Countable compactness and finite powers of topological groups without convergent sequences

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Abstract

We show under $\text{MA}_{\text{countable}}$ that for every positive integer $n$ there exists a topological group $G$ without non-trivial convergent sequences such that $G^n$ is countably compact but $G^{n+1}$ is not. This answers the finite case of Comfort’s Question 477 in the Open Problems in Topology. We also show under $\text{MA}_{\text{countable}} + 2^c < c = c$ that there are $2^c$ non-homeomorphic group topologies as above if $n \geq 2$. We apply the construction on suitable sets, answering the finite case of a question of D. Dikranjan on the productivity of suitability and in a topological game defined by Bouziad.

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1. Introduction

One of the best well-known results in topology is Tychonoff’s theorem on the productivity of compactness. A natural question is whether other compact-like properties are productive as well. There are countably compact spaces whose square are not even pseudocompact (due independently to Novák [21] and Terasaka [25]). Frolik [11] extended this study to finite products, showing that for every $n$ there exists a space $X$ such that

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$X^n$ is countably compact but $X^{n+1}$ is not. On the other hand, Scarborough and Stone [24] showed that if the $2^\gamma$th power of a space is countably compact then every power is countably compact. This last result was improved by Ginsburg and Saks [14] who showed that a space is countably compact for every power if and only if its $2^\gamma$th power is countably compact. The natural question whether $2^c$ was the best possible was answered under MA by Saks [23]. Recently, using an equivalence of Yang [30], M. Hrusak noted that in Shelah’s model [2], the best possible cardinal in Ginsburg and Saks theorem is not larger than $c$.

Topological groups is an important class of spaces in which productivity has been studied, resulting in interesting theorems and examples. A quite unexpected result was obtained by Comfort and Ross [6], who showed that pseudocompactness becomes productive for the class of topological groups. A natural question, asked by Comfort, was whether the same would be true for countably compact topological groups. A consistent negative answer was obtained by van Douwen [10] who showed under MA that there exist two countably compact topological groups whose product is not countably compact. A decade later, Hart and van Mill [17] showed that there exists under MA a countably compact topological group whose square is not countably compact.

Because of these results, Comfort asked the following question in the Open Problems in Topology:

**Question 1.1** [4]. Is there, for every (not necessarily infinite) cardinal number $\alpha \leq 2^c$, a topological group $G$ such that $G^\gamma$ is countably compact for all cardinals $\gamma < \alpha$, but $G^\alpha$ is not countably compact?

Under $\text{MA}_{\text{countable}}$, it was shown that two [17] and three [28] are such cardinals. Furthermore, there are infinitely many such natural numbers [27]. All the examples above contain convergent sequences.

We answer Comfort’s question for the finite case, under $\text{MA}_{\text{countable}}$, providing witnesses without non-trivial convergent sequences. In addition, if $2^{<\omega} = \omega$ is also assumed, then $2^\gamma$ non-homeomorphic examples can be obtained for $\gamma \geq 3$. Some kind of ‘basis property’ for sequences of length at most $n$ are used to obtain the construction. The motivation for this approach came from Steve Watson’s lecture on Kunen’s solution to a question of van Douwen on Bohr topologies [20], given at University of São Paulo.

In [7], Dikranjan studied cardinal numbers related to the productivity of a topological property, restating Comfort’s question in this setting. He also asked about the productivity of suitability, which was introduced by Hofmann and Morris. This concept, introduced in [18], is a natural generalization of monotheticity.

**Definition 1.2.** A subset $S$ of a topological group $H$ is a (closed) suitable set for $H$ if (a) $S$ is discrete in $H$ and (b) $S$ is closed in $H \setminus \{e\}$ (respectively $H$) and (c) the group generated by $S$ is dense in $H$.

A topological group is monothetic if it has a suitable set of size 1. Hofmann and Morris showed that every locally compact group has a suitable set; in [5] the study of the non-locally compact case was started and it was followed in [8,9,22,26,29]. It is easy to see that if a power of $H$ has a suitable set then every larger power also does. Dikranjan [7] asked...
then for what cardinal numbers \( \kappa \), there exists a group \( G \) which does not have a suitable set for \( G^\lambda \) if \( \lambda < \kappa \) but \( G^\kappa \) has a suitable set.

We show that under MA\textsubscript{countable}, Dikranjan’s question has an affirmative answer for every positive integer.

In the last section we present the \( G \)-spaces, defined through a game introduced by Bouziad [3] and show that under MA\textsubscript{countable} there is, for each positive integer \( n \), a topological group \( G \) such that \( G^n \) is a \( G \)-space but \( G^{n+1} \) is not.

2. The examples and some preliminaries

Let \( G \) be the algebraic sum of \( c \) copies of 2. We will topologize \( G \) to obtain our desired topological groups. The relation between suitability and countably compact groups without non-trivial convergent sequences was first noticed in [5]. We further explore this relation in finite products.

