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On the Number of Solutions for the Forced Pendulum Equation

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INTRODUCTION

Let $e = e(t) \in C[0, T]$ be a T-periodic function with $\int_0^T e(s)\,ds = 0$. The equation under consideration is the following:

$$\begin{aligned} &\ddot{x}(t) + \lambda \dot{x}(t) + k \sin x(t) = e(t) + c \\ &x \quad T\text{-periodic}, \end{aligned} \tag{1}$$

where $\lambda, k, c \in \mathbb{R}$ and $k > 0$.

A simple integration shows that a necessary condition for (1) to have a solution is that $\exists \xi \in [-\pi/2, \pi/2]$ such that $c = k \sin \xi$. On the other hand, if $|\xi| = \pi/2$ (i.e., $c = \pm k$) then (1) has a solution if and only if $e = 0$. This fact was first pointed out by Castro [3] for $\lambda = 0$ and in any case follows easily by integrating (1) in $[0, T]$.

Moreover, notice that if $x = x(t)$ is a solution for (1) so is $x + 2\pi l$, $l \in \mathbb{Z}$. In order to distinguish a solution among its 2π-translations we give the following:

DEFINITION 0.1. $x = x(t))$ is called a *true* solution for (1) if

$$\frac{1}{T}\int_0^T x(s)\,ds \in [0, 2\pi).$$

Now observe that if $\lambda \neq 0$ and $e = 0$, then (1) admits exactly *two* true (constant) solutions $x_1 < x_2$ if $c = k \sin \xi$ and $\xi \in (-\pi/2, \pi/2)$, and exactly *one* true (constant) solution if $c = \pm k$.

In this note we will extend this result $e \neq 0$ provided the coefficient of friction λ is large in absolute value. (Because of its physical meaning one usually takes $\lambda \geqslant 0$.) Roughly, we obtain the following two results:

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(A) For every $e = e(t)$, T-periodic, with $\int_0^T e(s)\,ds = 0$, there exists a positive $\lambda_0 = \lambda_0(k, e)$ depending on k and e, such that $\forall \lambda$: $|\lambda| > \lambda_0$ there exist two numbers

$$d = d(e, \lambda) < 0 < D = D(e, \lambda) \qquad \text{(depending on } \lambda \text{ and } e\text{)}$$

with the following properties:

(i) if $c \notin [d, D]$, then problem (1) has *no* solution;

(ii) if $c = d$ or $c = D$ then problem (1) has exactly *one* true solution;

(iii) if $c \in (d, D)$ then problem (1) has exactly *two* true solutions x_1 and x_2 with $x_1(t) < x_2(t)$ $\forall t \in [0, T]$.

(B) Let $e = e(t)$ as above and $\xi \in (-\pi/2, \pi/2)$. There exists a positive number $\lambda_1 = \lambda_1(k, e, \xi)$ depending on k, e, and ξ such that $\forall \lambda$: $|\lambda| > \lambda_1$ problem (1) with $c = k \sin \xi$ has exactly *two* true solutions $x_1(t) < x_2(t)$ $\forall t \in [0, T]$.

See Theorems 1.1 and 1.2 for the precise statements.

We also derive the corresponding results for the frictionless case (i.e., $\lambda = 0$) together with other remarks on the nature of the variational problem induced by (1) when $\lambda = 0$. These last results seem to suggest a suitable definition of "inflection point" for functionals defined on Hilbert spaces.

To conclude we mention that similar to the spirit of this work is the result of Castro [3] for $\lambda = 0$ and its subsequent improvement and generalization obtained in [4, 6]. More precisely, in [3] it was proved that if $\lambda = 0$ and $0 < k \leqslant 2\pi/T$ then (1) admits a solution if and only if c satisfies suitable bounds. The restriction $0 < k \leqslant 2\pi/T$ was removed independently in [9, 10]. The first multiplicity results were obtained in [4], where among other things it was shown that (1) with $\lambda = 0$ and $c = 0$ admits at least two true solutions. These results were extended and completed in [6]. There Willem and Mawhin showed for example that for given $e = e(t)$ and $k > 0$, if λ is sufficiently large then there exists a closed interval $I \subset [-k, k]$ with $0 \in I$ such that (1) admits zero, one, and two true solutions if c belongs respectively outside, at the boundary, or inside I. Thus Theorem 1.1 shows how this result becomes sharp for appropriate values of λ. Similar considerations apply to the frictionless case where restrictions are imposed on $e = e(t)$, k, and $\omega = 2\pi/T$.

