamma^*_d(\mathbb{P}^N(A) \times_A Y)\uparrow \otimes \mathbb{Z} \mathbb{Q}). \]  

(14)

Before completing the indication of proof of Proposition 3, we digress to prove a lemma and corollary.

One can deduce from [3, Corollary II.4.4.1, p. 206] (or rather, from its generalization with "\(\mathbb{A}\)" replacing "\(\mathcal{O}\)" given in [2]), the following lemma.

**Lemma 4.** If \( Y \) is an \( \mathcal{O}\)-space of finite presentation over \( A \), if \( D \) is a closed subset of \( Y \) and if \( N \) is any non-negative integer then there are induced canonical isomorphisms,

\[ H^h(\mathbb{P}^N(A) \times_A (Y, Y - D), \Gamma^*_d(\mathbb{P}^N(A) \times_A Y)\uparrow \otimes \mathbb{Z} \mathbb{Q}) \]

\[ \approx \bigoplus_{i=0}^N H^{h-2i}(Y, Y - D, (\Gamma^*_d(Y)\uparrow) \otimes \mathbb{Z} \mathbb{Q}). \]  

(15)

**Proof.** We first define a natural mapping from the right side of Eq. (15) into the left side. We must build mappings:

\[ H^{h-2i}(Y, Y - D) \rightarrow H^h(\mathbb{P}^N(A) \times_A (Y, Y - D)), \quad 0 \leq i \leq N, \quad h \geq 0. \]

In fact, take the composite:

\[ H^{h-2i}(Y, Y - D) \xrightarrow{p^*_1} H^{h-2i}(\mathbb{P}^N(A) \times_A (Y, Y - D)) \xrightarrow{\text{cupping with}} H^h(\mathbb{P}^N(A) \times_A (Y, Y - D)), \]

where \( p^*_1 \) and \( p^*_2 \) are induced by projections and where \( u_i \in H^{2i}(\mathbb{P}^N(A), \Gamma^*_d(\mathbb{P}^N(A))\uparrow \otimes \mathbb{Z} \mathbb{Q}) \) is the cohomology class of any linear subspace of \(\mathbb{P}^N(A)\) of dimension \( N - i \) (i.e., is the image of \( u_{\mathbb{P}^N(A),\mathbb{P}^N(A)-i}(A) \in H^{2i}(\mathbb{P}^N(A), \mathbb{P}^N(A) \rightarrow \mathbb{P}^N(A)-i(A)) \)). This defines a canonical mapping as indicated. Con-
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Considering the cohomology sequences of the pairs $(Y, Y - D)$ and $(\mathbb{P}^n(A) \times_A (Y, Y - D))$ [3, Theorem I.6.3, pp. 140–141] and the Five Lemma, the proof of Lemma 4 reduces to the case $D = \emptyset$. We have

$$A^N(A) = \mathbb{P}^n(A) - \mathbb{P}^{n-1}(A) \subset \mathbb{P}^n(A)$$

$$- \mathbb{P}^{n-2}(A) \subset \mathbb{P}^n(A) - \mathbb{P}^{n-3}(A) \subset \cdots,$$

where "$\mathbb{P}^i(A)$" denotes the closed subset $(T_0 = T_1 = \cdots = T_{i-1} = 0)$ of $\mathbb{P}^n(A)$ and $A^N(A)$ is affine $N$-space over $A$. Let $\mathbb{P}^i$ stand for $\mathbb{P}^i(A)$. Then excising the closed subset "$T_i = 0$" we see that the restriction is an isomorphism

$$H^k((\mathbb{P}^n - \mathbb{P}^{n-(i+1)}, \mathbb{P}^n - \mathbb{P}^{n-i}) \times_A Y)$$

$$\approx H^k((A^N, A^N - (T_0 = \cdots = T_{i-1} = 0)) \times_A Y),$$

where $A^N$ denotes the affine open subset, "$T_i = 0$" of $\mathbb{P}^n(A)$, so that $A^N \approx \text{Spec}(A[T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_N])$, $0 \leq i \leq N$. The cohomology sequence of a triple [3, p. 141, Note following Theorem I.6.3, Eq. (*), and iteration of Eq. (16), together with Eq. (17) and [3, Corollary II.4.4.1, p. 206], (as generalized in [2] with "$A$" replacing "$C$") completes the proof of Lemma 4.

**Corollary 4.1.** The hypotheses being as in Lemma 4, there is induced a natural mapping

$$H^k(\mathbb{P}^n(A) \times_A (Y, Y - D)) \to H^{k-2N}(Y, Y - D).$$

Completion of the sketch of proof of Proposition 3. Define $H^h_{\#}(\iota, f, (A^+) \otimes \mathbb{Q})$ to be the composite of: the isomorphism (7), followed by the isomorphism (8), the restriction mapping into (9), the natural mapping (10), followed by the mapping (11), the isomorphism (12), the transfer mapping (14) and the mapping (18). It remains to show that (i) if $(\iota, f): (C, \bar{X}) \to (D, \bar{Y})$ and $(j, g): (D, \bar{Y}) \to (E, \bar{Z})$ are maps in $\mathcal{E}_{\mathfrak{e}, A}$, then $H^h_{\#}(j, g \circ H^h_{\#}(\iota, f)) = H^h_{\#}(j \circ (g \circ f))$, all integers $h$, and (ii) if $(\text{id}_C, \text{id}_\bar{X})$ is the identity of $(C, \bar{X})$ in $\mathcal{E}_{\mathfrak{e}, A}$ then $H^h_{\#}(\text{id}_C, \text{id}_\bar{X})$ is the identity of $H^h_{\#}(C, \bar{X}, (A^+) \otimes \mathbb{Q})$, all integers $h$.

(To prove Eq. (i), one first considers the case in which $\bar{X}$ and $\bar{Y}$ are projective spaces over $A$, and argues similarly to the proof of Theorem III.1.5 in [3, pp. 244–246]. (The proof is in some ways easier, since one does not have to argue with Poincaré duality.) To prove Eq. (ii), one notes that it suffices to show that $H^h_{\#}(\text{id}_C, \text{id}_\bar{X})$ is an isomorphism for all integers $h$. And, by using the second Leray spectral sequence of relative hypercohomology [1] this problem becomes local in $C$. By such techniques, one reduces to the case: $C$ an affine scheme, $\bar{X} = a$ projective space over $A$, and then an explicit computation is made.)
COROLLARY 3.1. The same hypotheses as Proposition 3, but delete the hypothesis that "A is normal." Let $\mathcal{C}_{\nu, A}$ be the full subcategory of $\mathcal{C}_{\nu, A}$ generated by those pairs $(C, X) \in \mathcal{C}_{\nu, A}$ such that $X \cong \mathbb{P}^n(A)$, there exists an integer $N \geq 0$. Then the assignment, $(C, X) \mapsto H^n_t(C, X, (A^+) \otimes \mathbb{Z} \mathbb{Q})$, $h \geq 0$, is a functor on the category $\mathcal{C}_{\nu, A}$.

Proof. Similar to (but easier than) Proposition 3; simply delete Eqs. (10)–(14) from the proof of Proposition 3.

Remark. For a better generalization of Proposition 3 beyond the normal case, see Remark 2 following Proposition 7 below.

PROPOSITION 5. The hypotheses being as in Proposition 3, let $(C, X)$ and $(C, X')$ be two objects in the category $\mathcal{C}_{\nu, A}$ having the same $C$. Then there are induced canonical isomorphisms of $(A^+) \otimes \mathbb{Z} \mathbb{Q}$-modules

$$H^n_t(C, X, (A^+) \otimes \mathbb{Z} \mathbb{Q}) \cong H^n_t(C, X', (A^+) \otimes \mathbb{Z} \mathbb{Q}),$$

all integers $h$.

Sketch of proof. We have maps in the category $\mathcal{C}_{\nu, A}$,

$$(id_C, \pi_1): (C, X \times_A X') \to (C, X)$$

and

$$(id_C, \pi_2): (C, X \times_A X') \to (C, X'),$$

where $\pi_1$ and $\pi_2$ are the canonical projections. Therefore we need only show that both of these maps induce isomorphisms on lifted $p$-adic homology with compact supports, e.g., the first of these. Let $X$ and $X'$ be as in the proof of Proposition 3, and $X = X \times_k k$, $X' = X' \times_k k$. Using second Leray spectral sequences of relative hypercohomology [1], we have the first quadrant cohomological spectral sequence

$$E_2^{p,q} = H^p(C, H^q(X, X - C, (\Gamma^+_A(X)) \otimes \mathbb{Z} \mathbb{Q}))$$

$$\Rightarrow H^n(X, X - C, (\Gamma^+_A(X)) \otimes \mathbb{Z} \mathbb{Q})$$

and a similar spectral sequence abutting at $H^n(X \times_A X', X \times_A X' - C, (\Gamma^+_A(X \times_A X')) \otimes \mathbb{Z} \mathbb{Q})$. Therefore, tracing the proof of Proposition 3 and of Lemma 1 (notice that all the maps constructed generalize to the cochain level (as defined in [3, Chap. I]), and therefore induce maps of spectral sequences), this problem is local in $C$. Let $N$, respectively: $N'$, be the dimension of the fibers of $X$, respectively: $X'$, over $A$.

Case 1. $N = N'$. Then if we replace $C$ by a small enough open subset, then there exists $E$ over $A$ such that $C$ is $A$-isomorphic to a closed subset of
\( E = E \times_c k \) and such that there exist étale \( A \)-mappings of finite presentation: \( E \to X \) and \( E \to X' \) both of which induce the identity on \( C \). But then, by the first cohomology theorem in [2], we have that these mappings induce isomorphisms,

\[
H^{2N-h}(X, X - C, (\Gamma^*_A(X)\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong H^{2N-h}(E, E - C, (\Gamma^*_A(E)\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}),
\]

\[
H^{2N-h}(X', X' - C, (\Gamma^*_A(X')\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}) \cong H^{2N-h}(E, E - C, (\Gamma^*_A(E)\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}),
\]

all integers \( h \). (And it is easy to see that the isomorphisms thus obtained between \( H^{2N-h}(X, X - C) \) and \( H^{2N-h}(X', X' - C) \), \( h \in \mathbb{Z} \), coincide with the ones indicated above.)

**Case II.** \( N \neq N' \), say \( N' > N \). Let \( D = \overline{X} \times_A \mathbb{P}^{N'-N}(A) \) and let \( C \to X \times_A \mathbb{P}^{N'-N}(A) \) be the map whose first coordinate is the inclusion: \( C \to X \) and whose second coordinate is the composite of the structure map: \( C \to \text{Spec}(A) \) with the map \( \text{Spec}(A) \to \mathbb{P}^{N'-N}(A) \) whose coordinates are the identity of \( X \) and the composite \( \overline{X} \to \text{Spec}(A) \) (the section \( (T_1 = \cdots = T_{N'-N} = 0) \) of \( \mathbb{P}^{N'-N}(A) \)) both of which induce the identity on \( C \), whence by Lemma 2,

\[
H^{2N-h}(X, X - C) \cong H^{2N-h}(X \times_A \mathbb{P}^{N'-N}(A), \mathbb{P}^{N'-N}(A) - C),
\]

all integers \( h \). But by Case I, with \( \overline{X} \times_A \mathbb{P}^{N'-N}(A) \) replacing \( \overline{X} \), we have that

\[
H^{2N-h}(X \times_A \mathbb{P}^{N'-N}(A), \mathbb{P}^{N'-N}(A) - C) \cong H^{2N-h}(X', X' - C),
\]

all integers \( h \).

**Definition 1.** Let \( A \) be a ring and let \( A \) be a quotient ring of \( A \). A scheme \( C \) over \( A \) is properly embeddable over \( A \) if there exists \( \overline{X} \) proper of finite presentation over \( A \) such that \( C \) is \( A \)-isomorphic to a locally closed subscheme of \( \overline{X} \to \overline{X} \times_A A \), such that \( \overline{X} - C \) is quasicompact and such that \( C \) is contained in the set of simple points of \( \overline{X} \) over \( A \).

**Remark 1.** When this is the case, it is easy to see that \( \overline{X} \) can be taken so that there exists \( X \) open in \( \overline{X} \) and simple of finite presentation over \( A \) such that \( C \) is closed in \( X \) and such that the connected components of fibers of \( X \) over \( \text{Spec}(A) \) all have the same dimension.

**Remark 2.** By a well-known theorem of Nagata [4], it is easy to show that, \( C \) is properly embeddable over \( A \) iff \( C \) is \( A \)-isomorphic to a closed subscheme of a scheme \( X \) that is separated and simple of finite presentation over \( \text{Spec}(A) \), and such that \( X - C \) is quasicompact where \( X = \overline{X} \times_c k \).
Definition 1 is relative to $\mathcal{A}$ as well as $A$. Definition 2 below is independent of $\mathcal{A}$, depends on the $\mathcal{O}$-algebra structure of $A$, and is more restrictive.

**Definition 2.** Let $\mathcal{O}$ be a ring, let $A$ be an $\mathcal{O}$-algebra and let $C$ be a scheme over $\text{Spec}(A)$. Fix a polynomial algebra $P$ over $\mathcal{O}$ (in possibly infinitely many variables. That is, $P \cong \mathcal{O}[(T_i)]_{i \in I}$) as $\mathcal{O}$-algebras, there exists some set $I$ and an epimorphism of $\mathcal{O}$-algebras: $P \to A$. Then we say that $C$ is *polynomially properly embeddable* over $A$ iff $C$ is properly embeddable over $P$ in the sense of Definition 1. (It is easy to see that this definition is independent of the choice of such a $P$ and of an epimorphism: $P \to A$, since polynomial algebras are projective objects in the category of $\mathcal{O}$-algebras.)

**Example 1.** If $C$ is *quasiprojective* over $A$, then $C$ is properly embeddable over $\mathcal{A}$ for $\mathcal{A}$ as in Definition 1 (take $\mathfrak{X} = a$ suitable $\mathbb{P}^n(A)$) (and therefore $C$ is also polynomially properly embeddable over $A$).

**Example 2.** If $C$ is *simple* over $A_{\text{red}}$ and *liftable* over $\mathcal{A}$, in the sense that there exists $\mathfrak{X}$ separated and simple of finite presentation over $\mathcal{A}$ such that $X \cong X \times_A A$, then $C$ is properly embeddable over $A$.

**Example 3.** Not every algebraic variety over a field is properly embeddable over that field—for a counterexample, see G. Horrocks, Birationally ruled surfaces without embeddings in regular schemes, *Bull. London Math. Soc.* 3 (1971), 57–60.

The main theorem is Theorem 6. We return to the notation $\mathcal{O}$, $K$, $k$, $A$, $\mathcal{A}$ as at the beginning of the section.

**Theorem 6.** Suppose that the ring $A$ is normal (respectively: Make no additional hypothesis on $A$). Then let $\mathcal{C}_{\mathcal{O}, \mathcal{A}}$ be the category having for objects all schemes $C$ over $\text{Spec}(A)$ that are properly embeddable over $A$ (respectively: that are polynomially properly embeddable over $A$), and for maps all proper $A$-maps. Then we have functors, “lifted $p$-adic homology with compact supports”

$$C \rightsquigarrow H^e_h(C, (A^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}), \quad \text{all integers } h,$$

from the category $\mathcal{C}_{\mathcal{O}, \mathcal{A}}$ into the category of $(A^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}$-modules.

**Sketch of proof.** (We sketch the case in which $\mathcal{A}$ is normal. The other case is covered in Remark 2 following Proposition 7 below.) For every object $C \in \mathcal{C}_{\mathcal{O}, \mathcal{A}}$ by definition there exists $\mathfrak{X}$ such that $(C, \mathfrak{X}) \in \mathcal{C}_{\mathcal{O}, \mathcal{A}}$. Then by Proposition 3 we have

$$H^e_h(C, (\mathfrak{X}, (A^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}), \quad \text{all integers } h.$$
LIFTED p-ADIC HOMOLOGY WITH COMPACT SUPPORTS

But by Proposition 5 these groups are independent, up to canonical isomorphisms, of $X$. So define

$$H^h_c(C, (A^+) \otimes \mathbb{Q}) = H^h_c(C, X, (A^+) \otimes \mathbb{Z}) \otimes \mathbb{Q}.$$  \hfill (19)$$

for any such $X$, all integers $h$. Propositions 3 and 5 prove that $H^h_c(C, (A^+) \otimes \mathbb{Z})$ is a functor on the category $\mathcal{C}_{\mathcal{O}, \mathcal{D}}$, all integers $h$.

**Proposition 7.** Let $C \in \mathcal{C}_{\mathcal{O}, \mathcal{D}}$ and let $D$ be a reduced closed $A$-subscheme of $C$ such that $D \in \mathcal{C}_{\mathcal{O}, \mathcal{D}}$ (i.e., such that $C - D$ is quasicompact). Let $U$ be the open subset $C - D$ of $C$. Then there is induced a homomorphism of $(A^+) \otimes \mathbb{Z}$-modules, which we call the restriction,

$$H^h_c(C, (A^+) \otimes \mathbb{Z}) \rightarrow H^h_c(U, (A^+) \otimes \mathbb{Z}),$$

all integers $h$. Moreover, we have a long exact sequence:

$$\cdots \rightarrow H^h_c(D, (A^+) \otimes \mathbb{Z}) \xrightarrow{H^h_c(\iota)} H^h_c(C, (A^+) \otimes \mathbb{Z}) \rightarrow H^h_c(U, (A^+) \otimes \mathbb{Z}) \rightarrow \cdots,$$  \hfill (20)$$

where $\iota$ is the inclusion: $D \rightarrow C$.

**Proof.** Choose $X$ such that $(C, X) \in \mathcal{C}_{\mathcal{O}, \mathcal{D}}$, and let $X$ be as in the proof of Proposition 3. Then

$$H^h_c(D, (A^+) \otimes \mathbb{Z}) = H^{2N-h}(X, X - D),$$

$$H^h_c(C, (A^+) \otimes \mathbb{Z}) = H^{2N-h}(X, X - C)$$

and $H^h_c(U, (A^+) \otimes \mathbb{Z}) = H^{2N-h}(X - D, X - C)$, all integers $h$. Therefore the indicated long exact homology sequence is the cohomology sequence of the triple $[3, 1.6, \text{Note, pp. 141-142}] (X, X - D, X - C)$ with coefficients in $(\mathcal{P}^h_{\mathcal{D}}(X)^+) \otimes \mathbb{Z}$.

**Remark 1.** The hypotheses being as in Theorem 6, let $M$ be any module over the ring $(A^+) \otimes \mathbb{Z}$. Then for every $C \in \mathcal{C}_{\mathcal{O}, \mathcal{D}}$ we define the lifted $p$-adic homology with compact supports of $C$ with coefficients in $M$ as follows. Fix any $X$ such that $(C, X) \in \mathcal{C}_{\mathcal{O}, \mathcal{D}}$, and let $X$ be as in the proof of Proposition 3. Let $U$ be a finite set of affine open subsets of $X$ that is a covering (in the sense of $[3, 1.5, \text{p. 127}]$), and that is a refinement of the covering $\{X, X - C\}$ (in the sense of $[3, 1.5, \text{p. 127}]$), and define

$$C_h = C^{2N-h}(U, (X, X - C), (\mathcal{P}^h_{\mathcal{D}}(X)^+) \otimes \mathbb{Z}).$$

all integers $h$ (where $C^*(U, X, X - C)$ is as defined in $[3, 1.6, \text{p. 144}]$).
Then $C_\gamma$ is a chain complex of $((A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-modules such that $C_h$ is flat over $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}$, all integers $h$. Define, for every $((A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-module $M$,

$$H_h^e(C, M) = H_h(C_\gamma \otimes_{(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}} M).$$

(21)

Then by the definition in Theorem 6 and by [3, Theorem I.6.7, p. 152], in the special case $M = (A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}$ this coincides with the lifted $p$-adic homology with compact supports of $C$ with coefficients in $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}$. In general, we call the $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}$-modules defined in (21) the lifted $p$-adic homology groups with compact supports of $C$ with coefficients in the $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-module $M$. Then, since each $C_h$ is flat over $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}$, we have the universal coefficients spectral sequence (see [5, Chap. V, shortly after the definition of "percohomology"]), a homological spectral sequence confined to the region: $p > 0$, $2N - M_0 < q < 2N$, where $M_0$ is an integer such that $H^i(X, X - C, (\mathcal{P}_a^\circ(X) \dagger) \otimes \mathbb{Z} \mathbb{Q}) = 0$, all integers $i > M_0 + 1$,

$$\text{Tor}_{p}^{(A^\dagger) \otimes \mathbb{Z} \mathbb{Q}}(H^n_e(C, (A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}), M) \Rightarrow H^n_e(C, M).$$

(22)

The hypotheses being as in Theorem 6, for every $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-module $M$ we also define the lifted $p$-adic cohomology with compact supports of $C$ with coefficients in $M$ as follows. Let $C_\gamma$ be the chain complex constructed in the last paragraph, and let $D_\gamma$ be a chain complex such that $D_h$ is projective as $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-module, all integers $h$, and such that we have a mapping $\phi_\gamma : D_\gamma \rightarrow C_\gamma$ of chain complexes that induces an isomorphism on homology in all dimensions. Let $D^* = \text{Hom}_{(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}}(D^*, (A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$, a cochain complex of $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-modules. Let $'D^*$ be a cochain complex of $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-modules, indexed by all the integers, and $\tau^*: 'D^* \rightarrow D^*$ a mapping such that $'D^h$ is flat as $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-module, and $H^h(\tau^*)$ is an isomorphism, all integers $h$. Then for every $(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q})$-module $M$, define

$$H^h_e(C, M) = H^h('D^* \otimes_{(A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}} M),$$

(23)

all integers $h$.