**Lemma 2.1.** Fix \( n \in \omega \). Suppose that \( \{x_{m,j}: m < \omega \land j < n + 2\} \) is a subset of a topological group \( H \subseteq 2^\omega \) such that \( \{(x_{m,n},x_{m,n+1}): m \in \omega\} \) is dense in \( H \times H \) and \( T = \{(x_{m,j})_{j<n+1}: m \in \omega\} \cup \{(x_{m,j})_{j \in \omega \cup \{n+1\}}: m \in \omega\} \) is closed and discrete in \( H^{n+1} \). Then \( H^{n+1} \) has a closed suitable set. If \( H \subseteq 2^\omega \) has no non-trivial convergent sequences and \( H^n \) is countably compact then \( H^n \) has no suitable set.

**Proof.** Let \( S \) be the set of \((n+1)-\)uples that are permutations of elements of \( T \), that is, \( \{y_j\}_{j<n+1} \in S \) if there exists \( m \in \omega \) and a bijection \( \sigma \) from \( n+1 \) into \( n+1 \) or \( n \cup \{n+1\} \) such that \( y_j = x_{m,\sigma(j)} \) for each \( j < n+1 \). Then clearly \( S \) is closed discrete and the group generated by \( S \) contains all the \( n+1 \)-uples that are 0 in all coordinates but one and the remaining coordinate is of the form \( x_{m,n} + x_{m,n+1} \) and \( \{x_{m,n} + x_{m,n+1}: m \in \omega\} \) is dense in \( H \). Therefore, the group generated by \( S \) is dense in \( H^{n+1} \) and \( S \) is a closed suitable set.

The group \( H^n \) does not have a suitable set since it is a countably compact group of order 2 without non-trivial convergent sequences (see [5]). \( \Box \)

**Example 2.2.** (MA\textsubscript{countable} + \( 2^{<\omega} = \omega \)) For every positive integer \( n \) there exists a family of topological group topologies \( \{T_\alpha: \alpha < 2^\omega \} \) on the group \( G \) such that \( \langle G, T_\alpha \rangle \times \langle G, T_\beta \rangle \) is not countably compact for any pair \( \alpha < \beta < 2^\omega \), \( \langle G, T_\alpha \rangle^n \) is countably compact without non-trivial convergent sequences and \( \langle G, T_\alpha \rangle^{n+1} \) has a closed suitable set.

We will construct \( 2^\omega \) many topologies using the tree \( 2^{<\omega} \) assuming MA\textsubscript{countable} + \( 2^{<\omega} = \omega \), but from the construction, it is clear that one can construct the example below using the tree \( 2^{<\omega} \).

**Example 2.3.** (MA\textsubscript{countable}) For every positive integer \( n \) there exists a family of topological group topologies \( \{T_\alpha: \alpha < \omega \} \) on the group \( G \) such that \( \langle G, T_\alpha \rangle \times \langle G, T_\beta \rangle \) is not countably compact for any pair \( \alpha < \beta < \omega \), \( \langle G, T_\alpha \rangle^n \) is countably compact and does not have a suitable set; and \( \langle G, T_\alpha \rangle^{n+1} \) is not countably compact and has a closed suitable set.
The following is an easy consequence of Example 2.2:

**Example 2.4.** Assume \((M_{\text{countable}} + 2^{<c} = c)\). If \(n \geq 2\) then there exists \(2^c\) many non-homeomorphic group topologies on the Boolean group of size \(c\) that makes its \(n\)th power countably compact and its \((n+1)\)st power not countably compact.

**Proof.** Since \(n \geq 2\), given two topologies as in Example 2.2, their squares are countably compact but their products are not, thus they are not homeomorphic. \(\square\)

Throughout the construction, \(n\) will be a fixed positive integer. The groups will be indexed by a function in \(D_c := \{0\}^{\omega} \times 2^c \setminus \omega\). The topological groups will be denoted by \(G_f\) where \(f \in D_c\).

We will use \(U\)-limits, a concept introduced by Bernstein [1], that is useful in the study of countable compactness.

**Definition 2.5.** Fixed a free ultrafilter \(U\) on \(\omega\), we say that \(x \in X\) is the \(U\)-limit point of a sequence \(\{x_m: m \in \omega\} \subseteq X\) if \(\{m \in \omega: x_m \in U\} \in U\) for every neighbourhood \(U\) of \(x\).

The basic facts on \(U\)-limit points that will be used in this paper are the following:

**Lemma 2.6.** Assume that all spaces below are Hausdorff.

1. The \(U\)-limit point of a sequence is unique and for sequences in compact spaces, there exists a \(U\)-limit point for each free ultrafilter \(U\).
2. \(x\) is an accumulation point of \(\{x_n: n \in \omega\}\) if and only if there exists a free ultrafilter \(U\) over \(\omega\) such that \(x\) is the \(U\)-limit point of this sequence.
3. \(\{x_{\alpha,n}: \alpha \in I, n \in \omega\}\) is the \(U\)-limit point of \(\{x_{\alpha}: \alpha \in I, n \in \omega\}\) if and only if \(x_{\alpha,n}\) is the \(U\)-limit point of \(\{x_{\alpha}: \alpha \in I, n \in \omega\}\) for each \(\alpha \in I\).
4. In topological groups, if \(x\) is the \(U\)-limit point of \(\{x_m: m \in \omega\}\) and \(y\) is the \(U\)-limit point of \(\{y_n: n \in \omega\}\) then \(x + y\) is the \(U\)-limit point of \(\{x_m + y_n: m \in \omega\}\).