1. The Pendulum Equation with Damping

Set $\omega = 2\pi/T$. We have

LEMMA 1.1. *Assume*

$$\lambda^2 \geqslant \left(\frac{k}{\omega}\right)^2 - \omega^2. \tag{1.1}$$

Then for every $\xi \in \mathbb{R}$ *there exists a* unique *solution* x_ξ *for the problem*:

$$(\mathrm{P})_\xi \qquad \begin{aligned} &\ddot{x} + \lambda\dot{x} + k\sin x - \frac{k}{T}\int_0^T \sin x(s)\,ds = e \\ &x \quad T\text{-periodic} \\ &\frac{1}{T}\int_0^T x(s)\,ds = \xi. \end{aligned}$$

Moreover, the map

$$\begin{aligned} &\mathbb{R} \to C^2[0, T] \\ &\xi \to x_\xi \end{aligned}$$

is analytic *and* $x_{\xi+2\pi} = x_\xi + 2\pi$.

Proof. For the existence and uniqueness result we refer to [6, 4]. In order to prove analyticity we shall apply the implicit function theorem. To this end set

$$x_\xi(t) = u_\xi(t) + \xi \tag{1.2}$$

so that $\int_0^T u_\xi(s)\,ds = 0$.

Define the map F: $C^2_\#[0, T] \times \mathbb{R} \to C^0_\#[0, T]$ as follows:

$$F(u, \xi) = \ddot{u} + \lambda\dot{u} + k\sin(u + \xi) - \frac{1}{T}\int_0^T \sin(u(s) + \xi)\,ds - e,$$

where for $s = 0, 2$, $C^s_\#[0, T] = \{x \in C^s[0, T]: x - T \text{ periodic } \int_0^T x(s)\,ds = 0\}$ is a Banach space equipped with the standard sup norm. As is well known, since $\sin x$ is analytic, the map F is analytic in $C^2_\#[0, T] \times \mathbb{R}$.

Given $\xi_0 \in \mathbb{R}$, we have $F(u_{\xi_0}, \xi_0) = 0$; moreover, $\forall v \in C^2_\#[0, T]$

$$\frac{\partial F}{\partial u}(u_{\xi_0}, \xi_0)v = \ddot{v} + \lambda\dot{v} + k\cos x_{\xi_0} v - \frac{1}{T}\int_0^T \cos x_{\xi_0}(t)\,v(t)\,dt.$$

Thus if $(\partial F/\partial u)(u_{\xi_0}, \xi_0)v = 0$, following [4] we obtain:

$$\begin{aligned} 0 &= \int_0^T (\ddot{v} + \lambda\dot{v})^2 + k \int_0^T \cos x_{\xi_0} v(\ddot{v} + \lambda\dot{v}) \\ &\geqslant \int_0^T (\ddot{v} + \lambda\dot{v})^2 - k \left(\int_0^T v^2\right)^{1/2} \left[\int_0^T (\ddot{v} + \lambda\dot{v})^2\right]^{1/2} \\ &\geqslant \left[\int_0^T (\ddot{v} + \lambda\dot{v})^2\right]^{1/2} \left((\omega^2 + \lambda^2)^{1/2} - \frac{k}{\omega}\right) \left(\int_0^T \dot{v}^2\right)^{1/2}. \end{aligned}$$

But by assumtion (1.1), $k \leqslant \omega(\omega^2 + \lambda^2)^{1/2}$ so necessarily $v = 0$. In fact, if $v \neq 0$ then $\dot{v} \neq 0$ (since v has mean value zero) so the inequalities above become strict. Therefore by a standard application of the Fredholm alternative we conclude that $(\partial F/\partial u)(u_{\xi_0}, \xi_0)$ defines an invertible operator which maps $C^2_{\#}[0, T]$ onto $C^0_{\#}[0, T]$. Henceforth by the implicit function theorem (see [7]), there exists $\varepsilon > 0$ and an *analytic* map $\gamma: (-\varepsilon + \xi, \xi + \varepsilon) \to C^2[0, T]$ such that $\gamma(\xi_0) = u_{\xi_0}$ and $F(\gamma(\xi), \xi) = 0$ $\forall \xi$: $|\xi - \xi_0| < \varepsilon$. By the uniqueness of $(\mathrm{P})_\xi$, it then follows that

$$x_\xi = \gamma(\xi) + \xi$$

which in turn implies that the map $\xi \to x_\xi$ is analytic. Again by the uniqueness of $(\mathrm{P})_\xi$ it follows that $x_{\xi + 2\pi} = x_\xi + 2\pi$ $\forall \xi$. ∎

Set

$$\phi(\xi) = \frac{k}{T} \int_0^T \sin x_\xi(s)\, ds; \tag{1.3}$$

so $\phi(\xi)$ defines an anlytic, 2π-periodic function. This readily implies that given $e = e(t)$, T-periodic with zero mean value, *either* $\exists c_0 \in \mathbb{R}$ such that if $c = c_0$ then problem (1) has a (unique) solution with any prescribed mean value, and no solution if $c \neq c_0$, *or* for every $c \in \mathbb{R}$ problem (1) admits a *finite* (maybe zero) number of true solutions.