In general, if $R$ is a ring and if $N$ and $M$ are left $R$-modules, then let $P_\gamma$ be an acyclic projective resolution of $N$, and let $P^* = \text{Hom}_R(P_\gamma, R)$, a non-negative cochain complex of right $R$-modules. Let $'P^*$ be a cochain complex (indexed by all the integers) of right $R$-modules, and $\rho^*: 'P^* \rightarrow P^*$ a mapping such that $'P^h$ is flat as right $R$-module, and $H^h(\rho^*)$ is an isomorphism, all integers $h$.

Then define

$$\delta_R^h(N, M) = H^h('P^* \otimes_R M),$$

(24)

all integers $h$. Then, in the terminology of [5, Chap. V], $\delta_R^h(N, M)$ is the $h$th percohomology group of $H^h_R(P^*, M)$ of $P^*$ with coefficients in $M$; and
therefore, if $F_*$ is any acyclic flat resolution of $M$, and if $F^i = F_{-i}$, all integers $i$, then

$$\delta^h_R(N, M) \approx H^h(P^* \otimes_R F^*)$$

all integers $h$. Then $\delta^h_R$ is an additive functor of two variables from the category of left $R$-modules into abelian groups, contravariant in the first variable and covariant in the second variable, and preserving direct limits over directed sets in the second variable, such that $\delta^h_R(N, M) \rightarrow \text{Ext}^h_R(N, M)$ for all left $R$-modules $M$ such that there exists an acyclic resolution of $M$ by finitely generated projective left $R$-modules.* (This latter follows from Eq. (24.1) and the fact that $\text{Hom}_R( , R) \otimes_R F^i \approx \text{Hom}_R( , F^i)$ if $F^i$ is finitely generated projective). Also, from Eq. (24.1), for all left $R$-modules $N$ and $M$, we have the second quadrant cohomological spectral sequence

$$\text{Tor}^R_{-\text{ad}}(\text{Ext}^q_R(N, R), M) \Rightarrow \delta^{-q}_{R}^*(N, M).$$

Then from definitions (23) and (24), we deduce the universal coefficients spectral sequence, a cohomological spectral sequence confined to the horiz. strip $2N - M_0 \leq q \leq 2N$, where $M_0$ is an integer such that $H^h(X, X - C, (\Omega_1^*(X)) \otimes \mathbb{Q}) = 0$, all integers $i \geq M_0 + 1$, and where $N$ is the dimension of the fibers of $X$ over $\text{Spec}(A)$,

$$E_2^{p,q} = \delta^p_{\text{ad}} \otimes \mathbb{Q}(H^q_{\text{ad}}(C, (\mathcal{A}) \otimes \mathbb{Q}), M) \Rightarrow H^{p,q}(C, M).$$

(In the spectral sequence (25), if $M$ admits an acyclic resolution by fin. gen. projectives, then "$\delta^p_{\text{ad}} \otimes \mathbb{Q}$" can be replaced by "$\text{Ext}^p_{\text{ad}} \otimes \mathbb{Q}$"). Using these universal coefficients spectral sequences (22) and (25), the proofs of Lemma 1 and of Propositions 3 and 5 show that definitions (21) and (23) are independent of all choices (i.e., of $X$ and $U$, and in the case of definition (23), also of $D_{+}$, $D^{*}$, $\phi_{+}$ and $\tau^{*}$), and are functors on the category $\mathcal{C}_{e, \text{ad}}$, all integers $h$, all $(\mathcal{A}) \otimes \mathbb{Q}$-modules $M$ (the homology being covariant and the cohomology contravariant). And of course also the long exact sequence (20) of Proposition 7 goes through with "$M$" replacing "$(\mathcal{A}) \otimes \mathbb{Q}$" and similarly for the analogous long exact sequence for lifted $p$-adic cohomology with compact supports with coefficients in any fixed $(\mathcal{A} \otimes \mathbb{Q})$-module $M$. Also, from Definitions (21) and (23), it follows that, if the hypotheses are as in Theorem 6, and if $C$ is any fixed object in the category $\mathcal{C}_{e, \text{ad}}$, then the assignment: $M \rightsquigarrow H^h_{\text{ad}}(C, M), h \in \mathbb{Z}$, respectively: $M \rightsquigarrow H^h_{\text{ad}}(C, M), h \in \mathbb{Z}$, is a homological,

* A ring $R$ is left coherent iff every finitely generated left ideal is finitely presented. (For example, if $A$ is a finitely generated $\mathbb{Z}$-algebra, then $A^{\mathbb{Z}}$ and $(A^{\mathbb{Z}}) \otimes \mathbb{Q}$ are both Noetherian, and therefore also left coherent.) If the ring $R$ is left coherent, then this condition on $M$ is equivalent to: "$M$ is of finite presentation as left $R$-module." It follows readily that, if the ring $R$ is left coherent, then $\delta^h_R = 0$ for $h \leq 0$ (since then these functors commute with direct limits over directed sets in the second variable, and vanish when $M$ is finitely presented). In fact, if $R$ is a ring, then $R$ is left coherent iff $\delta^h_R = 0$ iff for all sets $I$, $R^I$ is flat as left $R$-module; see [5, Introduction].
respectively: cohomological, exact connected sequence of functors from the category of \((\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\)-modules into the category of \((\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\)-modules, and also that both of these connected sequences of functors preserve direct limits over directed sets.

Also, if the hypotheses are as in Theorem 6, then we also have a universal coefficients spectral sequence for lifted \(p\)-adic cohomology with compact supports. Namely, then for every \((\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\)-module \(M\) and every object \(C\) in \(\mathfrak{F}_{e,d}\), we have the cohomological spectral sequence

\[
E_2^{p,q} = \text{Tor}_{p+q}^{\mathbb{Z}}\left(H^p_c(C, (\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}), M\right) \Rightarrow H_*^c(C, M).
\]

This cohomological spectral sequence is confined to the region \(p \leq 0, q \geq 2N - M_0\), where \(M_0\) is an integer such that \(H^i(X, X - C, (\mathcal{I}_d^c(X))^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}) = 0\) for \(i \geq M_0 + 1\), and where \(N\) is the dimension of the fibers of \(X\) over \(\text{Spec} (\mathcal{A})\) (a region that resembles the second quadrant). The filtration on the abutment of (26) is a discrete filtration (i.e. the zero-submodule is a filtered piece of \(H_*^c(C, M)\), for each integer \(n\), and also for each integer \(n\), the union of the filtered pieces of \(H_*^c(C, M)\) is all of \(H_*^c(C, M)\).

Remark 2. We now prove the parenthetical case of Theorem 6 (and also Proposition 7). Let \(P\) be any polynomial algebra over \(\mathcal{C}\) such that \(\mathcal{A}\) is isomorphic to a quotient \(\mathcal{C}\)-algebra of \(P\), and fix any \(\mathcal{C}\)-homomorphism: \(P \to \mathcal{A}\) lifting the epimorphism: \(P \to \mathcal{A}\). Let \(P = P \otimes_{\mathcal{C}} k\). Then \(P\) being a polynomial algebra is normal, and therefore \(P\) and \(\mathcal{A}\) obey the hypotheses of the non-parenthetical case of Theorem 6, so we have the functors on the category \(\mathfrak{F}_{e,P}: C \leadsto H^*_c(C, P^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})\), and by Remark 1 above even the functors: \(C \leadsto H^*_c(C, M)\), all integers \(h\), all \((P^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\)-modules \(M\), on the category \(\mathfrak{F}_{e,P}\). But the category \(\mathfrak{F}_{e,d}\) (with polynomially properly embeddable objects) is a full subcategory of \(\mathfrak{F}_{e,P}\) (with properly embeddable objects over \(P\)). If now \(M\) is any \((\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\)-module, then regarding \(M\) as a \((P^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\)-module, the restriction of the functor \(C \leadsto H^*_c(C, M)\) to the full subcategory \(\mathfrak{F}_{e,d}\) of \(\mathfrak{F}_{e,P}\) proves Remark 1 (for the parenthetical hypotheses of Theorem 6). And the special case \(M = (\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\) proves the parenthetical part of Theorem 6. Similarly for Proposition 7.

Q.E.D.

Remark 3. Let \(D \in \mathfrak{F}_{e,d}\) and let \(d\) be the largest dimension of fibers of \(D\) over \(\mathcal{A}\). Then by theorems in [2], \(H^h_c(D, (\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}) = 0\) for \(h \geq 2d + 1\). By the universal coefficients spectral sequence (Remark 1 above, Eq. (22)), it follows that \(H^h_c(D, M) = 0\) for all flat \((\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}\)-modules \(M\), all integers \(h \geq 2d + 1\).

Remark 4. If we take \(\mathcal{A} = \emptyset\), and if \(D \in \mathfrak{F}_{e,e}\), then it is easy to see that \(H^h_e(D, K) = 0\), all integers \(h\) such that either \(h < 0\) or \(h > 2d, d = \dim D\). However,
Example. Let $A$ be an $\mathcal{O}$-algebra that is simple over $\mathcal{O}$ and that is a local ring such that $A$ is of dimension $n \geq 1$ and let $D$ be the closed point of $\text{Spec}(A)$. Then $D \in \mathcal{O} \mathcal{O}_A$ and $H_c^n(D, (A^\dagger) \otimes_\mathbb{Z} \mathbb{Q}) \approx H^n(\text{Spec}(A), \text{Spec}(A) - D, \mathcal{O}_{\text{Spec}(A)^\dagger} \otimes_\mathbb{Z} \mathbb{Q}) \neq 0$. Therefore negative homology groups with compact supports need not always vanish if $A \neq \mathcal{O}$.

Remark 5. A very special case of the considerations of this section, is the case in which $A = \mathcal{O} = \mathbb{C}$, the field of complex numbers. Then for every complex algebraic variety $C$ that is properly embeddable over $\mathbb{C}$, and every complex vector space $M$, we have defined $H^c_\mathbb{C}(C, M)$ and $H^c_\mathbb{C}(C, M)$, complex vector spaces, defined for all integers $h$, which vanish unless $0 \leq h \leq 2n$, $n = \dim C$. It is easy to see, going back to the definitions in Proposition 3 and Theorem 6 and in Eqs. (21) and (23), that, if $C_{\text{top}}$ denotes the set of closed points of $C$ with the classical topology, then there are induced canonical isomorphisms

$$H^c_\mathbb{C}(C, M) \approx H^c_\mathbb{C}(C_{\text{top}}, M), \quad \text{all integers } h \geq 0,$$

$$H^c_\mathbb{C}(C, M) \approx H^c_\mathbb{C}(C_{\text{top}}, M), \quad \text{all integers } h \geq 0,$$

where the complex vector spaces on the right are the usual singular homology with compact supports and singular cohomology with compact supports. (These, the reader will recall, are defined as follows: If $C$ is any locally compact, Hausdorff, paracompact topological space, and if $M$ is any abelian group, then define

$$H^c_\mathbb{C}(C, M) = H^c(C, \mathbb{C} - C, M),$$

$$H^c_\mathbb{C}(C, M) = H^c(C, \mathbb{C} - C, M),$$

for all integers $h \geq 0$, where $\overline{C}$ is any compactification of $X$, and the groups on the right are the usual relative singular homology and cohomology groups.) Therefore, in the only case in which the usual singular homology and cohomology with compact supports, and our lifted theory as defined in this section, both make sense—namely, in the case $A = \mathcal{O} = \mathbb{C}$—then ours and the usual singular theories coincide.

2. Finite Generation of Lifted $p$-Adic Homology with Compact Supports

Let $\mathcal{O}$ be a complete discrete valuation ring having a quotient field of characteristic zero, and with residue class field $k$. Let $C$ be an algebraic variety over $k$ that is properly embeddable (see Section 1) over $\mathcal{O}$ (e.g., it suffices that $C$ be quasiprojective), and let $K$ be the quotient field of $\mathcal{O}$. Then in this section we prove that the lifted $p$-adic homology with compact supports, $H^c_\mathcal{O}(C, K)$, as defined in Section 1, is finite dimensional over $K$, all
integers \( h \). In consequence if \( C \) is simple over \( k \) and embeddable \([6]\) over \( \mathcal{O} \), then the lifted \( p \)-adic cohomology of \( C \), \( H^h(C, K) \), as defined in \([6]\), is finite dimensional over \( K \), all integers \( h \). (Therefore if \( C \) is simple of finite type over \( k \), and if \( C \) should admit the simple lifting \( C \) over \( \mathcal{O} \), then the groups \( H^h(C, (\Gamma^\theta_\mathcal{O}(C)^\dagger) \otimes_{\mathcal{O}} K) \), \( h \geq 0 \), as defined in \([3]\), are finite dimensional over \( K \), all integers \( h \).) (The analogous theorems for \( q \)-adic homology with compact supports, and for \( q \)-adic cohomology, about finite generation, can also be proved by the same method, see Section 4 below.)

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Let \( \mathcal{O}, K, k, A, A, \) and \( \mathcal{O}_{\mathcal{O}, A} \) be as in Section 1.

**Lemma 1.** Suppose that \( A \) is normal and let \( C, D \in \mathcal{C}_{\mathcal{O}, A} \) be such that \( C \) is proper over \( \text{Spec}(A) \), and let \( f: C \to D \) be an \( A \)-map. (Therefore \( f \) is a map in \( \mathcal{C}_{\mathcal{O}, A} \).) Suppose that there exist \( X \) and \( Y \) finite presentation over \( A \), such that \( \overline{X}_K = \overline{X} \times_{\mathcal{O}} k \) is simple over \( A \otimes_{\mathcal{O}} K \), such that \( C \) (respectively: \( D \)) is a closed \( A \)-subscheme of \( \overline{X} = \overline{X} \times_{\mathcal{O}} k \) (respectively: of \( \overline{Y} = \overline{Y} \times_{\mathcal{O}} k \)), such that \( D \) is contained in the set \( Y \) of \( Y \) of simple points of \( Y \) over \( \text{Spec}(A) \) and such that the dimension of all connected components of fibers of \( \overline{X} \) (respectively: \( \overline{Y} \)) over \( \text{Spec}(A) \) is equal to a constant integer \( N \) (respectively: \( M \)). Let \( Y = Y \times_{\mathcal{O}} k \). Suppose also that there exists \( D \) closed \( A \)-subscheme of \( \overline{Y} \) such that \( D = (D \times_{\mathcal{O}} k)_{\text{red}} \), and that there exists \( f: \overline{X} \to \overline{D} \) a mapping of \( A \)-schemes such that the restriction of \( f \) to \( C \) is \( f \). Then there is induced a homomorphism of \( \mathcal{A}^d \otimes_{\mathcal{O}} K \)-modules

\[
H^{2M-h}(X, X - C, (\Gamma^\theta_d(\overline{X})^\dagger) \otimes_{\mathcal{O}} K) \\
\to H^{2M-h}(Y, Y - D, (\Gamma^\theta_d(\overline{Y})^\dagger) \otimes_{\mathcal{O}} K) = H^h_\mathcal{O}(D, (\mathcal{A}^d)^{\dagger} \otimes_{\mathcal{O}} K), \quad (1)
\]

all integers \( h \). Suppose, in addition, that any one of the following three technical conditions holds: Either

(a) \( \mathcal{A} = \mathcal{O} \) or

(b) the image of: \( u_{\mathcal{O} \times_d X, \mathcal{O} Y} \) (as defined below)) in

(b1) \( H^{2M}(\overline{X} \times_d (Y, Y - D), (\Gamma^\theta_d(\overline{X} \times_d Y) \otimes_{\mathcal{O}} K) \) comes from an element of

(b2) \( \bigoplus_{i=0}^{2M} [H^i(\overline{X}, (\Gamma^\theta_d(\overline{X}))] \otimes_{\mathcal{O}} [H^{2M-i}(Y, Y - D, (\Gamma^\theta_d(\overline{Y}))] \otimes_{\mathcal{O}} K ; \) or that

(c) \( H^p(\overline{X}, (\Gamma^\theta_d(\overline{X})^\dagger \otimes_{\mathcal{O}} K) \) is finitely generated as \( (\mathcal{A}) \otimes_{\mathcal{O}} K \)-module, all integers \( h, 0 \leq h \leq 2M \).

Then, if the ring \( \mathcal{A}^d \) is Noetherian, then the image of the homomorphism \( (1) \) is finitely generated as \( ((\mathcal{A}^d) \otimes_{\mathcal{O}} K) \)-module.
Note. The proof of the lemma shows that, if the hypothesis \( f: X \to Y \) is weakened to \( f: X \to Y' \), then one can still construct the mapping (1).

Proof. The construction of the map (1) is similar to the proof of Proposition 3 of Section 1. First, notice that since \( X \) and \( Y \) are simple of finite presentation over \( A \), by [2] (or, in the case \( A = C \), by [3, II.5, p. 231, just before Proposition 2], we have the canonical class

\[
\bar{u}_{X \times_A Y, f} \in H^{2M}(X \times_A Y, X \times_A Y - \Gamma_f, \Gamma_A^*(X \times_A Y)), \tag{2}
\]

where \( f = f \times_{O} K \) and \( \Gamma_f \subseteq X \times_A Y \) is the graph of \( f \). But

\[
H^{2M}(X \times_A Y, X \times_A Y - \Gamma_f, \Gamma_A^*(X \times_A Y) \otimes_{O} K) \approx H^{2M}(X \times_A Y, X \times_A Y - \Gamma_f, \Gamma_A^*(X \times_A Y)) \tag{3}
\]

(since \( X \times_A Y \) and the objects are of finite presentation over \( A \); see, e.g., [2]). Therefore the element (2) defines an element, call it \( u_{X \times_A Y, f} \), in the left side of Eq. (3). We also use the same notation for the image of this element in \( H^{2M}(X \times_A Y, X \times_A Y - \Gamma_f, \Gamma_A^*(X \times_A Y) \otimes_{O} K) \).

We define the \((A^+ \otimes_{O} K)\)-homomorphism (1), each integer \( h \), to be the composite of the sequence:

\[
H^{2N-h}(X, X - C, \Gamma_d^*(X) \otimes_{O} K) \xrightarrow{\pi_1} H^{2N-h}(X, X - C) \times_A Y, \Gamma_d^*(X \times_A Y) \otimes_{O} K) \xrightarrow{\text{cupping with}} H^{2N+2M-h}(X \times_A Y \times_A Y - \Gamma_f, \Gamma_d^*(X \times_A Y) \otimes_{O} K) \xrightarrow{\text{restriction}} H^{2N+2M-h}(X \times_A Y \times_A Y - \Gamma_f, \Gamma_d^*(X \times_A Y) \otimes_{O} K) = H^{2N+2M-h}(X \times_A (Y, Y - D)) \xrightarrow{\text{natural map}} H^{2N+2M-h}(X \times_A (Y, Y - D)) \xrightarrow{(1/d)-\text{natural map}} H^{2N+2M-h}(X \times_A (Y, Y - D)) \xrightarrow{\text{natural map}} H^{2N+2M-h}(Y, Y - D, \Gamma_d^*(Y) \otimes_{O} K),
\]

where \( X', X'', X', X'' \), and \( d \) and \( G \) are constructed as in the proof of Proposition 3 of Section 1.
Considering the second mapping and the fifth group in the sequence (4), we see that the image of any element \( x \in H^{2N-h}(X, X - C, \Gamma_d^*(\overline{X})) \) under the mapping (1) depends only on the value of the image of \( \pi_1^*(x) \) in \( H^{2N-h}(X \times_A Y, \Gamma_d^*(\overline{X}) \otimes \mathcal{O}_K) \) after cupping with the image of \( u^*_d \times Y, \tau \). In this case the image of the element \( x \) in the fifth group of the sequence (4) can be written as

\[
H^{2M}(X \times_A (Y, Y - D), (\Gamma_d^*(\overline{X} \times_A Y) \otimes \mathcal{O}_K). \tag{5}
\]

Condition (a) implies condition (b). Therefore it suffices to prove the lemma if either (b) or (c) holds.

