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Curvature and the Eigenvalues of the Laplacian for Elliptic Complexes

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INTRODUCTION

Let M be a compact m dimensional Riemannian manifold without boundary. There is a natural connection induced by the metric on the tangent bundle TM. By using this connection, we can construct forms in the DeRham cohomology which represent the Pontrjagin classes of the manifold. These forms can be computed in any coordinate system by functorial expressions in the first and second order derivatives of the metric tensor g_{ij}. The Pontrjagin classes give rise to local formulas in the derivatives of the Riemannian metric which are invariantly defined, i.e., which are independent of the coordinate system in which they are evaluated.

We obtain a similar representation of the Chern classes of a vector bundle in terms of the curvature tensor associated with a connection. We will say that a map from metrics and connections to p forms is given by a local formula if given any coordinate system and any frame for the bundle, we can compute it in terms of the derivatives of the metric and of the connection. For such a map, there is a natural notion of order. We will discuss such local formulas in the derivatives of a metric and of a connection greater detail in Section I.

We can construct maps from metrics and connections to p forms by taking combinations of Chern and Pontrjagin classes. These maps will be given by local formulas of order p. In this paper, we will show the following theorem.

THEOREM. *Let R be a map from connections and metrics to p forms which is given by local formula of order n. If $n < p$, $R = 0$ while if $n = p$, R can* *be expressed as a combination of Pontrjagin classes of the metric and Chern classes of the connection.*

Such maps R arise naturally from the study of the asymptotic behavior of the eigenvalues for an elliptic complex. We summarize as follows: let E_0 and E_1 be smooth vector bundles over M with a smooth inner product (,). Let $d\colon \Gamma(E_0) \to \Gamma(E_1)$ be a first order elliptic differential operator. We define the positive self-adjoint second order elliptic operators:

$$D_0 = d^*d \qquad \text{and} \qquad D_1 = dd^*.$$

Let $\mu_j{}^i$ denote the eigenvalues of D_i and $\Theta_j{}^i$ the corresponding eigensections to E_i for $j = 1, 2, \ldots$. We define:

$$f(t, x, D_i) = \sum_{j=1}^{\infty} \exp(-t\mu_j{}^i)(\Theta_j{}^i, \Theta_j{}^i)(x).$$

By using the techniques of pseudo-differential operators which depend upon a complex parameter developed by Seeley [6, 7], we can show that f is well defined for $\mathrm{Re}(t) > 0$ and has an asymptotic expansion as $t \to 0^+$ of the form:

$$f(t, x, D_i) \sim \sum_{n=0}^{\infty} B_n(x, D_i)\, t^{(n-m)/2} \ (B_n = 0 \text{ for } n \text{ odd}).$$

The functions $B_n(x, D_i)$ are smooth and can be computed functorially in terms of the derivatives of the total symbol of the operator in any local system. We define

$$B_n(x, d) = B_n(x, D_0) - B_n(x, D_1).$$

It is well known that it is possible to compute the index of the operator d by the formula

$$\int_M B_n(x, d)\, d\mathrm{vol} = \begin{cases} 0 & n \neq m, \\ \mathrm{index}(d) & n = m. \end{cases}$$

This gives a formula for computing the index of any elliptic complex by integrating a local formula in the derivatives of the total symbol of the operator.

For an arbitrary elliptic operator, there is no reason to expect that this formula will agree with the formula given by the index theorem of

Atiyah–Singer [2]. However, there are many elliptic complexes for which the symbol of the operator depends on the orientation of M, the Riemannian metric, and on a connection for some vector bundle E. For such a complex, we obtain a local formula in the derivatives of the metric and of the connection for computing the index.

If M is an orientable manifold, we can define the signature complex on M. Let E be a complex vector bundle and let D be a Riemannian connection on E. By using the connection D, we can define the signature complex with coefficients in E. Let $R^m_{n,r}$ denote the local formula which describes the asymptotic behavior of the eigenvalues for this complex; m is the dimension of the manifold and r is the dimension of the vector bundle. Since the roles of the positive and negative spaces of the signature complex are interchanged if we reverse the orientation of the manifold M, the local formulas $R^m_{n,r}$ depend upon the orientation of the coordinate system in which they are computed. Consequently, we can view the invariants $R^m_{n,r}$ as giving a map from connections and metrics to m-forms.