Since for every $\xi \in \mathbb{R}$ we have

$$\ddot{x}_\xi(t) + \lambda\dot{x}_\xi(t) + k \sin x_\xi(t) = e(t) + \phi(\xi) \tag{1.4}$$

if we denote by $\partial x_\xi/\partial\xi$ the derivative of the map $\xi \to x_\xi$ we have that for every fixed ξ, $\partial x_\xi/\partial\xi$ is T-periodic, $(1/T)\int_0^T (\partial x_\xi/\partial_\xi)(s)\, ds = 1$, and

$$\frac{d^2}{dt^2} \frac{\partial x_\xi}{\partial\xi} + \lambda \frac{d}{dt} \frac{\partial x_\xi}{\partial\xi} + k \cos x_\xi \frac{\partial x_\xi}{\partial\xi} = \phi'(\xi). \tag{1.5}$$

Similarly if we denote by $\partial^2 x_\xi/\partial\xi^2$ its second derivative, then $\partial^2 x_\xi/\partial\xi^2$ is T-periodic, $\int_0^T (\partial^2 x_\xi/\partial\xi^2)(t)\,dt = 0$, and

$$\frac{d^2}{dt^2}\frac{\partial^2 x_\xi}{\partial\xi^2} + \lambda\frac{d}{dt}\frac{\partial^2 x_\xi}{\partial\xi^2} + k\cos x_\xi \frac{\partial^2 x_\xi}{\partial\xi^2} - k\sin x_\xi \left(\frac{\partial x_\xi}{\partial\xi}\right)^2 = \phi''(\xi). \quad (1.6)$$

In particular if for some $\xi_0 \in \mathbb{R}$, $\phi'(\xi_0) = 0$, then

$$y(t) = \frac{\partial x_\xi}{\partial_\xi}\bigg|_{\xi=\xi_0}^{(t)}$$

defines a T-periodic function with mean value 1, which satisfies

$$\ddot{y} + \lambda\dot{y} + x\cos x_{\xi_0} y = 0. \quad (1.5)'$$

LEMMA 1.2. *Assume*

$$\left(\frac{\lambda}{2}\right)^2 > k - \omega^2. \quad (1.7)$$

Then $y(t) > 0$ $\forall t \in [0, T]$.

Proof. Arguing by contradiction, assume that y vanishes somewhere. Since y and $\dot{y}$ cannot vanish at the same time (otherwise $y = 0$) and y is T-periodic, there must exist $t_1, t_2 \in [0, T)$ with $0 < t_2 - t_1 \leqslant T/2$ such that $y(t_1) = y(t_2) = 0$. Thus $y = y(t)$ satisfies the boundary value problem

$$\ddot{y} + \lambda\dot{y} + k\cos x_{\xi_0} y = 0$$
$$y(t_1) = y(t_2) = 0.$$

Set

$$w(t) = \begin{cases} e^{t(-\lambda + \sqrt{\lambda^2 - 4k}/2)} & \text{if } |\lambda| \geqslant \sqrt{4k} \\ e^{-(\lambda/2)t}\sin[(\sqrt{4k - \lambda^2}/2)(t - t_1 + \varepsilon)] & \text{if } |\lambda| < \sqrt{4k}, \end{cases}$$

where $\varepsilon > 0$ is chosen so small to have

$$\frac{\sqrt{4k - \lambda^2}}{2}\left(\frac{T}{2} + \varepsilon\right) < \pi$$

(this is possible by (1.7)). Therefore $w(t) > 0$ $\forall t \in [t_1, t_2]$ and it satisfies

$$\ddot{w} + \lambda\dot{w} + k\cos x_{\xi_0} w \leqslant 0.$$

So the maximum principle applies to $y(t)/w(t)$ (see [8]) and gives $y = 0$ in $[t_1, t_2]$, and so $y = 0$ in $[0, T]$ (by the uniqueness of the Cauchy problem for (1.5)′). This gives a contradiction since y has mean value 1. ∎

Remark 1. By a similar use of the maximum principle one can obtain uniqueness for $(\mathrm{P})_\xi$ under the different assumption $(\lambda/2)^2 > k - (\omega/2)^2$ which improves (1.1) for large values of k.

LEMMA 1.3. *Let* (1.7) *be satisfied, and assume that* $\|u_\xi\|_{L^\infty} \leqslant \pi/4$ $\forall \xi \in [0, 2\pi]$ (u_ξ *as given in* (1.2)). *We have*:

(i) *if* ϕ *achieves a local maximum at* $\xi_M \in [0, 2\pi]$ *then* $\phi(\xi_M) > 0$;

(ii) *if* ϕ *achieves a local minimum at* $\xi_m \in [0, 2\pi]$ *then* $\phi(\xi_m) < 0$.