3. The sketch and some auxiliary lemmas

We assume that \(2^{<c} = c\). Given a family of functions \(\{z_t: t \in F\}\) with \(z_t \in 2^\beta\) for some fixed \(\beta\) and \(F\) finite, we denote by \(z_F\) the sum \(\sum_{t \in F} z_t\).

Define \(D^\alpha = [0]^\omega \times 2^\omega\) if \(\alpha \geq \omega\) and \(D^m = [0]^m\) if \(m\) is a natural number. The set \(D^{<\alpha}\) is defined as \(\bigcup_{\beta < \alpha} D^\beta\).

We will construct \(2^{<c}\) linearly independent elements of \(2^c\) which will be denoted by \(x_{f,j}\) where \(f \in D^\alpha\) for some \(\alpha < c\) and \(j < n + 2\). If \(f\) is the unique element of \(D^m\), we will also denote the function by \(x_m,f\).

Given \(f \in D^\alpha\) we define \(G_f = \langle \{x_{f,j}: \alpha < c \wedge j < n + 2\} \rangle\). Each \(G_f\) will satisfy the conditions of Lemma 2.1 and it will not contain any non-trivial convergent sequences. Furthermore, for distinct \(f, h \in D^\alpha\), the set \(G_f \cap G_h\) is an infinite subgroup of size \(< c\) thus, \(G_f \times G_h\) is not countably compact.
The countable compactness of the \( n \)th power of each \( G_f \) will be attained by producing accumulation points for a number of sequences in a product of at most \( n \) copies of \( G_f \). These sequences will be related to some kind of linear independence for sequences of \( k \)-tuples for \( k \) at most \( n \).

We fix the following enumeration to obtain the conditions of Lemma 2.1.

**Definition 3.1.** Let \( \{K_\alpha: \alpha < \epsilon, \alpha \text{ even}\} \) be an enumeration of all functions \( K: n + \rightarrow [D^{\leq \epsilon} \times (n + 2)]^{\leq \omega} \) and \( \{K_\alpha: \alpha < \epsilon, \alpha \text{ odd}\} \) be an enumeration of all functions \( K: n \cup \{n + 1\} \rightarrow [D^{\leq \epsilon} \times (n + 2)]^{\leq \omega} \) so that \( \bigcup_{i \in \text{dom}K} K_\alpha(i) \subseteq D^{\leq \epsilon} \times (n + 2) \) for every \( \alpha \in [\omega, \epsilon) \).

The proof of the Lemma below is straightforward and it is left to the reader.

**Lemma 3.2.** If for every \( \alpha \) even \( \{m \in \omega: x_{m,j}(\alpha) = x_{K_\alpha(j)}(\alpha) \forall j \in n + 1\} \) is finite and for every \( \alpha \) odd \( \{m \in \omega: x_{m,j}(\alpha) = x_{K_\alpha(j)}(\alpha) \forall j \in n \cup \{n + 1\}\} \) is also finite, then the set \( T \) of Lemma 2.1 is closed and discrete in \((G_f)^{n+1}\), for each \( f \in D^\epsilon\).

The following enumeration will be used to obtain countable compactness in the product and the non-existence of non-trivial convergent sequences.

**Notation.** Let \( \{F_\alpha: \omega \leq \alpha < \epsilon\} \) be an enumeration of all functions \( F: S \times \omega \rightarrow [D^{\leq \epsilon} \times (n + 2)]^{\leq \omega} \) such that \( S \in [n + 2]^{\leq \omega} \setminus \{\emptyset\} \) and for every \( F \subseteq D^{\leq \epsilon} \times (n + 2) \) finite, the family \( \{(g, j): (g, j) \in F \} \cup \{F(i, m): i \in S\} \) is linearly independent for all but finitely many \( m \in \omega \). We assume that \( \bigcup_{(i, m) \in \text{dom}F} F_\alpha(i, m) \subseteq D^\omega \times (n + 2) \). Denote by \( S_\alpha \) the unique subset of \( n + 2 \) such that \( \text{dom}F_\alpha = S_\alpha \times \omega \).

**Lemma 3.3.** (MA countable) Fix \( h \in D^\epsilon \). Suppose that \( (x_{h_\alpha,i})_{i \in S_h} \) is an accumulation point of \( \{(x_{F_\alpha(i,m)})_{i \in S_\alpha}: m \in \omega\} \) for every \( \alpha \in [\omega, \epsilon) \). Then the \( n \)-th power of \( G_h \) is countably compact and \( G_h \) has no non-trivial convergent sequences.