*Case 1.* Condition (b) holds. Then by condition (b) the image of \( u^*_d \times Y, \tau \) in the group (5) can be written in the form

\[
\sum_{i=0}^{2M} \sum_{j=1}^{B} \pi_1^*(e_{ij}) \cup \pi_2^*(f_{ij}), \tag{6}
\]

where \( B \) is an integer \( \geq 1 \), and where \( e_{ij} \in H^i(\overline{X}, \Gamma^*_d(\overline{X})) \otimes \mathcal{O}_K \) and \( f_{ij} \in H^{2M-i}(Y, Y - D, (\Gamma_d^*(\overline{Y}) \otimes \mathcal{O}_K), 1 \leq j \leq B, 0 \leq i \leq 2M \). Therefore in this case the image of the element \( x \) in the fifth group of the sequence (4) can be written as

\[
\sum_{i=0}^{2M} \sum_{j=1}^{B} [\pi_1^*(x \cup e_{ij})] \cup \pi_2^*(f_{ij}). \tag{7}
\]

Considering the maps leaving the fifth, sixth and seventh groups in Eq. (4), it follows that image of \( x \) under the composite mapping (4) depends only on the images, for all integers \( i, j \) such that \( 0 \leq i \leq 2M, 1 \leq j \leq B \),

\[
\alpha_{ij}(x) \in H^{2N-h+i}(P^N(A), \Gamma_d^*(P^N(A))) \otimes \mathcal{O}_K \tag{8}
\]

of the elements: \( \pi_1^*(x \cup e_{ij}) \) under the composite mappings:

\[
H^{2N-h+i}(X, (\Gamma_d^*(\overline{X})) \otimes \mathcal{O}_K) \rightarrow H^{2N-h+i}(X', (\Gamma_d^*(\overline{X}')) \otimes \mathcal{O}_K) \rightarrow H^{2N-h+i}(P^N(A), (\Gamma_d^*(P^N(A))) \otimes \mathcal{O}_K),
\]

for \( 0 \leq i \leq 2M \). In fact, considering the last mapping in the sequence (4), the image of \( x \in H^{2N-h}(X, X - C, (\Gamma_d^*(\overline{X})) \otimes \mathcal{O}_K) \) under the mapping (1) depends actually only on those \( \alpha_{ij}(x) \) in Eq. (8) such that \( i = h \), i.e., only on the value of the elements \( \alpha_{h, j}(x) \in H^{2N}(P^N(A), (\Gamma_d^*(P^N(A))) \otimes \mathcal{O}_K), 1 \leq j \leq B. \)
Since (by Lemma 4 of Section 1 with \( Y = \text{Spec}(\mathcal{A}) \)) this latter group is isomorphic to \((\mathcal{A}^+) \otimes_\mathfrak{m} K\), it follows that for each integer \( j \), \( 1 \leq j \leq B \), the assignment: \( x \mapsto \alpha_{n_j}(x) \) is a homomorphism of \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-modules from the \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-module

\[
H^{2N-h}(X, X - C, (\Gamma^*_\mathcal{A}(X))^+) \otimes_\mathfrak{m} K),
\]

(9)

into \((\mathcal{A}^+) \otimes_\mathfrak{m} K\), and that if \( \alpha_h(x) = (\alpha_{n_j}(x))_{1 \leq j \leq B} \) is the mapping from the group (9) into \(((\mathcal{A}^+) \otimes_\mathfrak{m} K)^B\), then the image of \( x \) under (1) is completely determined by the image of \( x \) under \( \alpha_h \), all elements \( x \) of the group (9). Therefore the image of the homomorphism (1) is naturally \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-isomorphic to a quotient module of the image of \( \alpha_h \), and therefore is finitely generated as \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-module. Q.E.D.

Case 2. Condition (c) holds. We have seen that the image of an element \( x \in H^{2N-h}(X, X - C, (\Gamma^*_\mathcal{A}(X))^+) \otimes_\mathfrak{m} K) \) under the mapping (1) depends only on the image of \( \pi^*_h(x) \) in \( H^{2N-h}(X \times_A Y, \Gamma^*_\mathcal{A}(X))^+) \otimes_\mathfrak{m} K) \). Therefore the image of \( x \) under (1) depends only on the image of \( x \) in \( H^{2N-h}(X, (\Gamma^*_\mathcal{A}(X))^+) \otimes_\mathfrak{m} K) \). Therefore the image of (1) is naturally \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-isomorphic to a quotient module of the image of the restriction map:

\[
H^{2N-h}(X, X - C, (\Gamma^*_\mathcal{A}(X))^+) \otimes_\mathfrak{m} K) \rightarrow H^{2N-h}(X, (\Gamma^*_\mathcal{A}(X))^+) \otimes_\mathfrak{m} K).
\]

But by hypothesis (c) the range of this latter mapping is finitely generated. Q.E.D.

Remark. If one drops the hypothesis that “the ring \( \mathcal{A}^+ \) is Noetherian,” then the proof of Lemma 1 shows that the image of the mapping (1) is a submodule of a finitely generated \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-module. Also, if \( M \) is any finitely generated \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-module, then the image of the analogous mapping

\[
H^{2N-h}(X, X - C, M) \rightarrow H^h_c(D, M),
\]

(11)

with coefficients in \( M \), is a submodule of a finitely generated \((\mathcal{A}^+) \otimes_\mathfrak{m} K\)-module, all integers \( h \).

Corollary 1.1. The hypotheses being as in the Note to Lemma 1, if \( U \) is any open subset of \( C \) such that \( U \in \mathcal{W}_{\mathfrak{m}, A} \) (i.e., such that \( U \) is quasicompact) and such that \( U \) is contained in the simple points of \( X \), and if \( V \) is an open subset of \( D \) such that \( V \in \mathcal{W}_{\mathfrak{m}, A} \) (i.e., such that \( V \) is quasicompact), such that \( f \) maps \( U \)
into \( V \), and such that the restriction \( f_U: U \to V \) is proper over \( A \), then the following diagram is commutative

\[
\begin{array}{ccc}
H^{2N-h}(X, X - C, \Gamma_d(X) \to \otimes K) & \xrightarrow{\text{map (1)}} & H^{2M-h}(Y, Y - D, \Gamma_d(Y) \to \otimes K) = H_h(D, A \to \otimes K) \\
\downarrow \text{restriction} & & \downarrow \text{restriction} \\
H^{2N-h}(X_0, X_0 - U, \Gamma_d(X) \to \otimes K) & \to & H^{2M-h}(Y_0, Y_0 - V, \Gamma_d(Y) \to \otimes K) = H_h(V, A \to \otimes K) \\
\downarrow & & \downarrow \\
H_k(U, A \to \otimes K) & & H_h(f_U)
\end{array}
\]

all integers \( h \), where \( X_0 \) (resp: \( Y_0 \)) is any open neighborhood of \( U \) (resp: \( V \)) in \( X \) (resp: \( Y \)) that is simple of finite presentation over \( A \) such that if \( X_0 = X_0 \times K \) (resp: \( Y_0 = Y_0 \times K \)) then \( U \) (resp: \( V \)) is closed in \( X_0 \) (resp: \( Y_0 \)).

**Proof.** The two “equalities” in the diagram are the definition of \( H^*_h \); commutativity of the upper right square is the definition of the “restriction” mapping in \( H^*_h \); and commutativity of the bottom square is by definition of \( H_h(f_U) \). Commutativity of the upper left square follows from the definition of the map (1) of Lemma 1 and the definition (see the Proof of Proposition 3 of Section 1) of \( H^*_h(f_U) \).

**Lemma 2.** Let \( \overline{X} \) and \( C \) be as in the hypotheses of Lemma 1. Let \( (C', \overline{X}') \in \mathcal{C}_{e,d} \) be such that \( C' \) is closed in \( \overline{X}' = \overline{X} \times e k \), and such that we have a mapping \( \rho: \overline{X} \to \overline{X}' \) of \( A \)-schemes that maps \( C \) into \( C' \). Suppose that we have \( \overline{X} \) an open subset of \( \overline{X} \) that is simple of finite presentation over \( A \), and \( E \) a closed subscheme of \( \overline{X} \) of finite presentation over \( A \), of Krull codimension \( N - N' \), such that \( \overline{E} \cap \rho^{-1}(C') \subset C \) as sets, and such that if \( \overline{E} = \overline{E} \times e k \), \( X = \overline{X} \times e k \), \( E = X \cap \overline{E} \) and \( U = X \cap C \), then \( E_{\text{red}} \), \( U \), and the closed subscheme \( \rho^{-1}(C') \cap X \) of \( X_{\text{red}} \) are simple over \( \text{Spec}(A_{\text{red}}) \), and \( E_{\text{red}} \) and \( \rho^{-1}(C') \cap X \) are in general position and intersect transverse regularly [3, Proposition 11.5.2, p. 231] over \( \text{Spec}(A_{\text{red}}) \). Suppose also that \( E \) is dense in \( \overline{E} \), and that the generic points of fibers of \( E_X = \overline{E} \times e K \) over \( \text{Spec}(A \otimes e K) \) are simple over \( \text{Spec}(A \otimes e K) \). Suppose that we have \( \overline{X}' \) open in \( \overline{X}' \) of finite presentation over \( \text{Spec}(A) \) such that \( \rho \) maps \( \overline{X} \) into \( \overline{X}' \), and such that \( \overline{X} \) is simple over \( \overline{X}' \), and such that if \( U' = C' \cap \overline{X}' \), then \( \rho^{-1}(U') \cap \overline{X} = U \), and the restriction \( \tau \) of \( \rho \) to \( U \) is a proper mapping from \( U \) into \( U' \). Suppose, for simplicity, that the \( A \)-mapping \( \tau: U \to U' \) is an isomorphism from \( U \) onto \( U' \). Then the following diagram is commutative:
where $X' = \mathcal{X}' \times_k k$. 
Remark. The hypothesis that "the $A$-mapping $\tau: U \to U'$ is an isomorphism from $U$ onto $U'$" can be eliminated, leaving a true statement (which we shall not prove). (The proof is similar.)

Proof. We first must define the mapping $\beta$. Since the generic points of fibers of $E_K$ over $\text{Spec}(A_K)$ are simple over $\text{Spec}(A_K)$ (where $A_K = A \otimes E_K$), by [2] we have the canonical class $u_{x_K, E_K} \in H^{2(N - N')}(X_K, X_K - E_K, I_{A_K}^*(X_K))$, where $X_K = \overline{X} \times_K E_K$. This latter cohomology group is isomorphic to $H^{2(N - N')}(\overline{X}, \overline{X} - \overline{E}, \Gamma_{A_K}^*(\overline{X}) \otimes_K E_K)$. Let $u_{x_K, E_K}$ denote the image of that element in this latter group, and let $u_{x, E}$ denote the image in $H^{2(N - N')}(\overline{X}, \overline{X} - \overline{E}, (\Gamma_{A_K}^*(\overline{X})) \otimes_K K)$. Then let $\beta$ be the composite:

$$H^{2N-h}(\overline{X}', \overline{X}' - C', (\Gamma_{A_K}^*(\overline{X}')) \otimes_K E_K) \xrightarrow{\hat{\rho}} H^{2N-h}(\overline{X}, \overline{X} - \rho^{-1}(C'), (\Gamma_{A_K}^*(\overline{X})) \otimes_K E_K)$$

$$\xrightarrow{\text{cupping with } u_{x, E}} H^{2N-h}(\overline{X}, \overline{X} - C, (\Gamma_{A_K}^*(\overline{X})) \otimes_K E_K).$$

Then if we define $\beta_0$ similarly, the upper left square clearly commutes. And, by the remark following Lemma 1 of Section 1, it is easy to see that the upper right square commutes. Commutativity of the bottom square and triangle are by definition. Q.E.D.

Theorem 3. Let $D$ be an algebraic variety (= scheme of finite type) over $k$ that is properly embeddable over $\mathcal{O}$. Then the lifted $p$-adic homology with compact supports $H^\varepsilon(D, K)$ of $D$ with coefficients in $K$ is finitely generated as $K$-vector space, all integers $h$.

Proof. The proof is by induction on the dimension of $D$. If $U'$ is a dense open subset of $D$, then by Proposition 7 of Section 1 we have the exact sequence:

$$\cdots \to \partial_h \to H^\varepsilon_h(D - U', K) \xrightarrow{H^{h+1}_\varepsilon} H^\varepsilon_h(D, K) \xrightarrow{\text{restriction}} H^\varepsilon_h(U', K) \xrightarrow{\partial_h} H^\varepsilon_{h-1}(D - U', K) \to \cdots. \quad (10)$$

Since $\dim(D - U') < \dim(D)$, by the inductive assumption to prove the theorem for $D$ it suffices to prove it for any variety birationally equivalent to $D$. Therefore the theorem easily reduces to the case in which $D$ is irreducible, projective and a hypersurface. Then choose $\overline{D}$ proper and flat over $\text{Spec}(\mathcal{O})$, and an irreducible, projective hypersurface over $\text{Spec}(\mathcal{O})$,
such that $D$ is $k$-isomorphic to $\overline{D} \times_k k$, and such that the general fiber, $D_K = \overline{D} \times_k K$, is simple over $K$ (this is easily done by the Jacobian criterion). Since $D$ is projective over $\mathcal{O}$, $\overline{D}$ is $\mathcal{O}$-isomorphic to a closed subscheme of $\mathbb{P}^N(\mathcal{O})$ for some integer $N$. Let $x$ be a point of $\overline{D}$ that is in $D$ and that is a simple point of $\overline{D}$ over $\mathcal{O}$. Then there exists an open neighborhood $W$ of $x$ in $\mathbb{P}^N(\mathcal{O})$ and $d$ functions $t_1, \ldots, t_d \in \Gamma(W, \mathcal{O}_W)$ where $d = \dim D$, such that if $U = W \cap \overline{D}$ and if $t'_1, \ldots, t'_d$ are the images of $t_1, \ldots, t_d$ in $\Gamma(U, \mathcal{O}_U)$, and if we let $s_i = t_i \otimes_k 1 - 1 \otimes_k t'_i \in \Gamma(W \times_k U, \mathcal{O}_W)$, $1 \leq i \leq d$, where $X = \mathbb{P}^N(\mathcal{O}) \times_k \overline{D}$, then the closed subset $s_1 = \ldots = s_d = 0$ of $W \times_k U$ intersected with $U \times_k U$ is the diagonal of $U \times_k U$. Let $\bar{E}$ be the closure in $\bar{X}$ of the closed subset $(s_1 = \ldots = s_d = 0)$ of $W \times_k U$, let $\bar{C}$ be the intersection of $\bar{E}$ and $\overline{D} \times_k \overline{D}$ in $\bar{X}$ ($= \mathbb{P}^N(\mathcal{O}) \times_k \overline{D}$), let $C = (\overline{C} \times_k k)_{\text{red}}$, let $\bar{X}' = \bar{Y} = \mathbb{P}^N(\mathcal{O})$, let $f : \bar{X} \to \overline{D}$ be the second projection and let $C' = D$. Let $U'$ be the open subset $U \times_k k$ of $D$, let $V = U'$ and let $U$ be the diagonal of $U' \times_k U'$. Then $U$ is an open subset of $C$ and the restriction $\tau$ of the first projection is an isomorphism from $U$ onto $U'$. Let $X_0 = \overline{U} \times_k U$, $X_0 = \overline{U}'$. Then all the hypotheses of Lemmas 1 and 2, and of Corollary 1.1 hold, so that we have the mappings $\gamma, \beta, \beta_0$ and $f_\bar{X}$ of Lemmas 1, 2, 2, and of Corollary 1.1, respectively (notice that $f_\bar{U}$ is an isomorphism), and we have the commutative diagram (the notation "$\Gamma^+_{\mathcal{X}}$" is as in [3]):
all integers \( h \). (Commutativity of the three squares to the bottom right is Corollary 1.1; commutativity of the other squares is Lemma 2.) By Lemma 1, the image of \( \gamma \) is a finite dimensional \( K \)-vector space. Diagram chasing in the above diagram, it follows that the restriction mapping: \( H_{v}^{h}(C', K) \to H_{v}^{h}(U', K) \) has a finite dimensional image, all integers \( h \). But \( C' = D \). Then considering the long exact sequence (10), we have that the mappings “restriction” in Eq. (10) have finite dimensional images. Since by the inductive assumption \( H_{v}^{h}(D - U', K) \) is finite dimensional over \( K \), from the exact sequence (10) we deduce that \( H_{v}^{h}(D, K) \) is finite dimensional, all integers \( h \).

Q.E.D.

The proof of Theorem 3 shows equally well

**Corollary 3.1.** Suppose that the \( \mathcal{C} \)-algebra \( A \) is a discrete valuation ring \( \mathcal{C} \) containing \( \mathcal{C} \) such that the maximal ideal of \( \mathcal{C} \) contracts to the maximal ideal of \( \mathcal{C} \). Let \( k' \) be the residue class field of \( \mathcal{C} \) and let \( K' \) be the quotient field of \( \mathcal{C} \). Then the lifted \( p \)-adic homology with compact supports \( H_{v}^{h}(D, K') \) is finite dimensional as \( K' \)-vector space, all integers \( h \).

**Remark 1.** One might hope to prove the analogue of Theorem 3 with “\( A \)” an arbitrary \( \mathcal{C} \)-algebra, replacing “\( \mathcal{C} \).” This is in general false, even if \( D \) is proper over \( A_{\text{red}} \) and \( A \) is a regular local ring, even a simple \( \mathcal{C} \)-algebra.

**Counterexample.** Take \( A \) a simple local \( \mathcal{C} \)-algebra such that \( A \) is of Krull dimension \( n \geq 1 \) and let \( D = (\text{the closed point of } \text{Spec}(A)) \). Then \( H_{n}^{c}(D, (A^\mathbb{G}) \otimes_{\mathcal{C}} K) \) is isomorphic to \( H^{n}(\text{Spec}(A), \text{Spec}(A) - \{ \text{closed point} \}, \mathcal{C}\text{Spec}(A)^\mathbb{G}) \otimes_{\mathcal{C}} K \), where \( n = \dim A \), and is not finitely generated over \( A^\mathbb{G} \otimes_{\mathcal{C}} K \); see the example following Remark 4 after Proposition 7 of Section 1.

**Remark 2.** Under the hypotheses of Lemma 1, the proof of Lemma 1 can be refined slightly to prove a bit more: namely, that the composite of the natural mapping:

\[
H^{2n-h}(\overline{X}, \overline{X} - C, (\Gamma_{s}^{*}(\overline{X})^\mathbb{G}) \otimes_{\mathcal{C}} K) \to H^{2n-h}(\overline{X}, \overline{X} - C, (\Gamma_{s}^{*}(\overline{X})^\mathbb{G}) \otimes_{\mathcal{C}} K)
\]

with the mapping (1) of Lemma 1 is such that the image of that composite mapping is finitely generated as \( (A^\mathbb{G}) \)-module.


Let \( \mathcal{C} \) be a c.d.v.r. of mixed characteristic with quotient field \( K \) and residue class field \( k \). Let \( C \) be an algebraic variety over \( k \) that is properly embeddable over \( \mathcal{C} \). (See Section 1. It suffices that \( C \) be quasiprojective.) If \( F: \mathcal{C} \to \mathcal{C} \)
induces the $p$th power endomorphism of $k$, $p = \text{char}(k)$, then we define the 
\textit{zeta matrices} $W^h(C)$, $0 \leq h \leq 2 \dim C$. These generalize the zeta function 
of varieties over finite fields. This uses the author's lifted $p$-adic cohomology with compact supports $H^h_k(C, K)$. (See Sections 1 and 2.) More generally, if $A$ is an o-algebra and $F: \mathcal{O} \to A$ induces the $p$th power endomorphism of 
$A_{\text{red}}$, where $A = A \otimes_{\mathcal{O}} k$, and if $A$ is integral over the subring $F(A)$, and 
if $C$ is a reduced scheme over $A_{\text{red}}$ that is polynomially properly embeddable 
(see Section 1) over $A_{\text{red}}$, then we define the \textit{zeta endomorphism} $\zeta_h$ of the 
\textit{lifted} $p$-adic cohomology with compact supports $H^h_k(C, A^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}$) 
(Section 1), all integers $h$. These essentially determine the zeta matrices of 
all the algebraic varieties (= fibers over $\text{Spec}(A_{\text{red}})$) in the algebraic family $C$.

If the field $k$ is finite then a generalization of the First Weil Conjecture, 
"Lefschetz Theorem," is stated and proved for $C$, $p$-adically in this Section, and 
$q$-adically in Section 4. Also, a generalization of the Third Weil Conjecture, "Riemann Hypothesis," is stated for $C$ (conjectured $p$-adically 
in this section, and proved $q$-adically in Section 4, $q \neq \text{char} k$).