Proof. Let us start by proving (i). Since ξ_M is a local maximum for ϕ we have

$$\phi'(\xi_M) = 0 \qquad \text{and} \qquad \phi''(\xi_M) \leqslant 0. \tag{1.8}$$

In particular this implies that there exists $y^* = y^*(t)$ T-periodic such that

$$\ddot{y}^* - \lambda \dot{y}^* + k \cos x_{\xi_M} y^* = 0 \tag{1.9}$$

with

$$y^*(t) > 0 \quad \forall t \in [0, T] \qquad \text{and} \qquad \frac{1}{T}\int_0^T y^*(t)\, dt = 1.$$

Now take $\xi = \xi_M$ in (1.6) and multiply it by y^*, then integrate to obtain

$$\begin{aligned} T\phi''(\xi_M) = & -k \int_0^T \sin x_{\xi_M}(t) \left(\frac{\partial x_\xi}{\partial_\xi}\Big|_{\xi=\xi_M}^{(t)} \right)^2 y^*(t)\, dt \\ & + \int_0^T (\ddot{y}^* - \lambda \dot{y}^* + k \cos x_{\xi_M} y^*) \frac{\partial^2 x_\xi}{\partial \xi^2}\Big|_{\xi=\xi_M}. \end{aligned}$$

That is,

$$\phi''(\xi_M) = -\frac{k}{T}\int_0^T \sin x_{\xi_M}(t)\, y^*(t) \left(\frac{\partial x_\xi}{\partial_\xi}\Big|_{\xi=\xi_M}^{(t)} \right)^2 dt.$$

Arguing by contradiction assume $\phi(\xi_M) \leqslant 0$. Thus the following have to hold simultaneously:

(a) $\phi(\xi_M) = (k/T) \int_0^T \sin(u_{\xi_M}(t) + \xi_M)\, dt \leqslant 0$;

(b) $\phi'(\xi_M) = (k/T) \int_0^T \cos(u_{\xi_M}(t) + \xi_M)(\partial x_\xi/\partial_\xi)|_{\xi=\xi_M}(t)\, dt = 0$;

(c) $\phi''(\xi_M) = -(k/T) \int_0^T \sin(u_{\xi_M}(t) + \xi_M)\, y^*(t)((\partial x_\xi/\partial_\xi)|_{\xi=\xi_M}(t))^2\, dt \leqslant 0$,

with $y^*(t)$, $(\partial x_\xi/\partial_\xi)|_{\xi=\xi_M}(t) > 0$ $\forall t \in [0, T]$.

Since by assumption $|u_{\xi_M}(t)| \leqslant \pi/4$ $\forall t \in [0, T]$, by (b) we must have $\xi_M > \pi/4$, which in turn implies

$$x_{\xi_M}(t) = u_{\xi_M}(t) + \xi_M > 0 \qquad \forall t \in [0, T].$$

So by (a) there must exist $t_0 \in [0, T]$ such that $x_{\xi_M}(t_0) > \pi$. Thus $\xi_M > \pi - \pi/4 = \frac{3}{4}\pi$, that is, $x_{\xi_M}(t) > \pi/2$ $\forall t \in [0, T]$. Again (b) then requires that $x_{\xi_M}(t_1) > \frac{3}{2}\pi$ for some $t_1 \in [0, T]$, which implies $\xi_M > \frac{5}{4}\pi$ and therefore $x_{\xi_M}(t) > \pi$ $\forall t \in [0, T]$. Notice that up to this point we only have used (a) and (b). Now because of (c) we must have $x_{\xi_M}(t_2) > 2\pi$ for some $t_2 \in [0, T]$ that, as above, implies

$$\tfrac{3}{2}\pi < x_{\xi_M}(t) \leqslant 2\pi + \pi/4$$

which contradicts (b).

A similar argument gives (ii). ∎

LEMMA 1.4. *Under the assumptions of Lemma* 1.3 *there exist unique* $\xi_M, \xi_m \in [0, 2\pi)$ *such that* $\phi(\xi_M)$ *and* $\phi(\xi_m)$ *are, respectively, the only (global) maximum and minimum of* ϕ *in* $[0, 2\pi)$. *Furthermore,* $\phi(\xi_m) < 0 < \phi(\xi_M)$.

Proof. We shall show that if ϕ achieves a local maximum at $\xi_M \in [0, 2\pi)$ and a local minimum at $\xi_m \in [0, 2\pi)$ then necessarily $\xi_M < \xi_m$. This fact together with the analyticity of ϕ implies that ϕ does not have local minima or maxima and that the global maximum and minimum of ϕ are uniquely achieved in $[0, 2\pi)$.