We will define the notion of order in Section I; it will be clear by dimensional analysis that the local formulas $R^m_{n,r}$ are of order n. This implies that $R^m_{n,r} = 0$ for $n < m$, while $R^m_{n,r}$ can be computed in terms of the Pontrjagin classes of M and the Chern classes of E.

Patodi has kindly pointed out that it is possible to use this result to prove the Atiyah–Singer index theorem: From K-theory, it suffices to prove the index theorem for the special case of the signature complex with coefficients in a vector bundle E. Since $R^r_{m,m}$ can be expressed in terms of the Chern and Pontrjagin classes, there must be a certain combination of Chern classes and Pontrjagin classes which integrate to give the index for any such complex. Since this formula must be unique, this proves the Atiyah–Singer index theorem for this special case and, hence, in general.

In addition to applying this result to obtain a global theorem like the index theorem, we can also obtain results concerning the local asymptotic behavior of the eigenvalues for certain complexes. For the classical elliptic complexes we will use the following notation for the invariants $B_n(x, d)$:

$P_n{}^m$ for the DeRham complex;

$P_n^{s,m}$ for the signature complex;

$P_n^{sp,m}$ for the spin complex; and

$P_n^{a,m}$ for the Dolbeault complex.

By dimensional analysis, we can show that all of these local formulas are of order n in the derivatives of the metric. Since any germ of a metric can be imbedded in a spin manifold, in order to identify these invariants of the metric, it suffices to restrict our attention to spin manifolds. By using suitably chosen vector bundles, we can express the DeRham and signature complexes in terms of the spin complex with coefficients in the vector bundle. As was done with the signature complex, this implies that these local formulas vanish identically for $n < m$, while for $n = m$, they are computable in terms of the Pontrjagin classes of the metric and the Chern classes of the bundle. Since these formulas integrate to give the index of the complex, they are necessarily unique, and we have obtained an identification of these invariants with the classical formulas for computing the index.

Let M be a complex manifold. It is possible to represent the Dolbeault complex in terms of the spin complex if the metric is Kaehler. For a Kaehler metric, therefore, $P_n^{a,m}(x, \text{metric})$ vanishes for $n < 2m$, and $P_m^{a,m}(x, \text{metric})$ is the Riemann–Roch invariant. Conversely, we have shown that there exist complex metrics (which are not Kaehler) such that $P_n^{a,m}(x, \text{metric})$ does not vanish for $n \geqslant m/2$ and such that $P_m^{a,m}(x, \text{metric})$ is not the Riemann–Roch invariant for $m > 2$. This result is proved by combinatorial methods which are not of essential interest. In the last section of this paper we present some other combinatorial results which we have obtained.

Patodi [4, 5] has obtained many of these results for the classical elliptic complex by studying the fundamental solution of the appropriate heat equation. Atiyah, Bott, and Patodi have also proved an equivalent classification theorem for local formulas of order q from metrics and connections to p forms. Their techniques of proof rely more heavily on the results of classical invariance theory but are essentially equivalent to the technique used in this paper. Their proof is to be published shortly.

1.

In this section we will describe the ring of local formulas in the derivatives of a Riemannian metric and of a connection. First we define these notions for germs of metrics and connections on R^m; later these notions will be extended to manifolds.

Let $\bar{G}$ be the germ of a Riemannian metric defined in a neighborhood of 0 in R^m. Let $\bar{X}$ be the canonical system of coordinates on R^m. We define $g_{ij}(\bar{G})(\bar{x})$ to be the germ of a function on R^m by the formula

$$g_{ij}(\bar{G})(\bar{x}) = G(\partial/\partial \bar{X}_i , \partial/\partial \bar{X}_j)(\bar{x}).$$

If ω is a multiindex, we define $g_{ij/\omega}(\bar{G})(\bar{x})$ by

$$g_{ij/\omega}(\bar{G})(\bar{x}) = \partial/\partial \bar{X}^\omega g_{ij}(\bar{G})(\bar{x}).$$

We use the notation $g_{ij/\omega}(\bar{G})$ to denote the evaluation $g_{ij/\omega}(\bar{G})(0)$. We regard the $g_{ij/\omega}$ as variables which we evaluate on germs of metrics on R^m.