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Let $\mathcal{O}$ be a complete discrete valuation ring with residue class field $k$ and 
quotient field $K$, such that $K$ is of characteristic zero. Then we define cate-
gories $\mathcal{C}_0$ and $\mathcal{C}_\text{normal}$.

The objects in $\mathcal{C}_\text{normal}$ (resp: $\mathcal{C}_0$) are the pairs $(A, C)$ where $A$ is a normal 
(resp: an) $\mathcal{O}$-algebra and where $C$ is a reduced scheme over $\text{Spec}(A_{\text{red}})$ where 
$A = A \otimes_{\mathcal{O}} k$, such that $C$ is properly embeddable (resp: polynomially 
properly embeddable) over $A$ (resp: over $A_{\text{red}}$). (See Section 1.) The maps in 
$\mathcal{C}_\text{normal}$ (resp: $\mathcal{C}_0$) from $(A, C)$ into $(B, D)$ are the pairs $(F, f)$ where $F: A \to B^\dagger$ 
is a homomorphism of rings (such that $F$ maps the image of $\mathcal{O}$ in $A$ into the 
image of $\mathcal{O}$ in $B^\dagger$) and where $f: (C \times_B A_{\text{red}}) \to D$ is a proper morphism of 
schemes over $\text{Spec}(A_{\text{red}})$, where $A = A \otimes_{\mathcal{O}} k$, $B = B \otimes_{\mathcal{O}} k$. (We call such 
a map a semi-linear proper map.) We also introduce the category $\mathcal{M}_\mathcal{O}$, having 
for objects the pairs $(A, M)$ where $A$ is an $\mathcal{O}$-algebra and $M$ is an $A$-module, 
and such that the maps from $(A, M)$ into $(B, N)$ are the pairs $(F, f)$ where $F: A \to B$ is a homomorphism of rings (such that $F$ maps the image of $\mathcal{O}$ in $A$ into the image of $\mathcal{O}$ in $B$), and $f: M \otimes_A B \to N$ is a homomorphism of 
$B$-modules (we call such a mapping $(F, f)$ a \textit{semi-linear homomorphism}). Then

\textbf{Theorem 1.} For each integer $h$ there is induced a covariant functor, the 
$h$th lifted $p$-adic homology with compact supports, from the category 
$\mathcal{C}_\text{normal}$ (resp: $\mathcal{C}_0$) into the category $\mathcal{M}_\mathcal{O}$, such that the restriction of this 
functor to $\mathcal{C}_{\mathcal{O}, \mathcal{A}}$ is the functor $H^h_k$ of Theorem 6 of Section 1, all normal (resp: 
all) $\mathcal{O}$-algebras $A$. 
Sketch of proof. Let \((F, f): (A, C) \to (B, D)\) be a morphism in \(\mathcal{C}_0^{\text{normal}}\). Then there exists \(X\) such that, in the notations of Section 1, \((C, X)\) is an object in \(\mathcal{C}_0^{\text{normal}}\). There is a natural epimorphism: \(B^\dagger \to \mathcal{B}^\dagger\) (see [2, Introduction]). Then \(H^h_{\mathfrak{c}}(C, A^\dagger \otimes_{\mathcal{C}_0} \mathbb{Q}) = H^{2N-h}(X, X - C, (\Gamma^*_A(X)^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q})\) and \(H^h_{\mathfrak{c}}((C \times_{\text{Spec}(A)} \text{Spec}(B))_{\text{red}}, (\mathcal{B}^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q}) = H^{2N-h}(X \times_A B, X \times_A B - C \times_A B, (\Gamma^*_B(X \times_A B)^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q} \otimes_{\mathcal{B}^\dagger} \mathcal{B}^\dagger\), where \(N, X, \mathcal{X}\) are as in Proposition 3 of Section 1. Define \(H^h_{\mathfrak{c}}(F, f)\) to be the composite:

\[
H^{2N-h}(X, X - C, (\Gamma^*_A(X)^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q})
\]

\[
\to H^{2N-h}(X \times_A B, X \times_A B - C \times_A B, (\Gamma^*_B(X \times_A B)^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q} \otimes_{\mathcal{B}^\dagger} \mathcal{B}^\dagger
\]

\[
\to H^h_{\mathfrak{c}}(D, (\mathcal{B}^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q}).
\]

The objects in \(\mathcal{C}_0^{\text{normal}}\) (resp: \(\mathcal{C}_0\)) are the same as the objects in \(\mathcal{C}_0^{\text{normal}}\) (resp: \(\mathcal{C}_0\)). The maps in \(\mathcal{C}_0^{\text{normal}}\) (resp: \(\mathcal{C}_0\)) from \((A, C)\) into \((B, D)\) are the pairs \((F, f)\) where \(F: B \to A^\dagger\) is a homomorphism of rings (such that \(F\) maps the image of \(\mathcal{O}\) in \(B\) into the image of \(\mathcal{O}\) in \(A^\dagger\)), and such that if \(F^\dagger: B^\dagger \to A^\dagger\) denotes the homomorphism induced by \(F\), then the ring \(A^\dagger\) is integral over the subring \((F^\dagger)(B^\dagger) \subset A^\dagger\), and where \(f: C \to (D \times_{\text{Spec}(B)} \text{Spec}(A))_{\text{red}}\) is a proper morphism of schemes over \(\text{Spec}(A_{\text{red}})\), where \(A = A \otimes_{\mathcal{C}_0} k, B = B \otimes_{\mathcal{C}_0} k\).

We call such a map an integral, semi-linear proper map.) Then

**Theorem 1'**. For each integer \(h\) there is induced a contravariant functor, the \(h\)th lifted p-adic cohomology group with compact supports, from the category \(\mathcal{C}_0^{\text{normal}}\) (respectively: \(\mathcal{C}_0\)) into the category \(\mathcal{A}_\mathcal{O}\), such that the restriction of this functor to \(\mathcal{C}_0^{\text{normal}}\), \(\mathcal{C}_0\) is the functor \(C \mapsto H^h_{\mathfrak{c}}(C, (A^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q})\) as defined in Eq. (23) of Section 1, all normal (respectively: all) \(\mathcal{O}\)-algebras \(A\).

**Sketch of proof.** Let \((F, f): (A, C) \to (B, D)\) be a morphism in \(\mathcal{C}_0^{\text{normal}}\). Then there exists \(Y\) such that, in the notations of Section 1, \((D, Y)\) is an object in \(\mathcal{C}_0^{\text{normal}}\). Let \(Y\) be an open neighborhood of \(D\) in \(Y\) such that \(Y\) is simple of finite presentation over \(\text{Spec}(B)\) and such that all the fibers of \(Y\) over \(\text{Spec}(B)\) have the same dimension \(M\), and let \(Y = Y \times_{\text{Spec}(B)} \text{Spec}(k)\).

Let \(U\) be a finite covering by affine open subsets of \(Y\) (in the sense of [3, I.5, p. 127]) that is a refinement of the covering \(\{Y, Y - D\}\) (in the sense of [3, I.5, p. 127]). Let \(\pi: Y \times_{\text{Spec}(B)} \text{Spec}(A) \to Y\) denote the natural projection, an affine map. Then the \((2M - h)\)th cohomology group of \(C^\ast(U, (Y, Y - D), (\Gamma^*_B(Y^\dagger \otimes_{\mathcal{C}_0} \mathbb{Q}))\otimes_{\mathcal{A}^\dagger} A^\dagger\) and of \(C^\ast(U, (Y, Y - D), \pi^\ast((\Gamma^*_B(Y \times_{\text{Spec}(B)} \text{Spec}(A^\dagger)))^\dagger \otimes_{\mathcal{C}_0} \mathbb{Q})\otimes_{\mathcal{A}^\dagger} A^\dagger\) are isomorphic, respectively, to \(H^{2M-h}(Y, Y - 0, (\Gamma^*_B(Y^\dagger \otimes_{\mathcal{C}_0} \mathbb{Q})))\otimes_{\mathcal{A}^\dagger} A^\dagger\) and to \(H^{2M-h}(Y \times_B A, Y \times_B A - D \times_B A, (\Gamma^*_B(Y \times_B A)^\dagger) \otimes_{\mathcal{C}_0} \mathbb{Q})\otimes_{\mathcal{A}^\dagger} A^\dagger\) (this follows from [3, Corollaries II.3.1.1 and II.3.1.2, pp. 190, 191]). But by the definition in Proposition 3 of Section 1, these latter
are equal, respectively, to \( H_c^h(D, B^\dagger \otimes \mathbb{Q}) \) and to \( H_c^h(D \times_B A, A^\dagger \otimes \mathbb{Q}) \). Therefore, if we define 
\[
C_\dagger^h = C^{2M-h}(U, (Y, Y - D), \Gamma^h(Y)^\dagger \otimes \mathbb{Q}) \quad \text{and} \quad C_\dagger^h = C^{2M-h}(U, (Y \times_B A, Y \times_B A - D \times_B A), \Gamma^h(Y \times_B A^\dagger)^\dagger \otimes \mathbb{Q}) 
\]
then \( C_\dagger^h \) and \( C_\dagger^h \) are chain complexes of flat \( B^\dagger \otimes \mathbb{Q} \)-modules, respectively, and their homologies are canonically isomorphic to \( H_c^h(D, (B^\dagger) \otimes \mathbb{Q}) \), and to \( H_c^h(D \times_B A, (A^\dagger) \otimes \mathbb{Q}) \), respectively. Since \( A^\dagger \) is integral over \((F^\dagger)(B^\dagger)\), by [3, II.2, p. 169, Eq. (1)], we have, for every \( B \)-algebra \( R \), that \((R \otimes_B A^\dagger) \approx R^\dagger \otimes_R A^\dagger \). Therefore \( C_\dagger^h \otimes_B A^\dagger \approx C_\dagger^h \). Let \( P_\dagger^h \) be a chain complex of \( B^\dagger \otimes \mathbb{Q} \)-modules such that each \( P_\dagger^h \) is a projective \( B^\dagger \otimes \mathbb{Q} \)-module, \( h \in \mathbb{Z} \), and such that we have a mapping: 
\[
P_\dagger^h \to C_\dagger^h \quad \text{that induces an isomorphism on homology.} 
\]
Let \( P_\dagger^h = P_\dagger^h \otimes_{(B^\dagger) \otimes \mathbb{Q}} (A^\dagger \otimes \mathbb{Q}) \). Then \( P_\dagger^h \) is a chain complex of \((A^\dagger) \otimes \mathbb{Q} \)-modules such that every \( P_\dagger^h \) is a projective \((A^\dagger) \otimes \mathbb{Q} \)-module, all integers \( h \), and such that we have a mapping: 
\[
P_\dagger^h \to C_\dagger^h \quad \text{that induces an isomorphism on homology (this latter since each \( P_\dagger^h \) and \( C_\dagger^h \) are flat over \((B^\dagger \otimes \mathbb{Q}) \) and therefore the universal coefficients spectral sequences apply). Then by definition (Eq. (23) of Section 1), we have that 
\[
H_c^h(D, (B^\dagger) \otimes \mathbb{Q}) = H^h(\text{Hom}_{(B^\dagger) \otimes \mathbb{Q}}(P_\dagger^h, B^\dagger \otimes \mathbb{Q})), 
\]
\[
H_c^h(D \times_B A, (A^\dagger) \otimes \mathbb{Q}) = H^h(\text{Hom}_{(A^\dagger) \otimes \mathbb{Q}}(P_\dagger^h, A^\dagger \otimes \mathbb{Q})), 
\]
all integers \( h \).

The composite of the sequence:

\[
\text{Hom}_{(B^\dagger) \otimes \mathbb{Q}}(P_\dagger^h, B^\dagger \otimes \mathbb{Q})
\]

\[
\overset{\text{Hom}_{B^\dagger \otimes \mathbb{Q}}(P_\dagger^h, P_\dagger^h \otimes \mathbb{Q})}{\longrightarrow} \text{Hom}_{B^\dagger \otimes \mathbb{Q}}(P_\dagger^h, A^\dagger \otimes \mathbb{Q}) 
\]

\[
\overset{\text{Hom}_{(A^\dagger) \otimes \mathbb{Q}}(P_\dagger^h, (A^\dagger) \otimes \mathbb{Q})}{\longrightarrow} \text{Hom}_{(A^\dagger) \otimes \mathbb{Q}}(P_\dagger^h, (A^\dagger) \otimes \mathbb{Q}) 
\]

is therefore a map of chain complexes, and the induced map on the \( h \)th homology groups is a map:

\[
H_c^h(D, (B^\dagger) \otimes \mathbb{Q}) \to H_c^h(D \times_B A, (A^\dagger) \otimes \mathbb{Q}), 
\]

semi-linear with respect to the ring homomorphism \((F^\dagger) \otimes \mathbb{Q}: (B^\dagger) \otimes \mathbb{Q} \to (A^\dagger) \otimes \mathbb{Q} \). The composite of this semi-linear mapping with the homomorphism of \((A^\dagger) \otimes \mathbb{Q} \)-modules,

\[
H_c^h(f, A^\dagger \otimes \mathbb{Q}): H_c^h(D \times_B A, A^\dagger \otimes \mathbb{Q}) \to H_c^h(C, A^\dagger \otimes \mathbb{Q}). 
\]
is therefore a homomorphism of abelian groups:

$$H_c^h(D, (B^\uparrow) \otimes \mathbb{Z} \mathbb{Q}) \to H_c^h(C, (A^\uparrow) \otimes \mathbb{Z} \mathbb{Q}).$$

semi-linear with respect to $(F^\uparrow) \otimes \mathbb{Q}$.

We define this latter composite map to be $H_c^h(F, f)$.

**Remarks.** In defining the categories "$\mathcal{C}_e^{\text{normal}}$" (and "$\mathcal{C}_e$"), we have defined the maps: $(A, C) \to (B, D)$ to be certain pairs $(F, f)$, where $F: B \to A^\uparrow$ is a homomorphism of rings (such that $F$ maps the image of $\mathcal{O}$ in $B$ into the image of $\mathcal{O}$ in $A^\uparrow$), and such that $A^\uparrow$ is integral over $(F^\uparrow)(B^\uparrow)$, where $F^\uparrow: B^\uparrow \to A^\uparrow$ denotes the map induced by $F$. Then, we have defined, for each integer $h$, the $h$th lifted $p$-adic cohomology group with compact supports, a contravariant functor on "$\mathcal{C}_e^{\text{normal}}"$ or on "$\mathcal{C}_e$", into the category $\mathcal{A}_e$.

One might wonder, can one eliminate the hypothesis on $F$ that "$A^\uparrow$ is integral over $(F^\uparrow)(B^\uparrow)$"? Meaning that, if one so generalizes the notion of "map" in "$\mathcal{C}_e^{\text{normal}}"$ and in "$\mathcal{C}_e$", by eliminating this hypothesis, then does the $h$th lifted $p$-adic cohomology with compact supports still remain a contravariant functor on the so enlarged category (inducing a map in $\mathcal{A}_e$, in the backwards direction, for every such generalized "map")? The answer, at the moment, is that I do not know (although I conjecture it). Tracing the proof of Theorem 1, we see at least that the condition on $F$, that "$A$ is integral over $(F^\uparrow)(B^\uparrow)$", can be replaced by the (more technical, but less restrictive) hypothesis: "Whenever $D$ is an affine scheme simple of finite presentation over $B$, if $C_{D^\uparrow}^h = \Gamma(D, \Gamma_D^h(D^\uparrow) \otimes \mathbb{Z} \mathbb{Q})$ and if $C_{A^\uparrow}^h = \Gamma(D \times_B A, \Gamma_D^h(D \times_B A^\uparrow) \otimes \mathbb{Z} \mathbb{Q}) \otimes_{A^\uparrow} A^\uparrow$, then the natural mapping of cochain complexes: $C_{D^\uparrow}^h \otimes_{B^\uparrow} A^\uparrow \to C_{A^\uparrow}^h$ induces isomorphisms on cohomology." (In the case that $A^\uparrow$ is integral over $(F^\uparrow)(B^\uparrow)$, this is true, since then the indicated map of cochain complexes is already an isomorphism on the cochain level.)

It is not difficult to show that every homomorphism $F: B \to A^\uparrow$ of rings (that maps the image of $\mathcal{O}$ into the image of $\mathcal{O}$) has this property, if $F$: the inclusion from $B = \mathcal{O}[T_1, \ldots, T_n]$ into $A^\uparrow$, where $A = \mathcal{O}[T_1, \ldots, T_{n+1}]$, has this property.

*(Note. In the special case in which $\mathcal{O}$ is a field of characteristic zero, this is of course true, since then the indicated map: $C_{B^\uparrow}^h \otimes_{B^\uparrow} A^\uparrow \to C_{A^\uparrow}^h$ is an isomorphism even on the cochain level.)*

**Example 1.** Suppose that char($k$) = $p \neq 0$, let $(A, C) \in \mathcal{C}_e^{\text{normal}}$ (or "$\mathcal{C}_e$") and let $F: A \to A^\uparrow$ be a homomorphism of rings, mapping the image of $\mathcal{O}$ into itself, such that (if $F^\uparrow: A^\uparrow \to A^\uparrow$ denotes the map induced by $F$, then) $A^\uparrow$ is integral over the subring $(F^\uparrow)(A^\uparrow)$, and such that the map induced by $F: A_{\text{red}} \to A_{\text{red}}$ is the $p$th power endomorphism of the ring in characteristic $p$ $A_{\text{red}}$ (that takes $x \in A_{\text{red}}$ into $x^p \in A_{\text{red}}$), where $A = A \otimes_k \mathbb{Q}_p$. (For example, if $\mathcal{O} = \mathbb{Z}_p$ and $A = W^-(A)$, see [7], or $A = W(A)$, then one
can take $F$ to be $W^-(\text{pth power map})$ or $W(\text{pth power map})$, respectively.\(^1\) Then $F$ maps this "$\mathcal{A}$" into itself.

Let $\alpha_c$ be the $p$th power endomorphism of the scheme $C$ in characteristic $p$ (that is set-theoretically the identity map, and that induces the $p$th power endomorphism of the local ring $\mathcal{O}_{C,c}$, all $c \in C$). Then for each integer $h$, we have the endomorphism

$$H_c^h(F, \alpha_c): H_c^h(C, (A^\dagger) \otimes \mathbb{Z} \mathbb{Q}) \to H_c^h(C, (A^\dagger) \otimes \mathbb{Z} \mathbb{Q})$$

of the abelian group $H_c^h(C, (A^\dagger) \otimes \mathbb{Z} \mathbb{Q})$, which is semi-linear with respect to the ring endomorphism

$$(F^\dagger) \otimes \mathbb{Z} \mathbb{Q}: (A^\dagger) \otimes \mathbb{Z} \mathbb{Q} \to (A^\dagger) \otimes \mathbb{Z} \mathbb{Q}$$

of the $\mathbb{Q}_p$-algebra $(A^\dagger) \otimes \mathbb{Z} \mathbb{Q}$. We call this map the $h$th zeta endomorphism of $H_c^h(C, (A^\dagger) \otimes \mathbb{Z} \mathbb{Q})$, for each integer $h$.