**Proof.** Let \( \{(a_{i,m})_{i<n}: m \in \omega\} \) be any sequence in the \( n \)-th power of \( G_h \).

**Claim 1.** There exists \( c_i \in G_h \) for each \( i < n, S \subseteq n \), \( E_i \subseteq S \) for each \( i < n \) and \( A \subseteq \omega \) infinite such that \( \{(a_{i,m})_{i \in S}: m \in A\} \) has an accumulation point in \((G_h)^S\) and \( c_i = a_{i,m} = \sum_{j \in E_i} a_{j,m} \) for each \( m \in A \).

The proof of Claim 1 is more technical, so we will first apply Claim 1 and prove it after Claim 2.

**Claim 2.** The sequence \( \{(a_{i,m})_{i<n}: m \in \omega\} \) has an accumulation point in \((G_h)^\omega\).

**Proof.** Let \( S \subseteq n, E_i \subseteq S \) for each \( i < n \) and \( A \subseteq \omega \) as in Claim 1. Let \( (b_j)_{j \in S} \in (G_h)^S \) be an accumulation point of the sequence \( \{(a_{i,m})_{i \in S}: m \in A\} \). Then there exists a free ultrafilter \( p \) on \( \omega \) such that \( b_j = p\text{-lim}(a_{i,m}: m \in \omega) \) for each \( i \in S \). Then, for each
that \{k\} is linearly independent for all but finitely many \(i<n\). Then there exists \(E\) by \((Gh)\omega/q\) and let \(\{(ai,m): m \in \omega\}\) has a p-limit point in \(G^h\). Therefore, the sequence \(\{(ai,m): i<n; m \in \omega\}\) has an accumulation point in \((G^h)\omega\).

**Proof of Claim 1.** Let \(q\) be a selective ultrafilter on \(\omega\) (such ultrafilter exist under MA\(_{\text{countable}}\) and we will mention the necessary properties in due time). We can define the following relation in \((G^h)\omega\): two functions \(f\) and \(g\) in \((G^h)\omega\) are equivalent if \(\{n \in \omega: f(n) = g(n)\} \in q\). Let \([f]\) denote the set of equivalent elements of \(f\) in \((G^h)\omega\) and let \((|(G^h)\omega)/q\) be the set of equivalent classes. Given two classes \([f]\) and \([g]\), denote by \([f] + [g]\) the class \([f + g]\). Then \((|(G^h)\omega)/q\) with this operation is a vector space over 2, since \(G^h\) is a vector space over 2. Given an element \(c \in G\), denote by \([c]\) the class of the constant function \(c\).

Now, let \(R\) be the vector subspace generated by \(\{(ai,m): m \in \omega\}: i<n\) and \(C \subseteq G^h\) be such that \(\{[c]: c \in C\}\) is a basis for \(R \cap \{(c): c \in G^h\}\). Let \(S \subseteq \omega\) be such that

(I) \(\{(ai,m): m \in \omega\}: i \in S\} \cup \{(c): c \in C\}\) is a basis for \(R\).

For each \(i<n\), let \(E_i \subseteq S\) and \(C_i \subseteq C\) be such that \(\{(ai,m): m \in \omega\} = \sum_{c \in C_i} [c] + \sum_{i \in E_i} [\{(ai,m): m \in \omega\}]\) for each \(i<n\). Then clearly, \(\{(ai,m): m \in \omega\}\) is equivalent to a constant sequence. Thus, there exists \(B \in q\) and \(c_i \in G^h\) for each \(i<n\) such that

(II) \(c_i = \sum C_j = ai_m = \sum_{j \in E_j} ai_m\) for each \(i<n\) and \(m \in B\).

We claim that

(III) \(\sum_{i \in T} [\{(ai,m): m \in \omega\}] \notin [d] \quad \forall T \subseteq \omega\) non-empty.

Indeed, let \(D \subseteq G\) be such that \(D \supseteq C\) and \(D\) is a basis of \(G\). Then, from (I), it follows that \(\{(ai,m): m \in \omega\}: i \in D\} \cup \{(c): c \in D\}\) is linearly independent. Therefore, \(\{(ai,m): m \in \omega\}: i \in \omega\}\) and (III) holds.

We claim that there exists \(A \in q\) with \(A \subseteq B\) such that

(IV) \(\{\sum_{i \in T} ai_m: m \in A\}\) is faithfully indexed for each \(T \subseteq \omega\) non-empty.

Indeed, since \(q\) is selective, there exists \(A^b \in p\) such that \(\{\sum_{i \in T} ai_m: m \in A^b\}\) is either constant or faithfully indexed. It follows from (III) that \(\{\sum_{i \in T} ai_m: m \in A^b\}\) cannot be constant, thus, it is faithfully indexed. Define \(A = B \cap \bigcap_{\emptyset \neq T \subseteq \omega} A^b \in q\). Then \(A\) is as required.