We will consider only germs of metrics $\bar{G}$ which are orthonormal at 0. Therefore, we assume that $g_{ij}(\bar{G}) = \delta_{ij}$ and we ignore the g_{ij}'s as variables. Let $\mathscr{P}_m$ be the polynomial ring generated by the variables $g_{ij/\omega}$ for $1 \leqslant i, j \leqslant m$ and $| \omega | > 0$. Since the $g_{ij/\omega}$ variables are symmetric in i, j, we introduce the equivalence relation $g_{ij/\omega} = g_{ij/\omega}$. $\mathscr{P}_m$ is a pure polynomial algebra over R. An element P of $\mathscr{P}_m$ is a local formula which is defined on germs of Riemannian metrics on R^m.

The invariants $R_{n,r}^{s,m}$ for the signature complex with coefficients in a vector bundle E are local formulas in the derivatives of the metric and in the derivatives of a connection. Let E be an r dimensional vector bundle with a smooth Hermitian inner product. We describe the ring $\mathscr{Q}_{m,r}$ of local formulas in the derivatives of a connection on E as follows: Let $\bar{E}$ be the complex r dimensional vector bundle $R^m \times C^r$, let $\bar{F}r$ be the canonical frame $(\bar{e}_1 ,..., \bar{e}_r)$, and let $\bar{D}$ be the germ of a connection on $\bar{E}$. We define the Hermitian inner product on $\bar{E}$ so that $(\bar{e}_i , \bar{e}_j) = \delta_{ij}$. We define $w_{stj/\omega}(\bar{D})(\bar{x})$ as the germ of a function on R^m by the formula:

$$w_{stj/\omega}(\bar{D})(\bar{x}) = \partial/\partial \bar{X}^\omega[(\bar{\nabla}_{\partial/\partial \bar{X}_j} e_s , e_t)]$$

Let $\mathscr{Q}_{m,r}$ be the complex polynomial algebra generated by the variables

$$w_{stj/\omega} \quad \text{for} \quad 1 \leqslant s, t \leqslant r; \qquad 1 \leqslant j \leqslant m,$$

and ω a multiindex. If Q is an element of $Q_{m,r}$, we can evaluate $Q(\bar{D}) = Q(\bar{D})(\bar{x})$ in the obvious fashion.

Let $\mathscr{R}_{m,r} = \mathscr{Q}_{m,r} \otimes \mathscr{P}_{m,r}$ be the ring of local formulas in the formal derivatives of a Riemannian metric and of a connection. We evaluate $R(\bar{G}, \bar{D}) = R(\bar{G}, \bar{D})(0)$ in the obvious fashion if R is an element of $\mathscr{R}_{m,r}$. We define the functions, order, degree, type, and length on the monomials A of this ring, as follows: Let A be a monic monomial of $\mathscr{R}_{m,r}$. We decompose $A = BC$ where $B \in \mathscr{P}_m$ and $C \in \mathscr{Q}_{m,r}$. Let $B = g_{i_1 j_1/\omega_1} \cdots g_{i_p j_p/\omega_p}$ and $C = w_{s_1 t_1 k_1/\bar{\omega}_1} \cdots w_{s_q t_q k_q/\bar{\omega}_q}$. We define

$$\operatorname{ord}(A) = \sum_{i=1}^{p} |\omega_i| + \sum_{j=1}^{q} (|\bar{\omega}_j| + 1),$$

$$L(A) = p + q,$$

$$t(A) = (|\omega_1|, ..., |\omega_p|, 0, ...),$$

where the ω_i are ordered so that $|\omega_1| \geqslant |\omega_2| \geqslant \cdots$

$$\deg_k A = \sum_{i=1}^{p} \omega_i(k) + \sum_{j=1}^{q} \bar{\omega}_j(k) + \text{the number of times}$$

the index k appears in the collection

$$(i_1, j_1, ..., i_p, j_p, k_1, ..., k_q).$$

We will also use the notation $g_{ij/k_1 \cdots k_p \omega}$ to denote the variable formally given by $\partial/\partial X_{k_1} \cdots \partial/\partial X_{k_p}(g_{ij/\omega})$. Thus, for example, if

$$A = g_{11/22} g_{12/33} w_{st3/11},$$

then

$$\operatorname{ord}(A) = 7, L(A) = 3, t(A) = (2, 2, 0, ...), \quad \text{and} \quad \deg_1(A) = 5.$$