**Example 1.1.** Let us give another example, besides the Witt vectors or the bounded Witt vectors on a ring, of an $\mathcal{O}$-algebra $A$ such that we have a map $F:A \to A^\dagger$, such that for any $C$, if $(A, C) \in \mathcal{C}_G^{\text{normal}}$ or $\mathcal{C}_G$, then the pair $(F, \alpha_C)$ is a map in $\mathcal{C}_G^{\text{normal}}$ or in $\mathcal{C}_G$. Namely, consider the case in which $A = \mathcal{O}[T_1, \ldots, T_N]_G$, the localization of the polynomial algebra in $N$ variables over $\mathcal{O}$ at some element $g$; and suppose that we have an endomorphism $F_G$ of the valuation ring $\mathcal{O}$ lifting the $p$th power endomorphism of $k$ (e.g., if $\mathcal{O} = \mathbb{F}_p$, the identity of $\mathcal{O}$ accomplishes this). Then there exists a unique homomorphism of rings $F:A \to A^\dagger$, extending $F_G$, such that $F(T_i) = T_i^p$, $1 \leq i \leq N$. That $F$, of course, lifts the $p$th power endomorphism of $A = A \otimes \mathcal{O} k = k[T_1, \ldots, T_N]_G$; and is such that $A^\dagger$ is integral over $(F^\dagger)(A^\dagger)$ (since the mapping $F^\dagger: A^\dagger \to A^\dagger$ is an integral mapping, since it is obtained, by throwing through the functor "$^\dagger$", the integral ring homomorphism: $\mathcal{O}[T_1, \ldots, T_N]_G \to \mathcal{O}[T_1, \ldots, T_N]_{F_G}$, that maps $T_i$ into $T_i^p$, $1 \leq i \leq N$, and $g^{-1}$ into $F(g)^{-1}$ (where $F(g) = g^{F_G}(T_1^p, \ldots, T_N^p) \in \mathcal{O}[T_1, \ldots, T_N]$ and $g^{F_G}$ is obtained by throwing the coefficients of $g$ through $F_G$). Therefore $F:A \to A^\dagger$ so defined also has the desired properties. (This example can be generalized, from the case in which $A = \mathcal{O}[T_1, \ldots, T_N]_G$, to the case: $A = S^{-1}(\mathcal{O}[T_i])_{i \in I}$, the localization of the polynomial ring in possibly infinitely many variables $T_i$, $i \in I$, at any multiplicatively closed subset $S$ of $\mathcal{O}([T_i])_{i \in I}$). (Another, further, generalization of this example is as follows: Let $A$ be any $\mathcal{O}$-algebra and let $F:A \to A^\dagger$ be such that all the hypotheses of Example 1 hold. Let $S$ be any multiplicatively closed subset of $A$. Then there exists a unique ring homomorphism, call it $S^{-1}F$, $S^{-1}F:A \to (S^{-1}A)^\dagger$, extending $F$. And

\(^1\) Notice that, if $A = W(A)$ or $W^-(A)$, then $x \in A$ implies $x^p = F(x) \pmod p$, so that $x^p = F(x) + py = F(x + V(y))$, $\exists y \in A$. Therefore $x$ is integral over $F(A)$. Therefore $A$ is integral over $F(A)$ in this case.
then the \( \mathcal{O} \)-algebra \( S^{-1}\mathcal{A} \), together with the ring homomorphism \( S^{-1}F : S^{-1}\mathcal{A} \rightarrow (S^{-1}\mathcal{A})\), also obey all the hypotheses of Example 1.) Therefore, by Example 1, if \( \mathcal{A} \) is this indicated \( \mathcal{O} \)-algebra, and we define \( F \) as in Example 1.1, then for any \( C \) such that \((\mathcal{A}, C)\) is an object in \( \mathcal{G}\) or in \( \mathcal{G}_v \), we have the \( h \)th \textit{zeta endomorphism} \( \zeta^h(C) = H_c^h(F, \alpha_C) \), an \((F\) \( \otimes \mathbb{Q})\)-linear endomorphism of the \((\mathcal{A}) \otimes \mathbb{Q})\)-module \( H_c^h(C, (\mathcal{A}) \otimes \mathbb{Q}) \), all integers \( h \).

**Example 1.2.** The notations being as in Example 1, suppose that \( F : \mathcal{A} \rightarrow \mathcal{A} \) is a homomorphism of rings that obeys all the hypotheses of Example 1, except possibly the hypothesis that "(if \( F:\mathcal{A} \rightarrow \mathcal{A} \) denotes the map induced by \( F \), then \( \mathcal{A} \) is integral over \((F\mathcal{A} = \mathcal{A} \) \( \mathcal{A} \) \), and \( F \) by an extension \( F' : \mathcal{A}' \rightarrow \mathcal{A}' \), such that all the hypotheses of Example 1 then hold? The answer is "yes." Namely, define \( \mathcal{A}' = \mathcal{A}'(F\mathcal{A}) \), the direct limit of the sequence: \( \mathcal{A} \rightarrow F\mathcal{A} \rightarrow F^2\mathcal{A} \rightarrow \cdots \). Then \((\mathcal{A}' \otimes \mathbb{Q})\) \( \mathcal{A}' \), the \textit{perfection of} \((\mathcal{Z}P\mathcal{Z})\)-algebra \( \mathcal{A} \), in the sense of [7], \( \mathcal{A} = \mathcal{A} \otimes \mathbb{Q} \). Then the \( \mathcal{O} \)-algebra \( \mathcal{A}' \), and the induced ring homomorphism \( F' : \mathcal{A}' \rightarrow \mathcal{A}' \), obey all the hypotheses of Example 1, including the condition that "\( \mathcal{A}' \) is integral over \((F\mathcal{A})\)\), since in fact \( \mathcal{A}' = (F\mathcal{A}) \), an automorphism of the ring \( \mathcal{A}' \). Therefore, by Example 1, if, e.g., \((\mathcal{A}, C) \in \mathcal{G} \), so that \((\mathcal{A}', C \otimes \text{Spec}(\mathcal{A} \otimes \mathbb{Q})) \in \mathcal{G} \), then (even if \( \mathcal{A}' \) is not integral over \((F\mathcal{A})\)\), for each integer \( h \), we have the \( h \)th \textit{zeta endomorphism} \( \zeta^h(C) = H_c^h(F', \alpha_C) \) of the algebraic family \( C' = C \otimes \text{Spec}(\mathcal{A} \otimes \mathbb{Q}) \) over \( \text{Spec}(\mathcal{A} \otimes \mathbb{Q}) \), an endomorphism of the \( h \)th lifted \( \mathbb{P} \)-adic cohomology group with compact supports: \( H_c^h(C, (\mathcal{A}') \otimes \mathbb{Q}) \), that is semi-linear with respect to the \textit{automorphism} \((F') \otimes \mathbb{Q})\) of the ring \((\mathcal{A}') \otimes \mathbb{Q})\).

**Example 2.** In Example 1, consider the special case in which \( \mathcal{A} = \mathcal{O}' \), a discrete valuation ring containing \( \mathcal{O} \) as a subring such that \( \mathcal{M} \cap \mathcal{C} = \mathcal{M} \). Let \( K' = (\mathcal{O}' \otimes \mathbb{Q}) \), and let \( k' \) be the residue class field of \( \mathcal{O}' \). Then by Section 2, \( H_c^h(C, K') \) is a finitely generated \( K' \)-vector space, all integers \( h \), \( 0 \leq h \leq 2 \text{ dim } C \) (and the groups vanish for \( h \) not in this range by Remark 4 of Section 1). By Example 1, we have the \textit{zeta endomorphism} \( H_c^h(F, \alpha_C) \) of this \( K' \)-vector space, an endomorphism semi-linear with respect to the endomorphism \((F') \otimes \mathbb{Q})\) (call this endomorphism \( F' \)) of the field \( K' \). Therefore if we fix a basis for the finite dimensional vector space \( H_c^h(C, K') \) over the field \( K' \), then the \( h \)th \textit{zeta endomorphism} defines a \( \beta_h \times \beta_h \) matrix \((\beta_h = \text{dim}_{K'} H_c^h(C, K'))\), unique up to \( F' \)-similarity. (Two \( \beta_h \times \beta_h \) matrices \( W, W_0 \) with coefficients in \( K' \) are \( F' \)-similar if there exists an invertible \((\beta_h \times \beta_h)\)-matrix \( B \) with coefficients in \( K' \) such that \( B F' \cdot W B^{-1} = W_0 \). where \( B F' \) is obtained by throwing the coefficients of \( B \) through \( F' \).) We
denote this matrix, unique up to $F'$-similarity, by $W^h(C)$, and call it the $h$th zeta matrix of the algebraic variety $C$ over the field $k'$ of characteristic $p \neq 0$, for $0 \leq h \leq 2 \dim C$.

**Example 3.** Let $k'$ be a finite field and for simplicity suppose that we take $\mathcal{O}' = W(k')$. Then there exists a unique endomorphism (in fact, automorphism) $F$ of the ring $\mathcal{O}'$ such that $F$ induces the $p$th power automorphism of the field $k'$. Then for every algebraic variety $C$ over the field $k'$ that is properly embeddable over $\mathcal{O}'$, we have that the hypotheses of Example 2 hold, where $K' = q.f. \mathcal{O}'$, a finite extension of $\mathbb{Q}_p$ of degree $r$, where $r = [k': \mathbb{Z}/p\mathbb{Z}]$. Therefore by Example 2 we have the zeta matrices $W^h(C)$, square $\beta_h \times \beta_h$ matrices with coefficients in $K'$ (each unique up to $F'$-similarity, where $F'$ is the automorphism of $K'$ induced by $F$) and where $\beta_h = \dim_{K'} H^h_e(C, K')$, $0 \leq h \leq 2 \dim C$. The composite of the $p$th power endomorphism $x_C$ of $C$ as defined in Example 1 with itself $r$ times is the standard Frobenius endomorphism $f$ of the algebraic variety $C$ over the finite field $k'$, an (ordinary) map of algebraic varieties over $k'$, where $r$ is the degree of $k'$ over $\mathbb{Z}/p\mathbb{Z}$: $\alpha_c^r = f$. It follows readily that

$$
(W^h)^{(F')}^{r-1} \cdot (W^h)^{(F')}^{r-2} \cdots (W^h)^{(F')}^{h} \cdots (W^h)^{F'} \cdot W^h
$$

(1)

the matrix of the linear transformation $H^h_e(id_{\mathcal{O}'}, f)$ of the $\beta_h$-dimensional $K'$-vector space $H^h_e(C, K')$ into itself,

$0 \leq h \leq 2 \dim C$, where $W^h = W^h(C)$, $0 \leq h \leq 2 \dim C$. (This latter linear transformation is an ordinary linear transformation, not merely semi-linear, since $F' = \text{identity of } \mathcal{O}'$.) That is, the product matrix (1) is the matrix of the linear transformation of the $h$th lifted $p$-adic cohomology group with compact supports of $C$ into itself induced by the Frobenius mapping, and is unique up to (ordinary) similarity.

**Remark.** It follows readily from the formalism of Witt vectors, as developed in [7], that: In the situation of Example 1, suppose that (1) the natural epimorphism: $\mathcal{A} \to A = \mathcal{A} \otimes_{\mathcal{O}} k$ maps the ($p$-torsion part of $\mathcal{A}$) into $\{\text{nilpotent elements of } A = \mathcal{A} \otimes_{\mathcal{O}} k\}$; and that (2) the endomorphism $F^\dagger: \mathcal{A}^\dagger \to \mathcal{A}^\dagger$ induces the $p$th power endomorphism of $\mathcal{A}^\dagger/p\mathcal{A}^\dagger$. (For example, this is the case if either $\mathcal{O} = \mathbb{Z}_p$, $\mathcal{A} = W(A)$ or $W^{-}(A)$ and $F$ is the Witt "$F$"; or, if, as in Example 1.1 above, $\mathcal{A} = \mathcal{O}[T_1, \ldots, T_N]$, and if the given endomorphism $F_\mathcal{O}$ of $\mathcal{O}$ is such that $F_\mathcal{O} \otimes_{\mathcal{O}}(\mathbb{Z}/p\mathbb{Z})$ is the $p$th power endomorphism of the ring $\mathcal{O}/p\mathcal{O}$ (this is a bit stronger than the corresponding hypothesis in Example 1.1), and if, as in Example 1.1, $F: \mathcal{A} \to \mathcal{A}^\dagger$ is the unique ring homomorphism extending $F_\mathcal{O}$ such that $F(T_i) = T_i^p$,}
1 \leq i \leq N$. Then for every prime ideal \( \mathfrak{p} \in \text{Spec}(A) \) there is induced a natural unramified discrete valuation ring \( \mathcal{O}'_{\mathfrak{p}} \) of mixed characteristic such that the residue class field of \( \mathcal{O}'_{\mathfrak{p}} \) is \( k(\mathfrak{p})^{p^{-\infty}} \), the purely inseparable algebraic closure of the field \( k(\mathfrak{p}) \) (it follows that the completion of \( \mathcal{O}'_{\mathfrak{p}} \) is naturally isomorphic to the Witt vectors on \( k(\mathfrak{p})^{p^{-\infty}} \), together with an endomorphism \( F_{\mathfrak{p}} \) of \( \mathcal{O}'_{\mathfrak{p}} \) that induces the \( p \)-th power endomorphism of \( k(\mathfrak{p}) = k(\mathfrak{p})^{p^{-\infty}} \), and such that we have a natural mapping of rings: \( A \to \mathcal{O}'_{\mathfrak{p}} \) that commutes with \( F \) and \( F_{\mathfrak{p}} \); and then the zeta endomorphisms of the algebraic family \( C \) over \( \text{Spec}(A) \) for all integers \( h \) determine, essentially, the \( h \)-th zeta endomorphism of the algebraic variety: \( C'_{\mathfrak{p}} = (C \times_A k(\mathfrak{p})^{p^{-\infty}})_{\text{red}} \) over the field \( k(\mathfrak{p})^{p^{-\infty}} \) (an endomorphism of the \( h \)-th lifted \( p \)-adic cohomology group with compact supports: \( H^{h}_{\text{c}}(C'_{\mathfrak{p}}, (\mathcal{O}'_{\mathfrak{p}})^{\dagger} \otimes_{\mathbb{Z}} \mathbb{Q}) \) of the algebraic variety \( C'_{\mathfrak{p}} \) over \( k(\mathfrak{p})^{p^{-\infty}} \), for each integer \( h \), all prime ideals \( \mathfrak{p} \subseteq A \). Also, in my seminar at Harvard in 1969–1970 it was shown that if, e.g., \( C \) is simple and proper over \( \text{Spec}(A_{\text{red}}) \) and liftable over \( A \), then \( H^{h}_{\text{c}}(C, (\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}) \) is projective and finitely generated as \((\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}\)-module. Therefore in this case the \( h \)-th zeta endomorphism of \( H^{h}_{\text{c}}(C, (\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}) \) can be expressed by a square matrix with coefficients in \((\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}\), unique up to \((F^{\dagger} \otimes_{\mathbb{Z}} \mathbb{Q})\)-similarity, which we called the \( h \)-th zeta matrix of the algebraic family \( C \) over \( \text{Spec}(A_{\text{red}}) \) with coefficients in \((\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}\), and denote by \( W^{h}(C) \), all integers \( h \). In this case, the zeta matrix \( W^{h}(C) \) of the algebraic family \( C \) over \( \text{Spec}(A) \) determines the \( h \)-th zeta matrix \( W^{h}(C'_{\mathfrak{p}}) \) of the algebraic variety \( C'_{\mathfrak{p}} \) over \( k(\mathfrak{p})^{p^{-\infty}} \) simply by throwing the coefficients of \( W^{h}(C) \) through the homomorphism: \( A \to \mathcal{O}'_{\mathfrak{p}} \), all integers \( h \), all prime ideals \( \mathfrak{p} \subseteq A \). Thus, the zeta matrices of such an algebraic family determine the zeta matrices of each of the varieties in the family.)

The following completely proved theorem is a generalization of the first Weil Conjecture [8], “Lefschetz Theorem.”

**Theorem 2 (Generalized Weil's Lefschetz Theorem Conjecture).** Let \( k' \) be a finite field and let \( C \) be an algebraic variety over \( k' \) that is properly embeddable over \( \mathcal{O}' = W(k) \). Then the zeta function \( Z_{C}(T) \) can be written as an alternating product:

\[
Z_{C}(T) = \frac{P_{1}(T) \cdot P_{3}(T) \cdot P_{5}(T) \cdots P_{2d-1}(T)}{P_{0}(T) \cdot P_{2}(T) \cdot P_{4}(T) \cdots P_{2d}(T)}
\]

(2)

2 In the applications below, a weaker condition than (2) that suffices in lieu of (2) is:

(2') \( F \) induces the \( p \)-th power endomorphism of \( A_{\text{red}} \); and \( x \in A, \ j \geq 1, \ p \mid x^{j}, \) implies that there exists \( i \geq 0 \) such that \( p \mid (F^{\dagger})^{i}(x) \) in \( A^{\dagger} \).

3 If \( H^{h}_{\text{c}}(C, (\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}) \) is not a free module, then choose \( M^{h} \) an \((\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}\)-module such that \( N^{h} = H^{h}_{\text{c}}(C, (\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus M^{h} \) is free of finite rank; and extend the zeta endomorphism \( \zeta^{h}(C) \) to the free \((\mathbb{A}_{\mathfrak{p}}^{\dagger}) \otimes_{\mathbb{Z}} \mathbb{Q}\)-module of finite rank \( N^{h} \) by requiring that the extension be zero on \( M^{h} \).
when \( d = \dim C \), and where

\[
P_h(T) = \det(f_h - T),
\]

(3)

the reverse characteristic polynomial of \( f_h \), where \( f_h \) is the map induced by the Frobenius endomorphism of \( C \) on the \( \beta_h \)-dimensional \( K' \) vector space \( H_h(C, K') \), the \( h \)th lifted homology group with compact supports of \( C \), and where \( f_h \) is the identity endomorphism of \( H_h(C, K') \), \( 0 \leq h \leq 2d \). In particular, \( P_h \) is a polynomial of degree \( \beta_h \) with coefficients in \( K' \) and with constant term 1, \( 0 \leq h \leq 2d \). (The fact that \( P_h \) has degree \( \beta_h \) is equivalent to the fact that the map \( f_h \) induced by the Frobenius on the \( h \)th homology group with compact supports is injective, and this is proved by arguing using the perfection \([7]\) \( C^{\text{perf}} \) of \( C \) in a manner similar to the analogous assertion proved in \([9]\).)

**Sketch of proof.** Case 1. \( C \) is a non-singular affine hypersurface, of the type \( C = \text{Spec}(k'[T_1, \ldots, T_{d+1}, g^{-1}]/(H)) \), \( d = \dim C \), where \( g, H \in k'[T_1, \ldots, T_{d+1}] \). Assume also that \( T_1, \ldots, T_{d+1} g \). Then let \( g, H \in \mathcal{O}'[T_1, \ldots, T_{d+1}] \) be elements that map into \( g, H \) and let \( \gamma = \text{Spec}(\mathcal{O}'[T_1, \ldots, T_{d+1}, g^{-1}]/(H)) \).

Then by definition (see Section 1) we have that

\[
H_h(C, K') = H^{2d-h}(C, (T_{0}^{*}(C) \otimes_{\mathbb{Z}} \mathbb{Q})), \quad 0 \leq h \leq 2d.
\]

(4)

Since \( C \) is affine, by \([3, II.3.1.1, p. 190]\), we have that these latter groups are isomorphic to the cohomology of the global sections cochain complex, call it \( (C^{*})^\dagger \), of \((T_{0}^{*}(C) \otimes_{\mathbb{Z}} \mathbb{Q}) \) over \( \gamma \),

\[
H_h(C, K') \approx H^{2d-h}(C^{*}), \quad \text{all integers } h.
\]

(4')

The group on the right side of Eq. (4) is the \((2d - h)\)th lifted \( p \)-adic cohomology group of \( C \) \([6]\) and under the isomorphism (4) the endomorphism \( f_h \) induced by the Frobenius of \( C \) on the left side of (4) corresponds to \( p^{rd-1} f_{2d-h} \), where \( f_{2d-h} \) is the endomorphism of the right side of Eq. (4), induced by the Frobenius mapping.

The key diagram is:

\[
\begin{align*}
H^{2d-h}(C) & \xrightarrow{\text{proj}} H^{2d-h}(C \times k', A) \cup \nu_{U_{T_{0}}} H^{2d-h}(C \times k', A, C \times k', A - \Gamma_{i}) \\
& \xrightarrow{f_{C \times k', \text{id}, A}} H^{2d-h}(C \times k', A) \cup \nu_{U_{T_{0}}} H^{2d-h}(C \times k', A, C \times k', A - \Gamma_{i}) \\
& \approx_{\text{exc.}} H^{2d-h}(C \times k', A, C \times k', A - \Gamma_{i}) \xrightarrow{\text{rest.}} H^{2d-h+2}(\mathcal{R} \times k', (A, A - C)) \\
& \xrightarrow{(f_{\mathcal{R} \times k', A})^{2d-h+2}} H^{2d-h+2}(\mathcal{R} \times k', A, C \times k', A - \Gamma_{i}) \xrightarrow{\text{rest.}} H^{2d-h+2}(\mathcal{R} \times k', (A, A - C)) \\
\end{align*}
\]
where $A = \text{Spec}(k'[T_1, \ldots, T_{d+1}, g^{-1}])$, $\overline{X}$ = closure of $C$ in $P^{d-1}(k')$, $\Gamma_\nu$ is the graph of the inclusion $\nu: C \to A$, $\Gamma_0$ is the graph of the composite, $f_0$, of the Frobenius $f_C$ of $C$ with the inclusion $\nu: C \to A$, $P = \mathbb{P}^d(k')$ and $f_X$, resp: $F_p$, is the Frobenius endomorphism of $\overline{X}$, resp: $P$, over $k'$. The composite of the top row of this diagram is $H^{d-h}_h(C, K')$, which is the identity of $H^{d-h}_h(C, K')$. The composite of the bottom row is $f_h = H^{d-h}_h(f, C, K')$. The leftmost vertical map is $j^\circ$, and the next-to-the-rightmost vertical map is $(f_p \times k' \text{id}_A)x_{d-h+2}$. Since $f^{2d-h}_p = (\text{multiplication by } p^{rd})$, considering the maps of Corollary 4.1 of Section I, commutativity of the diagram tells us that $$(f_C) \circ f^{2d-h}_C = (\text{multiplication by } p^{rd}).$$

as required.