Let \((m_j: k \in \omega)\) be an enumeration of \(A\). Let \(\mathcal{F}: S \times \omega \to \{D^{\lt \omega} \times (n + 2)\}^{\lt \omega}\) be such that \(x_{\mathcal{F}(i,k)} = ai_m\) for each \(i \in S\) and \(k \in \omega\).

We claim that there exists \(\alpha < \epsilon\) such that \(\mathcal{F}_\alpha = \mathcal{F}\). For that, it suffices to prove that if \(F \subseteq D^{\lt \omega} \times (n + 2)\) finite, then the family \(\{(c, j): (c, j) \in F\} \cup \{\mathcal{F}(i, k): i \in S\}\) is linearly independent for all but finitely many \(k \in \omega\). Suppose by contradiction that this is not the case. Then there exists \(E \subseteq F, \emptyset \neq T \subseteq \omega\) and \(K \subseteq \omega\) infinite such that \(\Delta(\mathcal{F}(i, k): i \in T) = E\) for each \(k \in K\).

Then, \(\sum_{i \in T} ai_m = x_E\) for each \(k \in K\). However, this contradicts (IV). Now, by hypothesis, the sequence \(\{x_{\mathcal{F}(i,k)}\}: k \in S\) has an accumulation point in \((G^h)S\). We are done with Claim 1, since the last sequence is a subsequence of \(\{(ai,m): i \in S; m \in \omega\}\). □
We will show now that $G_h$ has no non-trivial convergent sequences. Let $\{b_m: m \in \omega\}$ be a non-trivial sequence in the group $G_h$. We can assume without loss of generality that this sequence is an injection and that none of its elements is the identity in $G_h$. Then, there exists $\alpha_i \in [\omega, \omega)$ such that $S_{\alpha_i} = \{0\}$ and $b_{2m+i} = x_{F_{\alpha}, (0, m)}$ for every $m \in \omega$ and $i < 2$. Therefore, $\{b_m: m \in \omega\}$ has two accumulation points $x_{h|_{\omega}, 0}$ and $x_{h|_{\omega}, 1}$ in $G_h$. Since $G_h$ is Hausdorff, the sequence $\{b_m: m \in \omega\}$ does not converge. 

4. The inductive hypothesis and the partial order

The construction of the linearly independent set $\{x_{f,i}: f \in D^{<\omega} \land i < n + 2\}$ is by induction. It is left to the reader to check that if the inductive hypothesis below are satisfied and $x_{f,i} = \bigcup_{\alpha \in \omega} x_{f,i}\alpha$ for each $f \in D^{<\omega}$ and $i < n + 2$ then the conditions of Lemmas 2.1 and 3.3 are satisfied for the groups $G_h = \left\{\{x_{h|_{\alpha}, i}: \alpha < \omega, i < n + 2\}\right\}$, where $h \in 2^\omega$.

At an infinite stage $\alpha$ we will have defined $\{x_{f,i}|\alpha: f \in D^\omega\}$ such that the following are satisfied:

1. $x_{f,i}|\beta \in 2^\beta$ for each $\beta \leq \alpha$ and $f \in D^{\leq \alpha}$;
2. $x_{f,i}|\beta \subseteq x_{f,i}\alpha$ for each $\beta < \alpha$ and $f \in D^{\leq \beta}$;
3. if $\omega \leq \beta < \alpha$ and $K_{\beta}(0) \neq \emptyset$ then $x_{K_{\beta}(0)}(\beta) \neq 0$;
4. if $\omega \leq \beta \leq \alpha$ and $f \in D^\beta$ then $\{x_{f,i}|\alpha\}_{i \in S_{\beta}}$ is an accumulation point of the sequence $\{x_{f_{\alpha}(m,i)}|\alpha\}_{i \in S_{\beta}}$;
5. the set $\{(x_{m,n}|\alpha_m, x_{m,n+1}|\alpha_{m+1}) : m \in \omega\}$ is dense in $2^\omega \times 2^\omega$;
6. if $\omega \leq \beta < \alpha$ is even then $\{m \in \omega: x_{m,i}(\beta) = x_{K_{\beta}(i)}(\beta) \forall i \in n + 1\}$ is finite;
7. if $\omega \leq \beta < \alpha$ is odd then $\{m \in \omega: x_{m,i}(\beta) = x_{K_{\beta}(i)}(\beta) \forall i \in n \cup \{n + 1\}\}$ is finite.

At stage $\omega$, only condition (5) is not trivially satisfied. Choose $x_{m,i}|\alpha \in 2^\omega$ for each $m \in \omega$ and $i < n + 2$ such that the sequence $\{(x_{m,i})_{i < n + 2}: m \in \omega\}$ is dense in $(2^\omega)^{n+2}$.

At limit stage $\alpha$, define $x_{f,i}|\alpha = \bigcup_{\delta < \gamma \in \omega} x_{f,i}\gamma$ for each $f \in D^{<\omega}$ and $i < n + 2$.

Clearly, all seven inductive hypothesis will be satisfied.