If $R \in \mathscr{R}_{m,r}$ is any polynomial, then let $c_R(A)$ be the coefficient of A in R. If $c_R(A) \neq 0$, then we say that A is a monomial of R. We can express $R = \sum c_R(A)A$. The functions $c_R(A)$ are linear functions on the polynomial ring $\mathscr{R}_{m,r}$. We use these functions to prove various classifying theorems later in this paper. We say that a collection $A_1, ...$ classifies a subspace S of $\mathscr{R}_{m,r}$ if for every $R \in S$, $c_R(A_i) = 0$ for all these A_i implies that $R = 0$. This implies that $\dim(S) \leqslant$ the number of such A_i. The difficulty is discovering the correct minimal number of such classifying monomials for a given subspace.

This ring of local formulas gives maps from germs of connections and metrics to R. The following lemma states that such a formula is completely determined by its evaluation on connections and metrics. For technical reasons, it is often convenient to restrict attention to Riemannian connections.

LEMMA 1.1. *Let R be a nonzero polynomial of $\mathscr{R}_{m,r}$. Then there is a germ of a Riemannian metric $\bar{G}$ and a germ of a Riemannian connection $\bar{D}$ so that $R(\bar{G}, \bar{D}) \neq 0$.*

Proof. We define new variables $x_{stj/\omega}$ and $y_{stj/\omega}$:

$$x_{stj/\omega} = [w_{stj/\omega} - w_{tsj/\omega}]/2 \quad \text{for } s < t,$$

$$y_{stj/\omega} = [w_{stj/\omega} + w_{tsj/\omega}]/2i \quad \text{for } s \leqslant t.$$

We can express $w_{stj/\omega}$ and $w_{tsj/\omega}$ in terms of these new variables:

$$w_{stj/\omega} = x_{stj/\omega} + iy_{stj/\omega} \qquad \text{for } s < t,$$

$$w_{tsj/\omega} = -x_{stj/\omega} + iy_{stj/\omega} \qquad \text{for } s < t,$$

$$w_{sstj/\omega} = iy_{ssj/\omega}\,.$$

We can express R in terms of these new variables $R(g_{ij/\omega}, x_{stj/\omega}, y_{stj/\omega})$. Since $R \neq 0$, we can find *real* constants so that $R(g^0_{ij/\omega}, x^0_{stj/\omega}, y^0_{stj/\omega}) \neq 0$. By Taylor's theorem, we can find the germ of a metric $\bar{G}$ so that $g_{ij/\omega}(\bar{G}) = g^0_{ij/\omega}$. We can also find the germ of a connection $\bar{D}$ so that $x_{stj/\omega}(\bar{D}) = x^0_{stj/\omega}$ and so that $y_{stj/\omega}(\bar{D}) = y^0_{stj/\omega}$. We replace $\bar{D}$ by the Riemannian connection $(\bar{D} + \bar{D}^*)/2$; since $x^0_{stj/\omega}$ and $y^0_{stj/\omega}$ are *real* constants, we still have that $x_{stj/\omega}(\bar{D}) = x^0_{stj/\omega}$ and $y_{stj/\omega}(\bar{D}) = y^0_{stj/\omega}$. Clearly $R(\bar{G}, \bar{D}) \neq 0$.

We extend these local formulas to manifolds. If G is a Riemannian metric on a manifold M and if D is a connection on a complex vector bundle E, then there is no natural way to define $R(G, D)(x)$. First, we introduce a coordinate system for M and a frame for E.

Let $F: N \to M$ be a diffeomorphism. We pull back the metric G to N as follows: let $Y_1, Y_2 \in TN_y$. Then

$$F^*G(Y_1, Y_2)(y) = G(F_*Y_1, F_*Y_2)(Fy).$$

Since F_* is injective, F^*G is a Riemannian metric on N. We similarly define F^*E as a complex r dimensional vector bundle over N with a connection F^*D.