Let $\xi_m, \xi_M \in [0, 2\pi)$ be, respectively, a (local) minimum and maximum for ϕ. Hence $\phi'(\xi_m) = 0$ and $\phi(\xi_m) < 0$ (as follows by Lemma 1.3). As seen in the proof of Lemma 1.3 these two conditions imply $\xi_M > \frac{5}{4}\pi$. Now if $\xi_M > \xi_m > \frac{5}{4}\pi$ then

$$\frac{3}{2}\pi < x_{\xi_M}(t) < 2\pi + \frac{\pi}{4}$$

which is impossible since $\phi'(\xi_M) = 0$. Lemma 1.3 also gives $\min_{[0,2\pi]}\phi < 0 < \max_{[0,2\pi]}\phi$. ∎

Set

$$\mu^2(k) = \max\left\{\left(\frac{k}{\omega}\right)^2 - \omega^2, 4(k - \omega^2)\right\}.$$

THEOREM 1.1. *Given* $e \in C[0, T]$ *T-periodic with* $\int_0^T e(s)\,ds = 0$, *assume that*

$$|\lambda| > \max\left\{\mu(k), \frac{2}{\pi}\frac{\sqrt{T}}{\sqrt{3}}\|e\|_{L^2}\right\}. \tag{1.10}$$

Then $d=d(e,\lambda)=\min_{[0,2\pi]}\phi$ *and* $D=D(e,\lambda)=\max_{[0,2\pi]}\phi$ *are well defined and* $d<0<D$. *Furthermore,*

(i) *if* $c\notin[d,D]$ *then problem* (1) *has* no *solutions;*

(i) *if* $c=d$ *or* $c=D$ *then problem* (1) *has exactly* one *true solution;*

(iii) *if* $c\in(d,D)$ *then problem* (1) *has exactly* two *true solutions,* $x_1(t)<x_2(t)$ $\forall t\in[0,T]$.

Proof. Notice that if $x=x(t)$ is a true solution for (1) then for some $\xi_0\in[0,2\pi)$ we have $x=x_{\xi_0}$ and $c=\phi(\xi_0)$ where $x_\xi=x_\xi(t)$ and $\phi=\phi(\xi)$ are defined according to the above notations. So (i) follows immediately. In order to obtain the rest, we show how (1.10) ensures that $\|u_\xi\|_{L^\infty}\leqslant\pi/4$. Thus by Lemma 1.4 we conclude that $d<0<D$, $\phi^{-1}(c)\cap[0,2\pi)=\{1\text{ point}\}$ if $c=d$ or $c=D$ and $\phi^{-1}(c)\cap[0,2\pi)=\{2\text{ points}\}$ if $d<c<D$. By (1.4) we have

$$\int_0^T \ddot{x}_\xi\dot{x}_\xi+\lambda\int_0^T \dot{x}_\xi^2+k\int_0^T \sin x_\xi\dot{x}_\xi=\int_0^T e\dot{x}_\xi.$$

Thus

$$\lambda\int_0^T \dot{u}_\xi^2=\lambda\int_0^T \dot{x}_\xi^2=\int_0^T e\dot{x}_\xi=\int_0^T e\dot{u}_\xi,$$

that is,

$$\|\dot{u}_\xi\|_{L^2}\leqslant\frac{\|e\|_{L^2}}{|\lambda|}.$$

On the other hand, if we set $u_\xi(t)=\sum_{j\neq 0}c_je^{i(2\pi/T)jt}$ we have

$$\|u_\xi\|_{L^\infty}\leqslant\sum_{j\neq 0}|c_j|\leqslant\left(\sum_{j\neq 0}\frac{1}{j^2}\right)^{1/2}\left(\sum_{j\neq 0}j^2|c_j|^2\right)^{1/2}$$
$$=\frac{\pi}{\sqrt{3}}\frac{\sqrt{T}}{2\pi}\|\dot{u}_\xi\|_{L^2}\leqslant\frac{\sqrt{T}\,\|e\|}{2\sqrt{T}\,|\lambda|}L^2\leqslant\frac{\pi}{4}.$$

Finally in (ii) if $x_1(t)$ and $x_2(t)$ are the two given solutions with $\int_0^T x_1(t)\,dt<\int_0^T x_2(t)\,dt$, set $w=x_2-x_1$. Thus w is T-periodic and satisfies $\ddot{w}(t)+\lambda\dot{w}(t)+h(t)\,w(t)=0$ $\forall t\in[0,T]$ where

$$h(t)=\begin{cases}\dfrac{\sin x_2(t)-\sin x_1(t)}{x_2(t)-x_1(t)} & \text{if } x_2(t)\neq x_1(t)\\[2ex] 1 & \text{if } x_2(t)=x_1(t)\end{cases}$$

so $|h(t)| \leqslant 1$. Therefore as for Lemma 1.2, one sees via the maximum principle that w cannot change sign in $[0, T]$. This concludes the proof. ∎

THEOREM 1.2. *Let* $e = e(t)$ *as in Theorem* 1.1 *and* $\xi \in (-\pi/2, \pi/2)$. *Assume that*

$$|\lambda| > \max\left\{\mu(k), \frac{2\sqrt{T}}{\sqrt{3}\pi}\|e\|_{L^2}, \frac{\|e\|_{L^2}}{\omega[2T(1-\sin|\xi|)]^{1/2}}\right\}. \tag{1.11}$$

Then problem (1) *with* $c = k \sin \xi$ *admits exactly* two *true solutions* x_1 *and* x_2 *with* $x_1(t) < x_2(t)$ $\forall t \in [0, T]$.