At successor stage $\alpha = \gamma + 1$, define $x_{f,i}|\gamma < n + 2$ for every $f \in 2^\gamma$ so that $x_{f,i}|\gamma$ is an accumulation point of the sequence $\{x_{f_{\gamma}(i,m)}|\gamma\}_{i \in S_{\gamma}}$, $m \in \omega$. We will assume that $\alpha$ is odd (thus, $\gamma$ is even), as the other case is similar.

As in other constructions of countably compact groups without non-trivial convergent sequences (see [16,10,19]), some form of Martin’s Axiom will be used to split many subsequences of a sequence at a successor stage.

In order to apply MA$_{\text{countable}}$ rather than MA in the successor stage, we use the following definition (compare with [19]):

**Definition 4.1.** For each $\xi < \alpha$ define by induction a set $\mathcal{I}_\xi$ as $\omega$ if $\xi \leq \omega$ and $\bigcup\{\mathcal{I}_{\text{dom } f} \cup \{f\}: (\{f\} \times (n + 2)) \cap (\bigcup_{m \in \omega \cup \{i,j\}} \mathcal{F}_\xi(i, m) \neq \emptyset)\}.

An easy induction leads to the following result:

**Lemma 4.2.** The set $\mathcal{I}_\xi$ is countable for each $\xi < \alpha$. 

We define now the partial order for the successor stage \( \alpha \).

Throughout the remainder of this section, let \( \mathcal{I} = \bigcup_{(f,j) \in \bigcup_{\nu \in \omega} K_{\nu}(j)} I_{\text{dom } f} \) and fix a function \( r \) whose domain is \( F \times (n+1) \), where \( F \) is a finite subset of \( \mathcal{I} \), \( \text{dom } r \supseteq \bigcup_{i<\omega+2} K_{\nu}(i) \) and \( F \cap \omega \in \omega \). If \( K_{\nu}(0) \neq \emptyset \), choose \( r \) so that \( \sum_{(f,j) \in K_{\nu}(0)} r(f,i) = 1 \).

With the aid of \( \text{MA}_{\text{countable}} \) we will construct a function \( \psi : \mathcal{I} \times (n+2) \rightarrow 2 \) and define \( x_{f,i}(\gamma) = \psi(f,i) \) for each \( (f,i) \in \mathcal{I} \times (n+2) \) so that the inductive conditions restricted to \( \mathcal{I} \times (n+2) \) are satisfied. The definition of \( x_{f,i}(\gamma) \) for \( (f,i) \in (\alpha \setminus \mathcal{I}) \times (n+2) \) will be done by induction later and do not require \( \text{MA}_{\text{countable}} \).

**Definition 4.3.** Let \( \mathbb{P} \) be a partial order whose underlying set is the family of all functions \( p : F \times (n+2) \rightarrow 2 \) with \( F \in [\mathcal{I}]^{<\omega} \) and \( F \cap \omega \in \omega \). Let \( M_p \) be the unique integer such that \( F \cap \omega = M_p \). Given \( p, q \in \mathbb{P} \), define \( p \preceq q \) if and only if \( p \supseteq q \) and for each \( k \in [M_q, M_p) \) there exists \( i < n+1 \) such that \( p(k,i) = 1 - \sum_{(g,j) \in K_{\nu}(i)} r(g,j) \).

The dense sets defined below will be used to define \( x_{f,i}(\gamma) \) for each \( f \in \mathcal{I} \) and \( i < n+2 \). The proof of the next lemma is left to the reader.

**Lemma 4.4.** The set \( \{ p \in \mathbb{P} : \text{dom } p \supseteq \{ f \} \times (n+2) \} \) is dense in \( \mathbb{P} \) for each \( f \in \mathcal{I} \).

The following sets will be used to define the dense sets needed for condition 4.

**Definition 4.5.** For each \( f \in \mathcal{I} \) and \( F \subseteq \gamma \) finite, define \( \mathcal{E}(f,F) = \{ m \in \omega : \forall i \in S_{\text{dom } f} \left( \sum_{(g,j) \in \mathcal{F}_{\text{dom } f}(i,m)} x_{g,j} \cap F = x_{f,i}(F) \} \).\)

Note that by hypothesis, the sets defined above are infinite.

**Lemma 4.6.** The set \( \mathcal{D}(f,F,v,M) = \{ p \in \mathbb{P} : \exists m \in \mathcal{E}(f,F) \setminus M \text{ such that } \text{dom } p \supseteq \bigcup_{i \in S_{\theta}} \mathcal{F}_{\beta}(i,m) \text{ and } (\forall i \in S_{\theta}) \sum_{(g,j) \in \mathcal{F}_{\beta}(i,m)} p(g,j) = v(i) \} \) is dense for each \( f \in \mathcal{I} \), \( F \in [\gamma]^{<\omega} \), \( v \in 2^\mathcal{P} \) (where \( \beta = \text{dom } f \)) and \( M \in \omega \).