We define our local formulas on manifolds using this notation. Let M, G, E, D be as before, and let x_0 be a point of M. We let X be a coordinate system centered at x_0 which is normalized with respect to the metric at x_0, i.e., $G(\partial/\partial X_i, \partial/\partial X_j)(x_0) = \delta_{ij}$. X is a local diffeomorphism from $M \to R^m$. There is a unique germ of a metric $\bar{G}(G, X)$ on R^m such that $X^*[\bar{G}(G, X)] = G$ near x_0. If P is an element of $\mathscr{P}_m$, we define $P(G, X)(x_0) = P(\bar{G}(G, X))$. P generally depends upon both G and X. We say that P is invariant if P depends only on G and is independent of the coordinate system X. For such a P, we define $P(G)(x_0)$ as this invariant value. The scalar curvature K is one such example. We can express $K = g_{12/12} - (g_{11/22} + g_{22/11})/2 +$ lower order terms for a 2 dimensional manifold. For a general coordinate system, the formula is more complicated. Since we have normalized our coordinate system, we simplify the formula by assuming $g_{ij} = \delta_{ij}$ at x_0.

E has a Hermitian inner product. So as not to involve the inner product in our local formulas, we will work solely with orthonormal frames. Let Fr be an orthonormal frame for E in a neighborhood of x_0. There is a unique connection $\bar{D}(X, Fr, D)$ so that $X^*\bar{D} = D$ and $X^*(\bar{F}r) = Fr$. We define

$$Q(X, D, Fr)(x_0) = Q(\bar{D}(X, D, Fr)).$$

In a similar fashion, we define $R(X, G, D, Fr)(x_0)$ for $R \in \mathscr{R}_{m,r}$.

Let $F: N \to M$ be a diffeomorphism. Since our definitions were completely functorial, we have the identity,

$$R(F^*X, F^*G, F^*D, Fr)(y) = R(X, G, D, Fr)(Fy).$$

A polynomial R is said to be skew-invariant if $R(X, G, D, Fr)$ depends only on the orientation of the coordinate system X, on the metric G, and on the connection D. Such a polynomial can also be regarded as an invariantly defined map from metrics and connections to m forms.

We can construct examples of skew-invariant polynomials as follows: let P be a Pontrjagin class mapping metrics to $4k$ forms and let Q be a Chern class mapping connections to $m - 4k$ forms. The product PQ belongs to $\mathscr{R}_{m,r}$ and is an invariant map of order m from metrics and connections to m forms. PQ is skew-invariant as a map to C.

Let $\prod_r (m)$ denote the number of partitions of m into integers $\leqslant r$ and let $\prod (m) = \prod_m (m)$. The subspace of $\mathscr{R}_{m,r}$ which is generated by such products PQ has dimension $= \sum_k \prod_r ((m - 4k)/2) \prod (k)$. We will prove that the subspace of all skew-invariant polynomials of order m has dimension at most $\sum_k \prod_r ((m - 4k)/2) \prod (k)$. This will prove that all skew-invariant polynomials of order m can be expressed in terms of the Pontrjagin classes of the metric and the Chern classes of the connection. We will also show that there are no skew-invariant polynomials of order $<m$.

Unfortunately, there are a great many more *invariant* polynomials than there are skew-invariant polynomials. We will need to impose additional conditions on an invariant polynomial in order to obtain similar vanishing theorems and to obtain a corresponding characterization of the Euler class.

For technical reasons, it is much easier to work with Riemannian connections when considering the formula $R_{n,r}^{s,m}$. Consequently, we will assume that a polynomial R is skew-invariant when it is evaluated on Riemannian connections. We will prove that this implies that R is skew-invariant when evaluated on an arbitrary connection. We will give an alternate formulation of the notion of invariance in terms of an action by a group on the ring $\mathscr{R}_{m,r}$.