Therefore, $P_h = (\text{reverse characteristic polynomial of the endomorphism } p^{rd}(f^{2d-h})^{-1} \text{ of } H^{2d-h}(C^\nu))$, all integers $h$. But, in his original proof of “rationality of the zeta function” [10] (we can choose $H \in \mathbb{C}[T, \ldots, T_{d+1}]$ so that its coefficients are Teichmuller representatives) Dwork proves that, if $Q^{2d-h}$ is the reverse characteristic polynomial induced by $p^{rd}(f^{2d-h})^{-1}$ on $C^{2d-h}$ (in the sense of characteristic polynomials of endomorphisms of $p$-adic Banach spaces), then

$$Z_x(T) = \frac{Q_1 \cdot Q_3 \ldots Q_{2d-1}}{Q_0 \cdot Q_2 \ldots Q_{2d}}. \quad (5)$$

Since given a short exact sequence

$$0 \to B' \to B \to B'' \to 0$$

of $p$-adic Banach spaces, and an endomorphism of $B$ that maps $B'$ into itself, then the reverse characteristic polynomial $P_h$ of the endomorphism of $B$ is the product: $P_{B'} \cdot P_{B''}$ of the reverse characteristic polynomials $P_{B'}$, $P_{B''}$ of the endomorphisms induced on $B'$ and on $B''$, Eqs. (4') and (5) prove Eq. (2) (since the alternating product of the reverse characteristic polynomials on the cohomology is the alternating product of the reverse characteristic polynomials on the cochains).

Case 2. General case. If $U$ is a dense open subset of $C$, then we have (see Proposition 7 of Section 1) the long exact sequence:

$$\cdots \xrightarrow{\partial_{h+1}} H^{d}_h(C - U) \xrightarrow{\partial_h} H^{d}_h(C) \xrightarrow{\text{restriction}} H^{d}_h(U) \xrightarrow{\partial_h} H^{d}_h(C - U) \to \cdots. \quad (6)$$
Since dimension of $C - U < \dim C$, if we proceed by induction on $d = \dim C$, then the assertion is true for $C$ iff true for $U$. Therefore the inductive assumption implies that the verity of the assertion for $C$ depends only on the birational equivalence class of $C$. Since every irreducible algebraic variety is birationally equivalent to a non-singular affine hypersurface as in Case 1, we are therefore through. Q.E.D.

Remark. In the case that $C$ is simple, proper and liftable, the above theorem (which is then exactly the original Weil Lefschetz Theorem Conjecture [8] for $p$-adic cohomology) was first proved in [3] by a somewhat different method.

Remark 1. Let us return to the situation described in the Remark following Example 3. That is, let $\mathcal{O}$ be a complete discrete valuation ring of mixed characteristic, and let $\mathcal{A}$ be an $\mathcal{O}$-algebra such that condition (1) of that Remark holds, where $p = \text{char } k$, $k = k(\mathcal{O})$, and such that we have a ring homomorphism $F: \mathcal{A} \to \mathcal{A}^\dagger$, mapping the image of $\mathcal{O}$ into itself, such that $\mathcal{A}^\dagger$ is integral over $(F^\dagger)(\mathcal{A}^\dagger)$ (where $F^\dagger$ denotes the extension of $F$); and such that, condition (2) of that Remark holds—i.e., $F$ induces the $p$th power endomorphism of $\mathcal{A}/p\mathcal{A}$ (or, weaker, and such that the more technical condition (2') in the footnote to that Remark holds). Let $A = \mathcal{A} \otimes_\mathcal{O} k$. Then $A_{\text{red}} \approx (A/pA)_{\text{red}}$. Then, by the Remark following Example 3, for every prime ideal $\mathfrak{p} \in \text{Spec}(A)$, we have the discrete valuation ring $\mathcal{O}'_{\mathfrak{p}}$ of mixed characteristic such that $k(\mathcal{O}'_{\mathfrak{p}}) \approx k(\mathfrak{p})^{p^{\infty}}$. Let $K_{\mathfrak{p}} = (\mathcal{O}'_{\mathfrak{p}})^{\dagger} \otimes_\mathcal{Z} \mathcal{Q}$, a field of characteristic zero, for all $\mathfrak{p} \in \text{Spec}(A)$. Then let $\mathcal{C}$ be an algebraic family over $\text{Spec}(A_{\text{red}})$ such that $(\mathcal{A}, \mathcal{C})$ is an object in $\text{V}_{\text{Fr}}$ normal (or in $\text{V}_{\text{Fr}}$) (it is equivalent to say, "such that $\mathcal{A}$ is normal and such that $\mathcal{C}$ is properly embeddable over $\mathcal{A}$ (or no condition on $\mathcal{A}$, but such that $\mathcal{C}$ is polynomially properly embeddable over $\text{Spec}(A_{\text{red}})$"). Then the universal coefficients spectral sequence, Eq. (26) of Section 1 applies, and is of the form

$$E_{2}^{p,q} = \text{Tor}_{p}^{(\mathcal{A}^\dagger) \otimes_\mathcal{Q}}(H^{q}(\mathcal{C}, (\mathcal{A}^\dagger) \otimes_\mathcal{Z} \mathcal{Q}), K_{\mathfrak{p}}) \Rightarrow H^{q}(\mathcal{C}_{\mathfrak{p}}, K_{\mathfrak{p}}),$$

all prime ideals $\mathfrak{p} \subset A = (\mathcal{A} \otimes_\mathcal{O} k)$. This is a cohomological spectral sequence of vector spaces over the field of characteristic zero $K_{\mathfrak{p}}$, confined to the region $p \leq 0$, $q \geq 2N - M$ (where $M$ is an integer such that $H^{i}(X, X - C, T_{\mathcal{A}}^{\mathcal{O}^{p}}(X)^{\dagger} \otimes_\mathcal{Q} \mathcal{Z}) = 0$ for $i \geq M + 1$, where $X$ is such that $C$ is closed in $X$, and $X$ is simple and of finite presentation over $\text{Spec}(A)$ and such that all the connected components of fibers of $X = X \times_\mathcal{O} k$ over $\text{Spec}(A)$ are all of the same dimension $N$), a region that resembles the second quadrant. By Section 2, the groups in the abutment are finite dimensional over $K_{\mathfrak{p}}$. The zeta endomorphisms $\zeta^{q}(\mathcal{C})$ of $H_{\mathcal{C}}^{q}(\mathcal{C}, (\mathcal{A}^\dagger) \otimes_\mathcal{Z} \mathcal{Q})$, together with the endomorphism $F_{\mathcal{A}}^{\dagger} \otimes_\mathcal{Z} \mathcal{Q}$ (call this endomorphism simply "$F_{\mathcal{A}}^\dagger$") of the field of characteristic zero $K_{\mathfrak{p}}$, induces an endomorphism of the spectral sequence
of $K_{\mathfrak{p}}$-vector spaces (a) semi-linear with respect to the endomorphism $F_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$, all prime ideals $\mathfrak{p} \subset A$. Therefore, the zeta endomorphisms $\zeta^h(C)$, $h \geq 2N - M$, of the algebraic family $C$ over $\text{Spec}(A)$, determine the induced $F_{\mathfrak{p}}$-semi-linear endomorphism of $E_{2}^{p,q}$ of the spectral sequence (a) and therefore determine the $F_{\mathfrak{p}}$-semi-linear endomorphism of $E_{\infty}^{p,q}$. Notice that the $F_{\mathfrak{p}}$-semi-linear endomorphism of the abutment of the spectral sequence (a) is simply the zeta endomorphisms of the algebraic variety $C_{\mathfrak{p}}$ over the field $k(\mathfrak{p})$. It follows that, more or less, the zeta endomorphisms: $\zeta^h(C)$ of the algebraic family $C$ essentially determine the zeta endomorphisms of the algebraic variety $C_{\mathfrak{p}} = C \times_{\text{Spec}(A)} \text{Spec}(k(\mathfrak{p}^{\infty}))$, all $\mathfrak{p} \in \text{Spec}(A)$.

For example, consider the special case in which $\mathfrak{p} \in \text{Spec}(A)$ is such that the field $k(\mathfrak{p})$ is finite. Then the zeta function $Z_{C_{\mathfrak{p}}}(T)$ of the algebraic variety $C_{\mathfrak{p}} = C \times_{\text{Spec}(A)} \text{Spec}(k(\mathfrak{p}))$ makes sense, and is determined by the polynomials $P_{h}$, $0 \leq h \leq 2 \dim C$. $P_{h}$, in turn, is the reverse characteristic polynomial of the $h$th Frobenius endomorphism, which is the $r$th iterate of the $h$th zeta endomorphism of the algebraic variety $C_{\mathfrak{p}}$ over $k(\mathfrak{p})$. If we take the $r$th iterate of the $F_{\mathfrak{p}}$-semi-linear endomorphism of the spectral sequence (a), then this iterate gives an ordinary $K_{\mathfrak{p}}$-linear transformation of each of the groups in the spectral sequence (a). Since the abutment is finite dimensional over $K_{\mathfrak{p}}$, so is $E_{\infty}^{p,q}$. If we let $P_{p,q}$ be the reverse characteristic polynomial of the $K_{\mathfrak{p}}$-linear transformation of $E_{\infty}^{p,q}$ given by this $r$th iterate, then we have

$$P_{h}(T) = \prod_{p+q=h, p \leq 0 \atop q \geq 2N-M} P_{p,q}(T),$$

all integers $h$, $0 \leq h \leq 2 \dim C_{\mathfrak{p}}$, (the product in (b) being finite since for all but finitely many pairs of integers $p, q \in \mathbb{Z}$, we have $P_{p,q}(T) = 1$) and therefore by the Lefschetz theorem (Theorem 2 above), we have that

$$Z_{C_{\mathfrak{p}}}(T) = \frac{\prod_{p+q \text{ odd}} P_{p,q}(T)}{\prod_{p+q \text{ even}} P_{p,q}(T)},$$

where $p, q$ run through the (finite) set of all pairs of integers such that $0 \leq p + q \leq 2 \dim C$, $p \leq 0$, $q \geq 2N - M$, $E_{\infty}^{p,q} \neq 0$. Therefore, in this case (namely, in the case that the prime ideal $\mathfrak{p}$ is such that $k(\mathfrak{p})$ is finite), the zeta endomorphisms: $\zeta^h(C)$, $h \in \mathbb{Z}$ of the algebraic family $C$ over $\text{Spec}(A)$ determine completely the zeta function of each such specific algebraic variety $C_{\mathfrak{p}}$ in the family $C$.

A further special case is as follows: Supposing as above that $\mathfrak{p} \in \text{Spec}(A)$ is such that the field $k(\mathfrak{p})$ is finite, let us also suppose that $E_{\infty}^{p,q}$ in the spectral sequence (a) is such that: $E_{\infty}^{p,q}$ is finite dimensional over $K_{\mathfrak{p}}$, all $p, q \in \mathbb{Z}$, and such that $E_{\infty}^{p,q} = 0$ for all but finitely many pairs $(p, q)$ of integers.
Then let $r = [k(\phi); (\mathbb{Z}/p\mathbb{Z})]$, and let $\varphi_{p,q}$ be the reverse characteristic polynomial of the $K_\phi$-linear transformation, induced by the $r$th iterate of the $qth$ zeta endomorphism: $\xi^q(C)$, of the $K_\phi$-vector space:

$$\text{Tor}_p^{(d+1)\otimes \mathbb{Q}}(H_\phi^q(C, (\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}), K_\phi).$$

Then by Eq. (c), and the formalism of spectral sequences, in this special case (the one in which $k(\phi)$ is finite, and in which $E_2^{p,q}$ of the spectral sequence (a) is finite dimensional over $K_\phi$, and is such that $E_2^{p,q} = 0$ for all but finitely many pairs of integers $(p, q)$), we deduce that

$$Z_{C_{\phi}}(T) = \prod_{p+q \text{ odd}} \varphi_{p,q}(T) \prod_{p+q \text{ even}} \varphi_{p,q}(T),$$

where $(p, q)$ runs through the set of all pairs of integers such that $E_2^{-p,q} \neq 0$.

(Notice that Eqs. (b) and (c) “always” hold, in the sense that they hold whenever $Z_{C_{\phi}}(T)$ makes sense, i.e., whenever $k(\phi)$ is finite; but that for Eq. (d) to hold, or even to make sense, one requires a strong auxiliary finiteness assumption that definitely does not always hold.)

**Example 4.** As an example of the considerations in the Remark following Example 3, let us compute the zeta function of the typical elliptic curve: $Y^2 = 4X^3 - g_2X - g_3$, where $g_2^3 - 27g_3^2 \neq 0$, and $g_2, g_3$ lie in a finite field of characteristic $\neq 2, 3$, as a function of $g_2$ and $g_3$. Let $p$ be a fixed positive rational prime $\neq 2, 3$. Let $k = \mathbb{Z}/p\mathbb{Z}$, $\mathfrak{C} = \mathbb{Z}_p$, $K = \mathbb{Q}_p$. Let $A$ be the ring $k[g_2, g_3, \Delta^{-1}]$ where $\Delta = g_2^3 - 27g_3^2$. Then Spec($A$) is the open subset of Euclidean two space over $k$, “$g_2^3 - 27g_3^2 \neq 0.”$ Let $C$ be the closed subset of $\mathbb{P}^2(k) \times_k$ Spec($A$) given by the homogeneous equation of degree three: $-Y^2Z + 4X^3 - g_2XZ^2 - g_3Z^3 = 0$, in which the homogeneous coordinates in $\mathbb{P}^2(k)$ are $X$, $Y$, and $Z$. Then $C$ is an algebraic family over Spec($A$), and is simple and proper over Spec($A$). For every pair $(g_2^{(0)}, g_3^{(0)})$ of elements of a field $K$ of characteristic $p$ such that $(g_2^{(0)})^3 - 27(g_3^{(0)})^2 \neq 0$, let $\mathfrak{p} = \mathfrak{p}_{g_2^{(0)}, g_3^{(0)}}$ denote the prime ideal in the ring $A$ that is the kernel of the substitution homomorphism: $(\mathbb{Z}/p\mathbb{Z})[g_2, g_3] \rightarrow K$ mapping $g_2$ into $g_2^{(0)}$ and $g_3$ into $g_3^{(0)}$. Then the fiber $C_{\phi}$ of $C$ over $\mathfrak{p}$ is the projectivization of the affine curve: $Y^2 = 4X^3 - g_2^{(0)}X - g_3^{(0)}$. The algebraic family $C$ is called the Weirstrass family; its fibers include all elliptic curves over all fields of the fixed characteristic $p$.

Let $\mathcal{A} = \mathbb{Z}_p[g_2, g_3, \Delta^{-1}]$, where $\Delta = g_2^3 - 27g_3^2$. Then $\mathcal{A} \otimes_k k \approx A$ (and the scheme $C$ over Spec($A$) admits the lifting $C'$ over Spec($A$), where $C'$ is the closed subscheme of $\mathbb{P}^2(\mathbb{Z}_p) \times_{\text{Spec}(\mathbb{Z}_p)}$ Spec($A$) given by the homogeneous equation $-YZ^2 + 4X^3 - g_2XZ^2 - g_3Z^3 = 0$ (where the homogeneous coordinates in $\mathbb{P}^2(\mathbb{Z}_p)$ are $X$, $Y$, and $Z$) (–although, we do not have to use the lifting $C'$ or any assumptions of “liftability’’)).
Then in Example 1.1, we have proved that there exists a unique homomorphism of $\mathcal{D}$-algebras $F: \mathcal{D} \to \mathcal{D}^{\dagger}$ such that $F(g_2) = g_2^\prime$, $F(g_3) = g_3^\prime$.

Then, $(\mathcal{D}, C)$ is an object in the category $\mathcal{C}_p^{\text{normal}}$ (C is polynomially properly embeddable over $\mathcal{D}$, e.g., since $C$ is quasiprojective over $\text{Spec}(\mathcal{D})$.

Another reason that $C$ is properly embeddable over $\mathcal{D}$ is that $C$ is simple and proper over $\text{Spec}(\mathcal{D})$, and admits the flat lifting $C$ over $\mathcal{D}$, and the pair $(F, \alpha_C)$ is a map in $\mathcal{C}_p^{\text{normal}}$. Therefore, by Example 1, we have the induced zeta endomorphisms of the lifted $p$-adic cohomology groups with compact supports: $H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$, all integers $h$. Since $C$ is simple and proper over $\text{Spec}(\mathcal{D})$, and liftable over $\text{Spec}(\mathcal{D})$, by the observations in the Remark following Example 3, we have that $H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$ is projective and finitely generated as $\mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q}$-modules, for each integer $h$, and vanish unless $0 \leq h \leq 2$ (since the dimensions of the fibers of $C$ over $\text{Spec}(\mathcal{D})$ are all one).

Therefore, by the universal coefficients spectral sequences, Eq. (25) of Section 1, it follows that

$$H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q}) \approx \text{Hom}_{\mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q}}(H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q}), \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q}),$$

all integers $h$, and therefore vanishes unless $0 \leq h \leq 2$, and are projective finitely generated $(\mathcal{D}^{\dagger}) \otimes \mathbb{Z} \mathbb{Q}$-modules for all integers $h$.

Since $C$ is simple over $\text{Spec}(\mathcal{D})$, and admits the simple separated lifting of finite presentation $C$ over $\text{Spec}(\mathcal{D})$, it follows by definition that

$$H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q}) \approx H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \Gamma_{\mathcal{D}^{\dagger}}(C) \otimes \mathbb{Z} \mathbb{Q}),$$

all integers $h$. These latter groups are easily explicitly computed, and turn out to be free of finite rank. Namely, $H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$ are free as $(\mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$-modules of ranks 1, 2, and 1 for $h = 0, 1, 2$, respectively. It follows from the universal coefficients theorem, Eq. (25) of Section 1, that $H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$ are free as $\mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q}$-modules of ranks 1, 2, and 1 for $h = 0, 1, 2$, respectively.

By Example 1, the pair $(F, \alpha_C)$, where $\alpha_C$ is the $p$th power map of $C$, induces an endomorphism of $H^h_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$, $0 \leq h \leq 2$, semi-linear with respect to the endomorphism $(F^{\dagger}) \otimes \mathbb{Z} \mathbb{Q}$ of $(\mathcal{D}^{\dagger}) \otimes \mathbb{Z} \mathbb{Q}$, the $h$th zeta endomorphism of the Weierstrass family $C$ over $\text{Spec}(\mathcal{D})$. These are trivial to compute for $h = 0$ or 2; for $h = 1$, it is an $(F^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$-linear endomorphism of the free $(\mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$-module of rank two, $H^1_\mathcal{D}(C, \mathcal{D}^{\dagger} \otimes \mathbb{Z} \mathbb{Q})$. Putting all of this together, we deduce the following

**Theorem.** Let $p$ be a fixed positive rational prime $\neq 2, 3$. Then there exists a $2 \times 2$ matrix (which is the first zeta matrix in the sense of Example 1
of the Weierstrass family in the fixed characteristic $p$) with coefficients in $A^\uparrow \otimes \mathbb{Z} \mathbb{Q}$. $W^1(C) \in M_{2,2}(A^\uparrow \otimes \mathbb{Z} \mathbb{Q})$, say

$$W^1(C) = \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix},$$

where $A^\uparrow = \{ \sum_{i,j,k \geq 0} \alpha_{i,j,k} g_2^i g_3^j \Delta^{-k}; \alpha_{i,j,k} \in \mathbb{Z}_p \}$, and such that there exists $\epsilon > 0$ such that $\text{ord}_{\rho}(\alpha_{i,j,k}) \geq \epsilon(i + j + k)$ for all but finitely many triples $i, j, k \geq 0$ (the ideal generated by the single relation: $1 = \Delta^{-1}(g_2^3 - 27g_3^3)$); such that,

(a) If $k_0$ is any, for simplicity perfect, field of characteristic $p$, and if $g_2^{(0)}, g_3^{(0)} \in k_0$ are any elements such that $(g_2^{(0)})^3 - 27(g_3^{(0)})^3 \neq 0$, then the first zeta matrix of the elliptic curve $C_{g_2^{(0)}, g_3^{(0)}}$ (that is, the projective completion over $k_0$ of the affine curve given by the equation: $Y^2 = 4X^3 - g_3^{(0)}X - g_2^{(0)}$) is determined as follows. Namely, let $g_{2,1}, g_{3,1} \in k_0$ in the Witt vectors $W(k_0)$ of $k_0$. Then the first zeta matrix of the curve $C_{g_2^{(0)}, g_3^{(0)}}$, a two-by-two matrix with coefficients in $W(k_0) \otimes \mathbb{Z} \mathbb{Q}$, is the matrix:

$$W_1^{g_{2,1}, g_{3,1}} = \begin{bmatrix} f_{11}(g_{2,1}) & f_{21}(g_{2,1}, g_{3,1}) \\ f_{12}(g_{2,1}, g_{3,1}) & f_{22}(g_{2,1}) \end{bmatrix} \in M_{2,2}(W(k_0) \otimes \mathbb{Z} \mathbb{Q}),$$

where

$$W_1^{g_{2,1}, g_{3,1}} = \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix} \in M_{2,2}(A^\uparrow \otimes \mathbb{Z} \mathbb{Q}).$$

(b) In (a) above, suppose that the field $k_0$ is finite; so that $C_{g_2^{(0)}, g_3^{(0)}}$ is the most general elliptic curve over a finite field of the fixed characteristic $p$. Then the zeta function $Z_{g_2^{(0)}, g_3^{(0)}}(T)$ of the general elliptic curve $C_{g_2^{(0)}, g_3^{(0)}}$ is:

$$Z_{g_2^{(0)}, g_3^{(0)}}(T) = \frac{P_1(T)}{(1 - T)(1 - p^r T)},$$

where $p^r = \#(k_0)$, and where

$$P_1(T) = \text{det}(I - U \cdot T),$$

where $U = (W_1^{g_{2,1}, g_{3,1}})^{p^{r-1}} \cdots (W_1^{g_{2,1}, g_{3,1}})^{p} (W_1^{g_{2,1}, g_{3,1}})$, where $W_1^{g_{2,1}, g_{3,1}}$ is as in (a) above. (And where "$B^F$" denotes the matrix obtained by throwing all the coefficients of $B$ through $F$, all $B \in M_{2,2}(W(k_0))$).