**Proof.** Fix \( f, \beta, F, v \) and \( M \) as above and let \( q \) be an arbitrary element of \( \mathbb{P} \). Set \( E = \{(g,j) : (g,j) \in \text{dom } q \} \). By hypothesis, there exists \( m \in \mathcal{E}(\beta,F) \setminus M \) such that \( E \cup \{ (g,j) : i \in S_{\beta} \} \) is linearly independent. Indeed, \( E \cup \{ (M_q, j) : j < n+1 \} \) is a linearly independent set of size strictly bigger than \( E \cup \{ (g,j) : i < S_{\beta} \} \). Therefore, there exists \( k_{M_q} \in n+1 \) such that \( E \cup \{ (g,j) : i \in S_{\beta} \} \cup \{ ((t,k_i)) : t \in [M_q, N) \} \) is linearly independent. Indeed, \( E \cup \{ (M_q, j) : j < n+1 \} \cup \{ (M_q, k_{M_q}) \} \) is a linearly independent set of size strictly bigger than \( E \cup \{ (g,j) : i \in S_{\beta} \} \cup \{ (M_q, k_{M_q}) \} \). Proceeding by induction, one can find \( k_i \in n+1 \) for each \( t \in [M_q, N) \) as required.

Define \( \phi : E \cup \{ (f,i,m) : i \in S_{\beta} \} \cup \{ ((t,k_i)) : t \in [M_q, N) \} \rightarrow 2 \) as \( \phi((g,j)) = q(g,j) \) if \( (g,j) \in E \); \( \phi((f,i,m)) = v(i) \) if \( i \in S_{\beta} \) and \( \phi(((t,k_i))) = 1 - \sum_{(f,j) \in K_{\nu}(i)} r(f,j) \).
The dense sets below will be needed for condition (5).

**Definition 4.7.** For each $A \in [\gamma]^{\omega}$, $t \in 2^A \times 2^A$ and $(a_0, a_1) \in 2 \times 2$, let $C(t, a_0, a_1, M) = \{p \in P : \exists m \in \omega \setminus M : (x_{m,n}|A, x_{m,n+1}|A) = t$ and $p(n, m + j) = a_j$ for $j < 2\}$.

**Lemma 4.8.** The sets defined above are dense in $P$.

**Proof.** Similar to the proof of Lemma 4.6. □

We are now ready to finish the construction. Applying MA$_{\text{countable}}$, there exists a filter $\mathcal{G}$ over $P$ which intercepts every dense set defined in Lemmas 4.4, 4.6 and 4.8. As $\mathcal{G}$ intercepts the dense sets in Lemma 4.4, the set $\psi = \bigcup \mathcal{G}$ is a function from $I \times (n + 2)$ into $2$. Define $x_{f,i}(\gamma) = \psi(f, i)$ for every $(f, i) \in I \times (n + 2)$.

Clearly conditions (1) and (2) are satisfied for each $f \in I$ and $i \in (n + 2)$. Condition (3) is satisfied by the choice of $r$ and the fact that $\psi$ extends $r$. We leave for the reader to check that condition (4) for $f \in I$ and condition (5) follow from the fact that $\mathcal{G}$ intercepts the dense sets in Lemmas 4.6 and 4.8, respectively.

We will check condition (6). Let $p$ be an element of $\mathcal{G}$. Then, for each $m \in \omega \setminus M_p$ (where $M_p$ is defined with the partial order $P$), there exists $q \in \mathcal{G}$ such that $q \leq p$ and $m \in M_q$. Then, by the definition of the ordering, there exists $i < n + 1$ such that $x_{m,i}(\gamma) = q(m, i) \neq \sum_{(g,j) \in K_p(i)} r(g, j) = x_{K_p(i)}(\gamma)$. Thus, the set in condition (6) for $\beta = \gamma$ is contained in $M_p$, therefore, is finite. Condition (7) is trivially satisfied at this stage, since $\gamma$ is even. At stage $\alpha$, even, the proof of condition (7) is similar to the proof of condition (6).

We still have to define $x_{f,i}(\gamma)$ for $f \in D^\omega \setminus I$ and $i < n + 2$. Let $\eta$ be the least ordinal for which $x_{f,i}(\gamma)$ has not yet been defined for some $f \in D^\omega$. For each such an $f$, fix an ultrafilter $U$ over $\omega$ such that $(x_{f,i}(\gamma))_{i \in S_\eta}$ is the $U$-limit of the sequence $\{x_{f,i}(m) \mid m \in \omega \}$. Define $x_{f,i}(\gamma)$ arbitrarily if $i \in (n + 2) \setminus S_\eta$ and $x_{f,i}(\gamma) = U$-limit $\{(x_{f,i}(m) \mid m \in \omega) \mid i \in S_\eta \}$. Clearly conditions (1), (2) and (4) are now satisfied by any $(f, i) \in D^\omega \times (n + 2)$.