Let $\bar{F}$ be the germ of a diffeomorphism of R^m and $\bar{U}$ the germ of a map from R^m to the unitary group $U(r)$. We define an action $\bar{F}^*$ and $\bar{U}^*$ on the ring $\mathscr{Q}_{m,r}$: let $\bar{G}$ be the germ of a Riemannian metric and $\bar{D}$ the germ of a connection. We pull-back $\bar{G}$ and $\bar{D}$ to obtain $F^*\bar{G}$ and $F^*\bar{D}$. Let $R \in \mathscr{R}_{m,r}$ and let $\bar{F}^{-1}$ be a new coordinate system on R^m. Since $\bar{F}^*(\bar{F}^{-1}) = \bar{X}$, we have the identity:

$$R(\bar{F}^*\bar{G}, \bar{F}^*\bar{D}) = R(\bar{X}, \bar{F}^*\bar{G}, \bar{F}^*\bar{D}) = R(\bar{F}^*[\bar{F}^{-1}], \bar{F}^*\bar{G}, \bar{F}^*\bar{D})$$
$$= R(\bar{F}^{-1}, \bar{F}^*\bar{G}, \bar{F}^*\bar{D}).$$

The pull-back expresses $\bar{G}$ and $\bar{D}$ with respect to the new coordinate system $\bar{F}^{-1}$.

Since $\bar{F}_*(\partial/\partial\bar{X}_i) = \sum_s \partial\bar{F}_s/\partial\bar{X}_i\, \partial/\partial\bar{X}_s$, we compute that

$$g_{ij}(\bar{F}^*\bar{G}) = \bar{F}^*\bar{G}(\partial/\partial\bar{X}_i, \partial/\partial\bar{X}_j) = \bar{G}(\bar{F}_*\partial/\partial\bar{X}_i, \bar{F}_*\partial/\partial\bar{X}_j)$$
$$= \sum_{s,t} \partial\bar{F}_s/\partial\bar{X}_i \partial\bar{F}_t/\partial\bar{X}_j g_{st}.$$

With this in mind, we define

$$\bar{F}^*(g_{ij}) = \sum_{s,t} \partial\bar{F}_s/\partial\bar{X}_i(0)\, \partial\bar{F}_t/\partial\bar{X}_j(0)\, g_{st}.$$

In a similar way we define $F^*(g_{ij/\omega})$ and $F^*(w_{stj/\omega})$ by using the maps $d^nF(0)$ so that if $R \in \mathscr{R}_{m,r}$, then

$$\bar{F}^*R(\bar{G}, \bar{D}) = R(\bar{F}^*\bar{G}, \bar{F}^*\bar{D}) \qquad \text{for all } \bar{G}, \bar{D}.$$

By Lemma 1.1, this identity uniquely defines the action $\bar{F}^*R$. Since we are only considering germs of metrics with $g_{ij}(\bar{G}) = \delta_{ij}$, we restrict our attention to diffeomorphisms $\bar{F}$ such that $d\bar{F}(0)$ is an orthogonal rotation.

Let $\bar{U}$ be the germ of a mapping from $R^m \to U(r)$. Let $\bar{F}r = (\bar{e}_1, ..., \bar{e}_r)$ be the canonical frame for $R^m \times C^r$. We define a new frame $\bar{U}^*\bar{F}r = [(\sum_{t=1}^{r} \bar{U}_{st}\bar{e}_t)_{s=1,...,r}]$. We express the connection matrix relative to this new frame as follows:

$$w_{st}(\bar{U}^*\bar{F}r, \bar{D}) = \sum \bar{U}_{ss'} w_{s't'}(\bar{F}r, \bar{D})[U^{-1}]_{t't} + d\bar{U}_{ss'}[U]^{-1}_{s't}.$$

From this identity, we define an action $\bar{U}^*$ on $\mathcal{Q}_{m,r}$ so that

$$\bar{U}^*Q(\bar{D}, \bar{F}r) = Q(\bar{D}, \bar{U}^*\bar{F}r).$$

We set $U^* = 1$ on $\mathcal{P}_m$ and define a corresponding action on the tensor algebra $\mathcal{R}_{m,r}$.