Proof. We shall show that under assumption (1.11) necessarily

$$d(e, \lambda) < k \sin \xi < D(e, \lambda).$$

Assume first $\xi \in [0, \pi/2)$, so $d(e, \lambda) < 0 \leqslant \sin \xi$.

On the other hand, if for $\xi_0 = \pi/2$ we assume that

$$\sin \xi \geqslant \frac{1}{T}\int_0^T \sin x_{\xi_0}(t)\, dt,$$

then by Jensen's inequality applied to the convex function $\frac{1}{2}x^2 + \sin x$ we have

$$\begin{aligned}
1 - \sin \xi &\leqslant -\frac{1}{T}\int_0^T \sin x_{\xi_0} + 1 \\
&= 1 + \frac{1}{2}\int_0^T x_{\xi_0}^2 \frac{dt}{T} - \int_0^T \left[\frac{1}{2}x_{\xi_0}^2 + \sin x_{\xi_0}\right]\frac{dt}{T} \\
&\leqslant 1 + \frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \frac{1}{2T}\int_0^T u_{\xi_0}^2 \\
&\quad - \frac{1}{2}\left(\int_0^T x_{\xi_0}(t)\frac{dt}{T}\right)^2 - \sin\left(\int_0^T x_{\xi_0}(t)\frac{dt}{T}\right) \\
&= \frac{1}{2T}\int_0^T u_{\xi_0}^2 \leqslant \frac{1}{2T\omega^2}\int_0^T \dot{u}_{\xi_0}^2 \leqslant \frac{1}{2T\omega^2\lambda^2}\|e\|_{L^2}^2,
\end{aligned}$$

which contradicts (1.11). Therefore $k \sin \xi < \phi(\pi/2) \leqslant D(e, \lambda)$. A similar argument gives the result for $\xi \in (-\pi/2, 0)$. ∎

2. The Pendulum Equation without Friction

We are now concerned with problem (1) in the case $\lambda = 0$. Namely, we seek solutions for the following,

$$\begin{aligned} &\ddot{x}(t) + k \sin x(t) = e(t) + c \\ &x \quad T\text{-periodic}, \qquad k > 0, \end{aligned} \tag{2}$$

where $e = e(t)$ is T-periodic and $\int_0^T e(s)\, ds = 0$.

Since friction has the effect of stabilizing the motion of the mechanical system corresponding to (1), once friction is neglected the situation is more delicate. In fact, even to have that all possible solutions for (2) with $e = 0$ are the constants one must require

$$k \leqslant \omega^2$$

(see [4], the case $k < \omega^2$ was already done in [3]).

Given $e = e(t) \in C([0, T])$ set

$$|||e||| = \min \left\{ \frac{T}{12} \left(1 - \frac{k}{\omega^2}\right)^{-1} \|e\|_{L^1}, \frac{\sqrt{T}}{2\sqrt{3}\omega} \left(1 - \frac{k}{\omega^2}\right)^{-1} \|e\|_{L^2} \right\}. \tag{2.1}$$

We have:

Theorem 2.1. *Assume $k < \omega^2$. For every $e = e(t) \in C([0, T])$ T-periodic with $\int_0^T e(t)\, dt = 0$ and satisfying*

$$|||e||| \leqslant \pi/4 \tag{2.2}$$

there exist two numbers $d = d(e) < 0 < D = D(e)$ (depending on e) such that

(i) *if $c \notin [d, D]$ then (2) has* no *solutions;*

(ii) *if $c = d$ or $c = D$ then (2) has exactly* one *true solution;*

(iii) *if $c \in (d, D)$ then (2) has exactly* two *true solutions $x_1(t) < x_2(t)$ $\forall t \in [0, T]$.*

Furthermore given $\xi \in (-\pi/2, \pi/2)$ if

$$\|e\|_{L^2} < \omega^2 (1 - \sin|\xi|)^{1/2} \sqrt{2T}\, (1 - k/\omega^2) \tag{2.3}$$

then problem (2) with $c = k \sin \xi$ has exactly two *true solutions $x_1(t) < x_2(t)$ $\forall t \in [0, T]$.*

Proof. First of all, notice that assumptions (1.1) and (1.7) when $\lambda = 0$ give exactly $k < \omega^2$. Hence Lemmata 1.1–1.4 continue to hold in this case.