5. $\mathcal{G}$-spaces and finite products

Modifying a game due to Gruenhage [15], Bouziad [3] defined the following game:

Let $X$ be a space and $x \in X$. The game $\mathcal{G}(x, X)$ is played as follows: there are two players (I) and (II). Player (I) plays first and chooses a neighborhood $U_0$ of $x$. Player (II) then responds by choosing $x_0 \in U_0$. After choosing $U_0, \ldots, U_n$ and points $x_i \in U_i$, for each $i \leq n$, player (I) chooses a neighborhood $U_{n+1}$ of $x$ and player (II) chooses $x_{n+1} \in U_{n+1}$.

Player (I) wins if $\{x_n : n \in \omega\}$ has an accumulation point in $X$, otherwise player (II) wins.
The space $X$ is a $G$-space if player (I) has a winning strategy for each game $G(x, X)$.

In [12], the authors showed that for each $n \in \omega$ there exists a space $X$ such that $X^n$ is a $G$ space but $X^{n+1}$ is not. They asked whether there is such an example that it is a topological group. We give a consistent answer using Martin’s Axiom.

We use the following lemma:

**Lemma 5.1.** Let \( \{x\} \cup \{x_n: n \in \omega\} \subseteq X \) and \( \{y\} \cup \{y_n: n \in \omega\} \subseteq Y \) such that $x$ is an accumulation point of \( \{x_n: n \in \omega\} \) and $y$ is an accumulation point of \( \{y_n: n \in \omega\} \). If for every injections $\psi, \phi: \omega \to \omega$ with $\phi^{-1}[\omega] \cap \psi^{-1}[\omega] = \emptyset$, the sequence \( \{(x_{\phi(k)}, y_{\psi(k)})\}: k \in \omega \) is closed discrete then player (II) has a winning strategy for the game $G((x, y), X \times Y)$. In particular, the product $X \times Y$ is not a $G$-space.

**Proof.** Without loss of generality, player (I) chooses neighbourhoods $U_n$ and $V_n$ of $x$ and $y$, respectively, at the $n$th move. Suppose that player (II) has chosen $k_0, \ldots, k_{n-1}$ and $l_0, \ldots, l_n$ pairwise distinct such that $(x_{k_i}, y_{k_i}) \in U_i \times V_i$ for each $i \leq n$. Player (II) chooses $k_n \in \omega \setminus \{k_0, \ldots, k_{n-1}, l_0, \ldots, l_{n-1}\}$ and $l_n \in \omega \setminus \{l_0, \ldots, l_{n-1}, k_0, \ldots, k_n\}$ such that $(x_{k_n}, y_{k_n}) \in U_n \times V_n$. By hypothesis, the sequence \( \{(x_{k_n}, y_{k_n}): n \in \omega\} \) is closed discrete. Therefore, player (II) has a winning strategy and $X \times Y$ is not a $G$-space.

**Example 5.2.** (MAcountable) For each positive integer $n$, there exists a topological group $G$ such that $G^n$ is a $G$-space but $G^{n+1}$ is not.

**Proof.** Fix $f \in 2^\omega$. In the construction of Example 2.2, we can make the sequence \( \{|x_{m,j}: j < n\}: m \in \omega\} \) dense in $(2^\omega)^n$. There are $\epsilon$ many pairs of injections $\psi, \phi: \omega \to \omega$ whose ranges are disjoint, thus, we can modify the partial orders at successor stages to make all the sequences \( \{|x_{\phi(m),j}: j < n\} \cup \{|x_{\phi(m),n}\}: m \in \omega\} \) closed and discrete in $(G_f)^{n+1}$.

For example, for a fixed pair $\phi$ and $\psi$ of injective function with disjoint range and a function $r: E \times (n + 2) \to 2$ with $\psi^{-1}(\omega \cap E) = \psi^{-1}(\omega \cap F) \in \omega$, let $P$ be a partial order whose underlying set is the family of all functions $p: F \times (n + 2) \to 2$ with $F \in [I]^{<\omega}$ and $\phi^{-1}(\omega \cap F) = \psi^{-1}(\omega \cap F) \in \omega$. Given $p, q \in P$, define $p \leq q$ if and only if $p \supsetneq q$ and for each $k \in [M_q, M_p]$ either there exists $i < n$ such that $p(\psi(k), i) \neq \sum_{(\xi, j) \in \xi_i} r(\xi, j)$ or $p(\phi(k), n) \neq \sum_{(\xi, j) \in \xi_i} r(\xi, j)$.

Applying Lemma 5.1 above, $(G_f)^{n+1}$ is not a $G$-space.

The product $(G_f)^n$ is countably compact, therefore, a $G$-space.

**Note added in September 2003**

Recently, the authors of [13] showed the existence of two countably compact groups whose product is not countably compact from the existence of a selective ultrafilter and asked whether there is a countably compact group whose square is not countably compact from the existence of a selective ultrafilter.

We can extend their question to finite powers:
Question 5.3. Does the existence of a selective ultrafilter implies, for each $n \in \omega$, the existence of a countably compact topological group $G$ such that $G^n$ is countably compact but $G^{n+1}$ is not countably compact?

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