Remarks

1. The hypotheses being as in Example 4, part (b), above, we can describe explicitly the Witt vectors on the finite field $k_0$ of characteristic $p$, where $\#(k_0) = p^r$, as follows. Let $F_{p,r}$ be the cyclotomic extension of the
field \( \mathcal{Q}_p \) of \( p \)-adic numbers formed by adjoining a primitive \(( p^r - 1) \)st root of unity, \( \xi_{p^r-1} \). Then the Witt vectors \( W(k_0) \) of the ring \( k_0 \) are isomorphic to the \( \hat{\mathbb{Z}}_p \)-subalgebra of \( F_{p^r} = \mathcal{Q}_p[\xi_{p^r-1}] \) generated by \( \xi_{p^r-1} \),

\[
W(k_0) = \hat{\mathbb{Z}}_p[\xi_{p^r-1}].
\]

And \( W(k_0) \) is free as \( \hat{\mathbb{Z}}_p \)-module of rank \( r \), with basis \( 1, \xi_{p^r-1}, \xi_{p^r-1}^2, \ldots, \xi_{p^r-1}^{p^r-1} \). The Witt automorphism \( F \) of the ring \( W(k_0) \) is the unique \( \hat{\mathbb{Z}}_p \)-algebra automorphism such that \( F(\xi_{p^r-1}) = \xi_{p^r-1}^p \). Also, for every element \( g \in k_0 \) (e.g., \( g = g_2^{(0)} \) or \( g_3^{(0)} \)), the Teichmuller representative \( g' \) of \( g \) can be described explicitly as follows: Choose a generator \( \xi \) for the multiplicative group \( (k_0)^* \) of \( k_0 \). Then every non-zero element of \( k_0 \) can be written uniquely as \( \xi^i \), \( 0 \leq i \leq p^r - 1 \). Then if \( g = \xi^i \), the Teichmuller representative \( g' \) of \( g \) is the element \( g' = (\xi_{p^r-1})^i \in W(k_0) \), \( 0 \leq i \leq p^r - 1 \); while if \( g = 0 \), then \( g' = 0 \) in \( W(k_0) \).

2. The hypotheses as in Example 4, part (b), it is easy to see that

\[
P_1(T) = 1 - aT \pm p^rT^2,
\]

where \( a \) is an integer of absolute value \( \leq p^r \). Therefore, to compute \( P_1(T) \), and therefore also \( Z_{\omega',\omega}(T) \), explicitly, it actually suffices to know the coefficients of \( P_1(T) \) modulo \( p^r \). Therefore, one actually has to add up only finitely many terms in the four \( p \)-adic convergent power series:

\[
f_{11}, f_{12}, f_{21}, f_{22} \in A^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

In fact, it is easy to compute just how quickly \( f_{11}, f_{12}, f_{21}, f_{22} \) converge \( p \)-adically, and one can therefore write down explicitly, as a function of \( r \), exactly how many terms one must add to determine \( P_1 \) (mod \( p^r \)), and therefore to determine completely \( P_1 \) and \( Z_{\omega',\omega}(T) \). (It is a linear function of \( r \)).

3. My Ph.D. student, Goro Kato, has computed a recursive formula for \( \alpha_{ijk}^{11}, \alpha_{ijk}^{12}, \alpha_{ijk}^{21}, \alpha_{ijk}^{22} \in \hat{\mathbb{Z}}_p \), such that

\[
f_{a,b} = \sum_{i,j,k \geq 0} \alpha_{ijk}^{a,b} g_2^i g_3^j \Delta^{-i-k} \quad \text{in} \ A^\dagger,
\]

\( 1 \leq a, b \leq 2 \) (where \( \Delta = g_2^3 - 27g_3^2 \)), for his Ph.D. thesis. (He will publish when he makes it elegant enough.)

4. The zeta matrices of the Weirstrass family, as described in Example 4 above, have coefficients in \( A^\dagger \otimes_{\mathbb{Z}} \mathbb{Q} \). In fact, they actually have coefficients in \( A^\dagger \). This follows from a general observation, that: the hypotheses being as in Theorem 6 of Section 1, if \( C \) is simple over \( \text{Spec}(A) \), with fibers all of the same dimension \( n \) over \( \text{Spec}(A) \), then there exist \textit{lifted} \( p \)-adic cohomology groups of \( C \) with coefficients in \( A^\dagger, H^n(C, A^\dagger) \), such that \( H^*\text{Tor}(C, A^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \cong H^{2n-h}(C, A^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q}, \) all integers \( h \). Notice that this (ordinary, no supports)
cohomology theory has \( \otimes_{\mathbb{Z}} \mathbb{Q} \). This is done in my book, "Lifted p-Adic Cohomology," to appear. There, if \( \mathcal{O} \) is a c.d.v.r. with quotient field of characteristic zero, such that the maximal ideal is generated by the rational prime \( p \), if \( \mathcal{A} \) is any \( \mathcal{O} \)-algebra and if \( A = \mathcal{A} \otimes_{\mathcal{O}} k \), then whenever \( C \) is a prescheme over \( \text{Spec}(\mathcal{A}) \), such that there exists \( X \) over \( \text{Spec}(\mathcal{A}) \) such that \( C \) is closed in \( X \), where \( X \) is simple of finite presentation, but not necessarily separated, over \( \text{Spec}(\mathcal{A}) \), and where \( X \) admits a flat lifting of finite presentation over \( \text{Spec}(\mathcal{A}) \) (a \( C \) obeying this condition is called embeddable over \( \mathcal{A} \)), then for all integers \( h \) and all open subsets \( U \) of \( C \) we define in [6] the lifted p-adic cohomology \( H^h(C, U, \mathcal{A}^\dagger) \) of \( C \) modulo \( U \) with coefficients in \( \mathcal{A}^\dagger \). (If also \( C_{\text{red}} \subset \mathcal{O}_{\mathcal{A}, \mathcal{A}} \), and \( X \) is as above, then for all integers \( h \) there is induced a canonical isomorphism of \( (\mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q} \)-modules,
\[
H_h^c(C, \mathcal{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}) \cong H^{2n-h}(X, X - C, \mathcal{A}^\dagger) \otimes_{\mathbb{Z}} \mathbb{Q},
\]
where the lifted p-adic homology groups with compact supports on the left are as defined in Proposition 3 and Theorem 6 of Section 1 of this paper, and the groups on the right (which can be defined for any \( C \) embeddable over \( \mathcal{A} \), whether properly embeddable over \( \mathcal{A} \) or not) are as defined in the book "Lifted p-Adic Cohomology" [6], to appear.)

5. The observations in Example 4 above, of course, generalize to other familiar algebraic families than the Weirstrass family.

6. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \mathcal{O} \)-algebras, let \( A = \mathcal{A} \otimes_{\mathcal{O}} k \), \( B = \mathcal{B} \otimes_{\mathcal{O}} k \), and suppose that we have \( F \) a homomorphism of rings from \( B \) into \( \mathcal{A}^\dagger \) (mapping the image of \( \mathcal{O} \) into the image of \( \mathcal{O} \)). Let \( C \) (resp.: \( D \)) be a prescheme over \( \text{Spec}(\mathcal{A}) \) (resp.: over \( \text{Spec}(\mathcal{B}) \)), such that \( C \) (resp.: \( D \)) is embeddable over \( \mathcal{A} \) (resp.: over \( \mathcal{B} \)) in the sense of [6]. And suppose that we have \( f: C \to D \) a morphism of preschemes such that the diagram:
\[
\begin{array}{ccc}
C & \xrightarrow{f} & D \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{A}^\dagger) & \xrightarrow{f} & \text{Spec}(\mathcal{B})
\end{array}
\]
is commutative. Let \( U \) (resp.: \( V \)) be any open subset of \( C \) (resp.: \( D \)) such that \( f(U) \subset V \). Then from [6], we have an induced mapping:
\[
H^n(D, V, \mathcal{B}^\dagger) \to H^n(C, U, \mathcal{A}^\dagger), \quad \text{all integers } h \geq 0,
\]
semi-linear with respect to the ring homomorphism induced by \( F \), \( F^\dagger: \mathcal{B}^\dagger \to \mathcal{A}^\dagger \). (Notice that we do not need, in this case, that \( \mathcal{A}^\dagger \) be integral over \( (F^\dagger)(\mathcal{B}^\dagger) \).) In particular, if \( F: \mathcal{A} \to \mathcal{A}^\dagger \) is as in Example 1, and we take \( D = C, U = V, f = \alpha_C \), then we obtain an \((F^\dagger)\)-semi-linear endomorphism
\[
H^n(F, \alpha_C) \quad \text{of} \quad H^n(C, U, \mathcal{A}^\dagger),
\]
all integers \( h \geq 0 \). These are analogous to the zeta endomorphisms as defined in Example 1; but do not have as much to do with the important zeta function unless, e.g., \( C \) is simple over Spec(\( A \)) (since, if, e.g., \( C \) is simple over \( A \) and with fibers of constant dimension, then by Remark 4 above these cohomology groups are isomorphic to the lifted \( p \)-adic homology with compact supports of \( C - U \), reindexed.) This is an important advantage of the lifted \( p \)-adic cohomology with compact supports over the (ordinary) lifted \( p \)-adic cohomology defined in [6].

**Conjecture 3** (Generalized Weil’s “Riemann Hypothesis” Conjecture). The hypotheses being as in Theorem 2, if we fix any complex embedding, \( K' \subset C \), then we can write

\[
P_h(T) = \prod_{i=1}^{\beta_h} (1 - \alpha_{h,i} T),
\]

where

\[
|\alpha_{h,i}| = p^{r h'/2}, \quad 1 \leq i \leq \beta_h, \quad 0 \leq h \leq 2d,
\]

and where \( h' = h'(i, h) \), depending on \( h \) and \( i \), is an integer between 0 and \( h \), all integers \( i, h, 1 \leq i \leq \beta_h, 0 \leq h \leq 2d \). Moreover, each \( \alpha_{h,i} \) is an algebraic number, and \( \alpha_{h,i} \) and all its conjugates over \( \mathbb{Q} \) have the same absolute value, \( 1 \leq i \leq \beta_h, 0 \leq h \leq 2d \).

**Remark 1.** In the special case that the algebraic variety \( C \) is complete, non-singular and liftable, so that the functional equation [3, III.2, p. 254] holds, (for the non-liftable case, see [6]) so that the sequences: \( (p^{d_1} |_{x_{h,1}}, \ldots, p^{d_\beta_h} |_{x_{h,\beta_h}}) \) and \( (\alpha_{d-h,1}, \ldots, \alpha_{d-h,\beta_h}) \) are identical up to permutation, \( 0 \leq h \leq 2d \), then Eqs. (7) and (8) would imply \( |\alpha_{h,i}| = p^{r h'/2}, 1 \leq i \leq \beta_h, 0 \leq h \leq 2d \), the usual form of the original Weil “Riemann Hypothesis” Conjecture [8]. (Of course, if one deletes “liftable,” then this remark remains valid, see [6].)

**Remark 2.** Conversely, if Conjecture 3 is true in dimensions \( \leq d - 1 \), and if \( d = \text{dim } C \), then from the exact sequence (6) we see that Conjecture 3 is true for \( C \) iff Conjecture 3 is true for any algebraic variety birationally equivalent to \( C \). Therefore, if Conjecture 3 were known, e.g., in the complete non-singular case, and if resolution of singularities in characteristic \( p \) were known, then Conjectures 3 would hold in general.

**Remark 3.** Therefore, Conjecture 3 is at the moment slightly stronger than the original Weil “Riemann Hypothesis” Conjecture [8] (but would be essentially equivalent if one had some form of resolution of singularities in characteristic \( p \)).
4. q-ADIC HOMOLOGY WITH COMPACT SUPPORTS AND THE ZETA FUNCTION

Many of the results of this paper go through to q-adic cohomology. The main difference is, of course, that q-adic cohomology does not give anything interesting for an algebraic family (Example 1 of Section 3 does not go through) and there are no q-adic analogues of zeta matrices or of the zeta endomorphism. But the Frobenius map still acts on q-adic cohomology, and we obtain the following theorems.

**Theorem 1.** Let \( k \) be an algebraically closed field and let \( q \) be a rational prime \( \neq \text{char}(k) \). Let \( \mathcal{C}_k \) be the category having for objects all reduced algebraic varieties \( C \) over \( \text{Spec}(k) \) that are properly embeddable (see Section 1) over \( k \), and for maps all proper maps of varieties over \( k \). Then for every integer \( h \geq 0 \), we have a covariant functor: \( C \rightsquigarrow H^{2n-h}_h(C, \mathbb{Z}_q) \), the q-adic homology of \( C \) with compact supports, from the category \( \mathcal{C}_k \) into the category of finitely generated \( \mathbb{Z}_q \)-modules,\(^4\) such that, whenever \( h \) is a positive integer and \( X \) is a non-singular, separated algebraic variety over \( k \), of constant dimension \( n \), containing \( C \) as a closed subvariety, then there is induced a canonical isomorphism

\[
H^{2n-h}_h(X, X - C, \mathbb{Z}_q) \cong H^{2n-h}_h(C, \mathbb{Z}_q)
\]

all integers \( h \), where the cohomology groups are the q-adic cohomology groups as defined in [9].

**Sketch of proof:** One proceeds as in the proofs of Lemmas 1 and 2, and of Propositions 3 and 5 and Theorem 6 of Section 1 to define the indicated functors. The proof that the groups are finitely generated over \( \mathbb{Z}_q \) proceeds exactly as in Section 2.\(^4\)

**Remark.** The proof of Theorem 1 is slightly easier than its p-adic analogue (Theorem 6 of Section 1). The reason is that certain maps can be constructed for q-adic cohomology, using the combinatorial definition given in [9, Chap. I], which cannot be constructed directly for p-adic cohomology, which allows a simplification in the proofs of e.g., the q-adic analogues of Lemmas 1 and 2 and Proposition 5 of Section 1. (However, the proof of Theorem 1 above can be given exactly as in the p-adic case, if one wishes.)

**Theorem 2** (q-Adic Generalized Weil's Lefschetz Theorem). Let \( k' \) be a finite field and let \( C \) be an algebraic variety over \( k' \) that is properly embeddable over \( k' \). Let \( \bar{k} \) be the algebraic closure of \( k' \) and let \( C_{\bar{k}} = C \times_{k'} \bar{k} \). Then the zeta function \( \zeta_C(T) \) can be written as an alternating product, as in Eq. (2) of

\(^4\) One first proves the corresponding theorem for coefficients in \( \mathbb{Z}/q\mathbb{Z} \), and then uses Theorem 1, Chapter V, of [5]. Finite generation of q-adic cohomology was first proved by Michael Artin, by a different method.
Section 3, where \( d = \dim C \), and where, as in Eq. (3) of Section 3, \( P_a(T) \) is the reverse characteristic polynomial of the endomorphism of the \( \beta_k \)-dimensional \( \mathbb{Q}_a \)-vector space, the \( h \)th homology group with compact supports \( H_h(C_k, \hat{\mathbb{Z}}_a) \otimes \hat{\mathbb{Z}}_a \mathbb{Q}_a \) of \( C_k \), induced by the Frobenius map of \( C \) over \( k' \).

The proof is entirely similar to that of Theorem 2 of Section 3. Finally,

**Theorem 3** (\( p \)-Adic Generalized Weil's "Riemann Hypothesis" Theorem). The hypotheses being as in Theorem 2, if we fix any complex embedding: \( \hat{\mathbb{Q}}_a \subset \mathbb{C} \), then Eqs. (7) and (8), and the observations immediately following Eq. (8), of Section 3 hold.

**Remark.** Notice that Theorem 6 is a theorem, rather than a conjecture, as is the case for the \( p \)-adic analogue (Conjecture 3 of Section 3).

**Proof.** As in Remark 2 following Conjecture 3 of Section 3, we see that to prove Theorem 6, if one proceeds by induction on \( d = \dim C \), it suffices to prove the result for a variety birationally equivalent to \( C \). Therefore one reduces to the case in which \( C \) is an affine non-singular hypersurface; and then the result follows from theorems in [11].

**Remark.** Analogues of the results of Section 3 of this paper can also be established for \( p \)-adic cohomology using the bounded Witt vectors [7].

5. Lifted \( p \)-Adic Homology with Compact Supports on Affines. \( p \)-Adic "Riemann Hypothesis" in Special Cases

In this section we study vanishing of the lower dimensional lifted \( p \)-adic homology groups with compact supports, when evaluated at an affine. Then we prove the \( p \)-adic "Riemann Hypothesis" for projective, non-singular, liftable varieties over finite fields.

**Theorem 1.** Let \( \mathcal{O} \) be a complete discrete valuation ring with residue class field \( k \) and quotient field \( K \), let \( A \) be an \( \mathcal{O} \)-algebra, let \( B \) be a simple \( A \)-algebra of finite presentation, and let \( B = B \otimes_{\mathcal{O}} k \). Suppose that all of the connected components of fibers of \( \text{Spec}(B) \) over \( \text{Spec}(A) \) are of the same fixed dimension \( N \). Let \( C \) be a reduced closed subscheme of \( \text{Spec}(B_{\text{red}}) \), such that we have an integer \( d \geq 0 \) such that the ideal of \( C \) on \( \text{Spec}(B_{\text{red}}) \) is the radicle of an ideal that is generated by \( d \) elements.

Assume also the technical hypothesis, that either the ring \( A \) is normal, or else that \( A \) can be represented as a quotient of a polynomial ring \( \mathcal{O}[(T_i)_{i \in I}] \) over \( \mathcal{O} \) such that there exists \( B_P \) simple of finite presentation over \( P \) such that \( (B_P) \otimes_P A \approx B \) as \( A \)-algebras.
Let $H^h(C, \mathcal{A} \uparrow \otimes \mathbb{Z} \mathbb{Q})$ be the lifted $p$-adic homology of $C$ with compact supports, with coefficients in $(\mathcal{A} \uparrow) \otimes \mathbb{Z} \mathbb{Q}$, as defined in Theorem 6 of Section 1. Then

$$H^h(C, \mathcal{A} \uparrow \otimes \mathbb{Z} \mathbb{Q}) = 0, \quad \text{all integers } h \leq N - d - 1.$$ 

Proof. First, notice that, since $C$ is closed in $\text{Spec}(B_{\text{red}})$, $C$ is affine. Since also the ideal of $C$ on $B_{\text{red}}$ is the radicle of a finitely generated ideal, we have that $C$ is properly embeddable over $\mathcal{A}$ (taking $X = \text{Spec}(B)$ proves this); and in the case $\mathcal{A}$ is not normal, $C$ is polynomially properly embeddable over $\mathcal{A}$ (take $X = \text{Spec}(B_{\mathcal{A}})$). Therefore, in all cases, $H^h(C, \mathcal{A} \uparrow \otimes \mathbb{Z} \mathbb{Q})$, as defined in Theorem 6 of Section 1, makes sense, all integers $h$. The proof reduces immediately to the case in which $\mathcal{A}$ is normal, which we assume.

The proof is by induction on $d$.