So with the notation of the previous section, we need to show that (2.2) implies $\|u_\xi\|_{L^\infty} \leqslant \pi/4\ \forall \xi \in \mathbb{R}$, and that (2.3) implies

$$d(e) = \min_{[0,2\pi]} \phi < k \sin \xi < D(e) = \max_{[0,2\pi]} \phi.$$

This will follow by basically known estimates which, for the sake of completeness, we will derive anyway. Thus if $\ddot{x}_\xi + k \sin x_\xi = e + \phi(\xi)$ and $x_\xi = u_\xi + \xi$, then

$$\int_0^T \dot{u}_\xi^2 - k \int_0^T [\sin(u_\xi + \xi) - \sin \xi] u_\xi = \int_0^T u_\xi e.$$

That gives

$$\left(1 - \frac{k}{\omega^2}\right) \int_0^T |\dot{u}_\xi|^2 \leqslant \int_0^T |e u_\xi|. \tag{2.4}$$

So $\forall \xi \in [0, 2\pi)$ we have

$$\|u_\xi\|_{L^\infty} \leqslant \frac{\sqrt{T}}{2\sqrt{3}} \left(\int_0^T |\dot{u}_\xi|^2\right)^{1/2} \leqslant \frac{\sqrt{T}}{2\sqrt{3}} \left(1 - \frac{k}{\omega^2}\right)^{-1/2} \|e\|_{L^1}^{1/2} \|u_\xi\|_{L^\infty}^{1/2},$$

that is,

$$\|u_\xi\|_{L^\infty} \leqslant \frac{T}{12} \left(1 - \frac{k}{\omega^2}\right)^{-1} \|e\|_{L^1}$$

or

$$\|u_\xi\|_{L^\infty} \leqslant \frac{\sqrt{T}}{2\sqrt{3}} \left(\int_0^T |\dot{u}_\xi|^2\right)^{1/2} \leqslant \frac{\sqrt{T}}{2\sqrt{3}\omega} \left(1 - \frac{k}{\omega^2}\right)^{-1} \|e\|_{L^2}.$$

In conclusion, we have obtained

$$\|u_\xi\|_{L^\infty} \leqslant |||e|||.$$

Finally, as for Theorem 1.2, if, for example, we take $\xi \in [0, \pi/2)$ and assume $\sin \xi \geqslant (1/T) \int_0^T \sin x_{\zeta_0}(t)\, dt$ with $\xi_0 = \pi/2$, then

$$\begin{aligned}
(1 - \sin \xi) &\leqslant -\int_0^T \sin x_{\xi_0}(t) \frac{dt}{T} + 1 \\
&= \int_0^T -\left(\frac{1}{2} x_{\xi_0}^2(t) + \sin x_{\xi_0}(t)\right) \frac{dt}{T} \\
&\quad + \frac{1}{2} \int_0^T x_{\xi_0}^2(t) \frac{dt}{T} + 1
\end{aligned}$$

$$\leqslant -\frac{1}{2}\left(\int_0^T x_{\xi_0}(t)\,\frac{dt}{T}\right) - \sin\left(\int_0^T x_{\xi_0}(t)\,\frac{dt}{T}\right)$$

$$+\frac{1}{2T}\int_0^T u_{\xi_0}^2(t)\,dt + \frac{1}{2}\xi_0^2 + 1 \leqslant \frac{1}{2T}\|u_{\xi_0}\|_{L^2}^2;$$

and by (2.4),

$$\|u_{\xi_0}\|_{L^2}^2 \leqslant \frac{1}{\omega^2}\|\dot{u}_{\xi_0}\|_{L^2}^2 \leqslant \frac{1}{\omega^2}\left(1-\frac{k}{\omega^2}\right)^{-1}\|e\|_{L^2}\|u_{\xi_0}\|_{L^2}.$$

We conclude

$$(1-\sin\xi) \leqslant \frac{1}{2T\omega^2}\left(1-\frac{k}{\omega^2}\right)^{-2}\|e\|_{L^2}^2$$

which contradicts (2.3). The case $-\pi/2 < \xi < 0$ is treated similarly. ∎

We would like to conclude with some remarks on the nature of the variational problem induced by (2), mostly because it seems to suggest a suitable notion of "inflection point" for functionals defined in infinite dimensional spaces. To this end let H be a Hilbert space with scalar product $\langle\ ,\ \rangle$ and $I \in C^3(H, \mathbb{R})$. We give the following

DEFINITION. Let $x_0 \in H$ be a critical point for I. We call x_0 an *inflection-type* point for I, if x_0 is a *fold singularity* for $I' \in C^2(H, H)$. That is,

(a) $\dim \ker I''(x_0) = 1$;

(b) if $v_0 \in \ker I''(x_0)$ and $v_0 \neq 0$ then $I'''(x_0)(v_0, v_0) \notin \operatorname{Range} I''(x_0)$ (see [1, 2]).

Remark. If $I''(x_0)$ is self-adjoint, condition (b) reduces to (b)′ $\langle I'''(x_0)(v_0, v_0), v_0\rangle \neq 0$.

As is well known, solutions of (2) are the critical points for the functional

$$I(x) = \int_0^T \left[\frac{1}{2}\dot{x}^2(t) + k\cos x(t) + (e(t)+c)\,x(t)\right] dt$$

defined on the Hilbert space $H = \{x \in H^1[0, T]: x(0) = x(T)\}$. It is easily verified that $I \in C^k(H, \mathbb{R})$ $\forall k \in \mathbb{N}$. We have:

THEOREM 2.2. *Assume $k < \omega^2$. If x_0 is a critical point for I then x_0 is a local minimum or a point of mountainpass type or of inflection type.*

Recall that a critical point x_0 is said to be of mountainpass type if for every sufficiently small neighborhood U of x_0 the set $U \cap \{x \in H: I(x) < I(x_0)\}$ is neither empty nor path-connected (see [5]).

Proof. Denote by $\lambda_1 < \lambda_2 < \cdots$ the sequence of eigenvalues of $I''(x_0)$ under T-periodic boundary conditions. If $\lambda_1 > 0$, then x_0 is a local minimum for I. Hence assume $\lambda_1 \leqslant 0$. We claim that necessarily $\lambda_2 > 0$. To see this notice first that if v is an eigenvector for $I''(x_0)$ with corresponding eigenvalue λ and $\int_0^T v(s)\, ds = 0$ then $\lambda > 0$. Arguing by contradiction, let v_1 and v_2 be two eigenvectors of $I''(x_0)$ corresponding to the eigenvalues $\lambda_1 < \lambda_2 \leqslant 0$. Since $\int_0^T v_1, \int_0^T v_2 \neq 0$ we can assume $\int_0^T v_1(t)\, dt = \int_0^T v_2(t)\, dt$ and $\int_0^T v_1(t) \cdot v_2(t)\, dt = 0$. Set $w = v_2 - v_1$. We have

$$\left(1 - \frac{k}{\omega^2}\right) \int_0^T \dot{w}^2 \leqslant \int_0^T \dot{w}^2 - (\lambda_2 + k \cos x_0) w^2$$

$$= +(\lambda_2 - \lambda_1) \int_0^T v_1 w = -(\lambda_2 - \lambda_1) \int_0^T v_1^2$$

which is impossible. Now if $\lambda_1 < 0$ then x_0 is a point of mountainpass type. Finally assume that $\lambda_1 = 0$; i.e., x_0 is a singular point for I'. By the arguments above we have $\dim \ker I''(x_0) = 1$. Furthermore since x_0 is a solution for (2), with the notation of the previous section, there exists $\xi_0 \in [0, 2\pi)$ such that: $x_{\xi_0} = x_0 + 2\pi l$ for some $l \in \mathbb{Z}$, $\phi(\xi_0) = c$, $\phi'(\xi_0) = 0$, and $(\partial x_\xi / \partial \xi)|_{\xi = \xi_0} \in \ker I''(x_0)$. Since

$$\frac{d^2}{dt^2} \frac{\partial^2 x_\xi}{\partial \xi^2}\bigg|_{\xi = \xi_0} + k \cos x_0 \frac{\partial^2 x_\xi}{\partial \xi^2}\bigg|_{\xi = \xi_0}$$

$$- k \sin x_0 \left(\frac{\partial x_\xi}{\partial \xi}\bigg|_{\xi = \xi_0} \right)^2 = \phi''(\xi_0)$$

we conclude that

$$\phi''(\xi_0) \neq 0 \Leftrightarrow \int_0^T \sin x_0 \left(\frac{\partial x_\xi}{\partial \xi}\bigg|_{\xi = \xi_0} \right)^3 \neq 0$$

which gives exactly the fold condition at x_0; so in this case x_0 defines an inflection-type point. Finally, if $\phi''(\xi_0) = 0$, i.e.,

$$\int_0^T \sin x_0 \left(\frac{\partial x_\xi}{\partial \xi}\bigg|_{\xi = \xi_0} \right)^3 = 0,$$

then

$$I\left(x_0+s\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)=I(x_0)+sI'(x_0)\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}$$
$$+\frac{s^2}{2}I''(x_0)\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^2$$
$$+\frac{s^3}{3!}I'''(x_0)\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^3$$
$$+\frac{s^4}{4!}I^{(4)}(x_0)\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^4+O(s^5)$$
$$=I(x_0)+\frac{s^4}{4!}I^{(4)}(x_0)\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^4+O(s^5)$$

and

$$I^{(4)}(x_0)\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^4=k\int_0^T \cos x_0\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^4$$
$$=-\int_0^T \frac{d^2}{dt^2}\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^3$$
$$=3\int_0^T\left(\frac{d}{dt}\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^2\left(\left.\frac{\partial x_\xi}{\partial \xi}\right|_{\xi=\xi_0}\right)^2>0.$$

Thus x_0 is a (degenerate) local minimum in this case. ∎

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