Case I. $d = 0$. Then $C = \text{Spec}(B_{\text{red}})$. Then a simple, separated lifting of finite presentation of $C$ over $\mathcal{A}$, having all fibers over $\text{Spec}(A)$ of the same dimension, is $X = \text{Spec}(B)$. Therefore in this case if $X$ is any scheme proper over $\mathcal{A}$ containing $X = C$ as an open subscheme, then $(C, X) \in \mathcal{C}_{\mathcal{A}, \mathcal{A}}$, and

$$H^h(C, \mathcal{A} \uparrow \otimes \mathbb{Z} \mathbb{Q}) = H^{2N-h}(C, \mathcal{I}_d^*(C) \uparrow \otimes \mathbb{Z} \mathbb{Q}),$$

all integers $h$. But clearly $\mathcal{I}_d^*(C) \uparrow = 0$ for $p \geq N + 1$ (since $B$ is simple over $\mathcal{A}$, and the fibers of $\text{Spec}(B)$ over $\text{Spec}(A)$ are all of dimension $N$). Also, since $(C, \mathcal{O}_C | C)$ is an affine $\mathcal{O}$-space, $H^q(C, \mathcal{I}_d^*(C) \uparrow \otimes \mathbb{Z} \mathbb{Q}) = 0$ for $q \geq 1$ all integers $p \geq 0$ (by [3, Chap. II, Sect. 3, Theorem 1, p. 174]). Therefore, in the first spectral sequence of hypercohomology [3, Chap. 1], which is of the form

$$E^{p,q}_1 = H^q(C, (\mathcal{I}_d^*(C) \uparrow) \otimes \mathbb{Z} \mathbb{Q}) \Rightarrow H^p(C, (\mathcal{I}_d^*(C) \uparrow) \otimes \mathbb{Z} \mathbb{Q}),$$

we have that $E^{p,q}_1 = 0$ unless $q = 0$ and $0 \leq p \leq N$. Therefore, the groups in the abutment vanish for $n \geq N + 1$. This proves Case I.

Case II. $d > 0$. We assume the assertion has been established for $d - 1$; to establish it for $d$.

Let $f_1, \ldots, f_d$ be $d$ elements of the ideal $I_C$ of $C$ on $\text{Spec}(B_{\text{red}})$ such that the radicle of the ideal generated by $f_1, \ldots, f_d$ is all of $I_C$. Let $C_{d-1}$ be the closed subset: $(f_1 = \cdots = f_{d-1} = 0)$ of $\text{Spec}(B_{\text{red}})$, endowed with its induced reduced structure. Then $C_{d-1}$ (respectively: $C_{d-1} - C_d$) is a reduced closed subscheme of $\text{Spec}(B_{\text{red}})$ (respectively: of $\text{Spec}(B_{d-1})_{\text{red}}$), and is such that the ideal of $C_{d-1}$ (respectively: of $C_{d-1} - C_d$) on $\text{Spec}(B_{\text{red}})$ (respectively: on $\text{Spec}(B_{d-1})_{\text{red}}$) is the radicle of an ideal generated by $d - 1$ functions, namely the functions $f_1, \ldots, f_{d-1}$. Let $f_d$ be any element of $B$ that maps into $f_d \in B_{\text{red}}$. Then by the inductive assumption, applied to the simple $\mathcal{A}$-algebra $B$ (respectively: $B_{d-1}$) of finite presentation over $\mathcal{A}$, such that the fibers
of \((B \otimes k)_{\text{red}}\) (respectively: of \((B_{\text{red}} \otimes k)_{\text{red}}\)) over \(A_{\text{red}}\) are all of the same dimension \(N\), and to the reduced closed subset \(C_{d-1}\) (respectively: \(C_{d-1} - C_d\)) of \(\text{Spec}(B_{\text{red}})\) (respectively: of \(\text{Spec}((B_{\text{red}})_{\text{red}})\)), we have, \((d - 1\) replacing \(d\), that

\[
H_h^c(C_{d-1}, (A^\dagger) \otimes \mathbb{Z} \mathbb{Q}) = 0, \quad \text{all integers } h \leq N - d,
\]

(respectively: that

\[
H_h^c(C_{d-1} - C_d, (A^\dagger) \otimes \mathbb{Z} \mathbb{Q}) = 0, \quad \text{all integers } h \leq N - d.
\]

We have the long exact sequence for lifted \(p\)-adic homology with compact supports:

\[
\cdots \rightarrow H_{h+1}^c(C_{d-1} - C_d) \xrightarrow{\delta_{h+1}} H_h^c(C_d) \xrightarrow{H_h^c(i)} H_h^c(C_{d-1}) \rightarrow H_h^c(C_{d-1} - C_d) \rightarrow \cdots,
\]

where the coefficients are in \(A^\dagger \otimes \mathbb{Z} \mathbb{Q}\). (Section 1, Proposition 7, with \(D = C_d\), \(C = C_{d-1}\), \(U = C_{d-1} - C_d\), and \(i: C_d \rightarrow C_{d-1}\) the inclusion.)

Substituting Eqs. (1) and (2) into the long exact sequence (3) completes the induction.

Q.E.D.

EXAMPLE 1. Let \(\mathcal{O}\) be a c.d.v.r. with residue class field \(k\), let \(A\) be an \(\mathcal{O}\)-algebra, let \(A = A \otimes k\), and let \(C\) be a reduced affine scheme over \(\text{Spec}(A_{\text{red}})\). Let \(n\) be the supremum of the dimensions of fibers of \(C\) over \(\text{Spec}(A_{\text{red}})\). Suppose that \(C\) is a (weak) set-theoretic complete intersection over \(\text{Spec}(A_{\text{red}})\)—that is, suppose that there exists \(B\) simple of finite presentation over \(A\), such that the connected components of the fibers of \(\text{Spec}(B_{\text{red}})\) over \(\text{Spec}(A_{\text{red}})\) are all of the same dimension \(N\), where \(B = B \otimes k\), and such that \(C\) is isomorphic to a reduced closed subscheme of \(\text{Spec}(B_{\text{red}})\), in such a way that the ideal of \(C\) on \(\text{Spec}(B_{\text{red}})\) is the radicle of an ideal generated by exactly \(N - n\) functions.

(Notice that the definition of “(weak) set-theoretic complete intersection” that we have given here is indeed less restrictive than the usual definition. In the usual definition of “(ordinary) set-theoretic complete intersection” over \(\text{Spec}(A_{\text{red}})\), one would insist, in addition, that \(B = B[T_1, \ldots, T_N]\).)

Then

COROLLARY 1.1. Let \(\mathcal{O}\) be a c.d.v.r., let \(A\) be an \(\mathcal{O}\)-algebra and let \(C\) be a (weak) set-theoretic complete intersection over \(\text{Spec}(A_{\text{red}})\). Then

\[
H_h^c(C, A^\dagger \otimes \mathbb{Z} \mathbb{Q}) = 0, \quad \text{all integers } h \leq n - 1.
\]

where \(n\) is the maximum dimension of all fibers of \(C\) over \(\text{Spec}(A_{\text{red}})\).
Proof. We can take \( d = N - n \) in Theorem 1. Then by Theorem 1 the indicated groups vanish for \( h \leq N - d - 1 = N - (N - n) - 1 = n - 1 \).

**Example 2.** Let \( \mathcal{A} = \mathcal{O}' \), a discrete valuation ring such that \( \mathcal{M}_{\mathcal{O}'} \cap \mathcal{O} = \mathcal{M}_0 \) and let \( C \) be an affine absolutely non-singular algebraic variety of constant dimension over the field \( k' = k(\mathcal{O}') \). Then usually \( C \) is a (weak) set-theoretic complete intersection over \( k' \). (For example, this is the case if either \( C \) admits a flat lifting over \( \mathcal{O}' \) or if \( C \) is an (ordinary) set-theoretic complete intersection over the field \( k' \).)

If this is the case, then by Example 1, for the lifted \( p \)-adic homology with compact supports of \( C \), we have

\[
H^h_\pi(C, K') = 0 \quad \text{for} \quad h \leq n - 1,
\]

where \( K' = (\mathcal{O}')\uparrow \otimes_\mathbb{Z} \mathbb{Q} \), and \( n \) is the dimension of the affine algebraic variety \( C \).

**Example 3.** Let \( \mathcal{A} \) and \( D \) be as in the Example following Remark 4, near the end of Section 1. Then \( C = D \) obeys the hypotheses of Theorem 1, with \( B = \mathcal{A} \), \( N = 0 \) and \( d = n \). Therefore by Theorem 1,

\[
H^h_\pi(C, (\mathcal{A}\uparrow) \otimes_\mathbb{Z} \mathbb{Q}) = 0 \quad \text{for} \quad h \leq -n - 1,
\]

which is indeed easily seen directly. Note that as we have observed in the Example in Section 1, \( H^h_\pi(C, (\mathcal{A}\uparrow) \otimes_\mathbb{Z} \mathbb{Q}) \neq 0 \), so this result is "best possible" in this case.

Note also that, in Example 3 above, the maximum dimension of fibers of \( C \) over \( \text{Spec}(\mathcal{A}_{\text{red}}) \) is zero, but that Eq. (4) of Example 1 of course fails very strongly if one takes \( n = 0 \) (since the \((-n)\)th group does not vanish). Therefore, in this example, \( C \) is indeed very far from being a "(weak) set-theoretic complete intersection" over \( \text{Spec}(\mathcal{A}_{\text{red}}) \) in the sense of Example 1.

The next result in this section is somewhat special. Quite possibly a slightly longer study (than I have made at the moment) might yield more.

**Proposition 2.** Let \( X \) be a projective, non-singular liftable algebraic variety over a finite field \( k \). Then the \( p \)-adic "Riemann hypothesis" (Conjecture 3 of Section 3; see [8]) is true for \( X \).

**Proof.** We prove the result using our \( p \)-adic cohomology for complete absolutely non-singular, liftable algebraic varieties over fields, as defined in [3, Chap. III, Sect. 1, Theorem 8, p. 250]. This functor, evaluated on \( X \) as in the hypotheses of this Proposition, is the direct product of the functor evaluated on the connected components of \( X \). Therefore we can assume that \( X \) is connected.

The proof is by induction on \( n = \dim X \). The proposition is trivial for \( n = 0 \). Suppose that the proposition is known in dimension \(< n \), where
$n > 0$. To prove it in dimension $n$. Let $X$ be connected, non-singular, complete and liftable of dimension $n$ over $k$.

Let $\overline{X}$ be a simple, projective lifting of $X$ over a complete discrete valuation ring $\mathcal{O}$ of mixed characteristic having $k$ for residue class field. Let $K$ be the quotient field of $\mathcal{O}$. Fix a complex embedding: $K \subset \mathbb{C}$.

Let $D$ be a generic hyperplane section of $\overline{X}$ and let $D = D \times_{\mathcal{O}} k$. Then $D$ obeys all the hypotheses of the proposition, and is of dimension $n - 1$. Therefore, by the inductive assumption, we know that

For every eigenvalue $\alpha$, of the endomorphism $H^b(f_D, K)$ of $H^n(D, K)$ induced by the Frobenius endomorphism $f_D$ of the algebraic variety $D$ over the finite field $k$, on the lifted $p$-adic cohomology $H^b(D, K)$ (as defined in [3, Chap. III, Sec. 1, Theorem 8, p. 250]), we have that $|\alpha| = p^{\log p}$ (where $\#(k) = p^r$ and $p$ is the characteristic of $k$).

Let $D_K = D \times_{\mathcal{O}} K$, $X_K = X \times_{\mathcal{O}} K$. Then taking $D$ for $\overline{X}$, $X$ for $Y$, the inclusion $i: D \to X$ for $f$ and the inclusion $i: D \to \overline{X}$ for $f$ in Eq. (2) of Theorem 8 of [3, Chap. III, Sect. 1, p. 250], we have the commutative diagram:

\[
\begin{array}{ccc}
H^*(X, K) & \xrightarrow{H^*(i, K)} & H^*(D, K) \\
\downarrow & & \downarrow \\
H^*(X_K, \Gamma^{\text{et}}_K) & \xrightarrow{i^*_K} & H^*(D_K, \Gamma^{\text{et}}_K),
\end{array}
\]

where $i_K = i \times_{\mathcal{O}} K$. Moreover, if $(X_C)_{\text{top}}$, resp: $(D_C)_{\text{top}}$, denotes the set of points rational over $\mathbb{C}$ of the projective, non-singular complex algebraic variety $X_C = X_K \times_{K} \mathbb{C}$, respectively: $D_C = D_K \times_{K} \mathbb{C}$, together with the classical topology, then by Theorem 1, conclusion (4), of [3, Chap. III, Sect. 1, pp. 238–239], we have that, for $i^*_K$ in the diagram (6) above, that, under the isomorphisms (4) of [3, Chap. III, Sect. 1, Theorem 1, pp. 238–239], we have that

\[
i^*_K \otimes_K \mathbb{C} \text{ is identified with } H^*(i_C, \text{top}, \mathbb{C}),
\]

in which the rightmost map is the mapping induced on classical complex cohomology: $H^*(i_C, \text{top}, \mathbb{C}) \to H^*(D_C, \text{top}, \mathbb{C})$ by the set-theoretic inclusion

$$(i_C)_{\text{top}}: (D_C)_{\text{top}} \hookrightarrow (X_C)_{\text{top}}.$$
Solomon Lefschetz, about generic hyperplane sections of projective varieties over \( \mathbb{C} \), we have that the mappings on the right side of Eq. (7) are

\[
\begin{cases}
\text{isomorphisms in dimension } \leq n - 2; \\
\text{a monomorphism in dimension } n - 1.
\end{cases}
\]

From the commutative diagram (6), and the identification (7), it follows, likewise, that

\[
H^h(\mu, K) \text{ is an isomorphism for } h \leq n - 2,
\]

\[
\text{a monomorphism for } h = n - 1.
\]  

(For an alternative proof, and a more general such observation, see Remarks 1 and 2 below.)

But then for \( h \leq n - 1 \), \( H^h(X, K) \) is a vector subspace of the \( K \)-vector space \( H^h(D, K) \), and is such that the \( K \)-endomorphism \( H^h(f_X, K) \) of \( H^h(X, K) \), induced by the Frobenius, \( f_X \), of \( X \) over \( k \), is induced by the endomorphism \( H^h(f_D, K) \) of \( H^h(D, K) \), where \( f_D \) is the Frobenius of \( D \) over \( k \). By the inductive assumption we have Eq. (5) for \( D \). Therefore,

For every eigenvalue \( \alpha \) of \( H^h(f_X, K) \) we have that \( |\alpha| = p^{rh/2} \), all integers \( h \leq n - 1 \).  

(9)

The cohomology groups: \( H^*(X, K) \) come equipped with cup products [3, Chap. I, Sect. 7], which are preserved by maps over \( k \) [3, Chap. III, Sect. 1, Theorem 6, p. 247], and in particular by \( H^*(f_X, K) \). And \( H^*(X, K) \) obeys Poincaré duality [3, Chap. III, Sect. 1, Theorem 1, pp. 238–239]. We therefore have the functional equation [3, Chap. III, Sect. 2, Proposition 2, pp. 253–254]:

If \( (\alpha_{h,1}, \ldots, \alpha_{h,h}) \) are the eigenvalues with multiplicities of \( H^h(f_X, K) \), where \( \beta_h = \dim_K H^h(X, K) \), then for every integer \( h \), \( 0 \leq h \leq n \), there exists a permutation \( \pi_h \) of \( \{1, \ldots, \beta_h\} \) such that

\[
\text{the sequences: } (p^{rh/\alpha_{h,1}}, \ldots, p^{rh/\alpha_{h,h}}) \text{ and } (\alpha_{h, \pi_h(1)}, \ldots, \alpha_{h, \pi_h(h)})
\]

coincide.  

Equations (9) and (10) imply

For every eigenvalue \( \alpha \) of \( H^h(f_X, K) \) we have that \( |\alpha| = p^{rh/2} \), all integers \( h \geq n + 1 \).  

(11)

Therefore, by Eqs. (9) and (11), for every integer \( h \neq n \), we have that every eigenvalue of \( H^h(f_X, K) \) has absolute value \( p^{rh/2} \). It remains to prove the corresponding assertion for \( h = n \).
But $\beta_h = \dim_K H^h(X, K) = \text{the } h\text{th Betti number, in the sense of combinatorial topology, of the complex algebraic variety } (X_{\text{top}})$ (The latter equality by [3, Chap. III, Sect. 1, Theorem 1, pp. 238–239]). If $P_h$ is the reverse characteristic polynomial of $H^h(f_X, K)$, then $P_h$ is of degree $\beta_h$, all integers $h$, and we have seen, in [3, Chap. III, Sect. 2, p. 253], that the zeta function $Z_X(T)$ of the algebraic variety $X$ over the finite field $k$ can be written as the alternating product:

$$Z_X(T) = \frac{P_1(T) \cdot P_3(T) \cdots P_{2n-1}(T)}{P_0(T) \cdot P_2(T) \cdots P_{2n}(T)}.$$

(12)

Let $q$ be any rational prime $\neq p$ and let $Q_h$ be the reverse characteristic polynomial induced by the Frobenius over the field $k$, of the $h$th $q$-adic cohomology group $H^h(X_{\bar{k}}, \bar{Q}_q)$ with coefficients in $\bar{Q}_q$, of the algebraic variety over $\bar{k}$, $X_{\bar{k}} = X \times k \bar{k}$, where $\bar{k}$ is the algebraic closure of the field $k$, $0 \leq h \leq 2n$. Then we have seen in [9] that the dimension of the $\bar{Q}_q$-vector space $H^h(X_{\bar{k}}, \bar{Q}_q)$ is $\beta_h$, the $h$th Betti number in the sense of combinatorial topology of $(X_{\text{top}})$, and that

$$Z_X(T) = \frac{Q_1(T) \cdot Q_3(T) \cdots Q_{2n-1}(T)}{Q_0(T) \cdot Q_2(T) \cdots Q_{2n}(T)}.$$

(13)

Therefore $\deg(Q_h) = \beta_h = \deg(P_h)$, all integers $h$. Fix a complex embedding, $\bar{Q}_q \subset \mathbb{C}$. Then in [11], it is shown that the inverse roots of $Q_h$ have absolute value $p^{h/2}$, all integers $h$, $0 \leq h \leq 2n$. By Eqs. (9) and (11), we have likewise that the inverse roots of $P_h$ have absolute value $p^{h/2}$, for all integers $h$, $0 \leq h \leq 2n, h \neq n$. Therefore, by Eqs. (12) and (13) (and unique factorization in $\mathbb{C}[T]$) it follows that $P_h$ must divide $Q_h$, for all integers $h$, $0 \leq h \leq 2n, h \neq n$. But since $\deg P_h = \deg Q_h$, and both $P_h$ and $Q_h$ have constant term 1 ($0 \leq h \leq 2n$), this implies that $P_h = Q_h$, all integers $h$, $0 \leq h \leq 2n, h \neq n$. But then from Eqs. (12) and (13) it follows that $P_n = Q_n$. Therefore the inverse roots of $P_n$ have absolute value $p^{n/2}$, completing the proof. Q.E.D.

Remark 1. Another way of proving Eq. (8) of Proposition 2 is as follows. Let $U = X - D$. Then $U$ obeys the hypotheses of Example 2 above, and therefore

$$H^h_c(U, K) = 0, \quad h < n - 1.$$

From the long exact sequence of lifted $p$-adic homology with compact supports:

$$\cdots \to H^h_{c-1}(U, K) \overset{\partial_{h+1}}{\longrightarrow} H^h_{c}(D, K) \overset{H^h_{c}(\iota, K)}{\longrightarrow} H^h_{c}(X, K) \overset{\text{restriction}}{\longrightarrow} H^h_{c}(U, K) \overset{\partial_h}{\longrightarrow} \cdots$$
(see Section 1, Proposition 7), it follows that

\[ H_h^\alpha(i, K) : H_h^\alpha(D, K) \to H_h^\alpha(X, K) \]

is \( \{ \)
\begin{align*}
\text{an isomorphism,} & \quad h \leq n - 2, \\
\text{an epimorphism,} & \quad h = n - 1.
\end{align*}

(14)

But \( H_h^\alpha(D, K) = H^{2n-h-2}(D, K) \), and \( H_h^\alpha(X, K) = H^{2n-h}(X, K) \), all integers \( h \), and, with these identifications, \( H_h^\alpha(i, K) \) corresponds, under Poincaré duality, to the dual of the mapping \( H^{2n-h-2}(i, K) \). Therefore Eq. (14) implies (and in fact is equivalent to) Eq. (8).

**Remark 2.** The proof given in Remark 1 above of Eq. (14) generalizes to the case in which \( X \) is an arbitrary projective, absolutely non-singular algebraic variety over a field \( k \), liftable or not, and \( D \) is a generic hyperplane section of \( X \), assuming that \( X - D \) is a (weak) set-theoretic complete intersection, as defined in Example 1. (And similarly, one proves the analogue of Eq. (8), for the lifted \( p \)-adic cohomology, (as defined in [6]), for such an \( X \).)

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