LEMMA 1.2. *Let $R \in \mathcal{R}_{m,r}$. Then the following conditions which define the notion of skew-invariance are equivalent.* (1) *$R(X, G, D, Fr)(x_0)$ depends only on D, G, and the orientation of X for all connections D.*

(2) *$R(X, G, D, Fr)(x_0)$ depends only on D, G, and the orientation of X when D is Riemannian.*

(3) *$\bar{U}^*R = R$ for all such $\bar{U}$. If $\bar{F}$ is the germ of a diffeomorphism such that $d\bar{F}(0)$ lies in $O(m)$, then $\bar{F}^*R = \det(d\bar{F}(o))\, R = \pm R$.*

Proof. We restrict our attention to the case $M = R^m$ and $x_0 = 0$ since $R(X, G, D, Fr)$ was defined in terms of these cases. Pull-back by $\bar{F}^*$ and $\bar{U}^*$ is equivalent to changing the canonical coordinate system for R^m and to changing the canonical frame for $R^m \times C^r$. Since $\bar{F}^*R(\bar{G}, \bar{D}) = R(X, \bar{F}^*\bar{G}, \bar{F}^*\bar{D}) = R(\bar{F}^{-1}, \bar{G}, \bar{D})$ and since $\bar{U}^*R(\bar{G}, \bar{D}) = R(\bar{G}, \bar{D}, \bar{U}^*\bar{F}r)$, it suffices to prove that the following three conditions are equivalent:

(1′) $\bar{F}^*R(\bar{G}, \bar{D}) = \det(d\bar{F}(o))\, R(\bar{G}, \bar{D})$ and $\bar{U}^*R(\bar{G}, \bar{D}) = R(\bar{G}, \bar{D})$ for all $\bar{F}$, $\bar{U}$, $\bar{G}$, $\bar{D}$;

(2′) Condition (1′) for only Riemannian connections $\bar{D}$; and

(3′) $\bar{F}^*\bar{R} = \det(d\bar{F}(0))R$ and $\bar{U}^*R = R$ for all $\bar{E}$, $\bar{U}$. It is clear that (3′) implies (1′) implies (2′). We use Lemma 1.1 to prove (2′) implies (3′).

The third condition expresses the notion of skew-invariance as skew-invariance under the action of a group on the ring $\mathcal{R}_{m,r}$. In the next section, we will use this lemma to show that skew-invariance implies that the form of our local formulas are skew-invariant.

2.

In this section, we study polynomials which are invariant under the action of $SO(m)$. The results of this section are closely related to the famous theorem of Weyl [8] on the invariants of $SO(m)$. These results are formally equivalent to the contraction of indices used by Atiyah, Bott, and Patodi. We can also give another proof of Weyl's theorem by using these methods.

We use these results to separate variables. We will reduce the classification of all skew-invariant polynomials of order m in the tensor algebra $\mathscr{R}_{m,r}$ to the corresponding classification problems for the rings $\mathscr{P}_m$ and $\mathscr{Q}_{m,r}$. This decomposition is the decomposition of a polynomial into sums of products of Pontrjagin and Chern classes.

We adopt the following notation for the generators of $SO(m)$: let F_{abij} for $a^2 + b^2 = 1$ denote the linear rotation of R^m so that $F^*_{abij}(\partial/\partial X_i) = a\partial/\partial X_i + b\partial/\partial X_j$, $F^*_{abij}(\partial/\partial X_j) = -b\partial/\partial X_i + a\partial/\partial X_j$ and $F^*(\partial/\partial X_k) = \partial/\partial X_k$ for $k \neq i, j$. We compute, for example,

$$\begin{aligned} F^*_{abij}(g_{ii/j}) &= -b[a^2 g_{ii/i} + 2ab g_{ij/i} + b^2 g_{jj/i}] \\ &\quad + a[a^2 g_{ii/j} + 2ab g_{ij/j} + b^2 g_{jj/j}]. \end{aligned}$$

In the following lemmas, we assume that R is a polynomial which is invariant under the action of a subgroup of $SO(m)$. We use the fact that the *form* of R is invariant—$F^*_{abij}R = R$—to gain information concerning some of the monomials which must occur in R. The triangular form theorem, Theorem 2.2, forms the basis for our separation of variables in the next section.

We describe in some detail the action of $O(m)$ on $\mathscr{R}_{m,r}$. We study the symmetric tensor algebra on variables $X_1, \ldots, X_m$ to obtain a simplified model for this action. Similar, but more complicated notation will be used for the ring $R_{m,r}$.

Let T be the complete tensor algebra on R^m and let S be the symmetric tensor algebra. Let F_{abij} act on T and S by sending $X_i \to aX_i + bX_j$, $X_j \to -bX_i + aX_j$, and $X_k \to X_k$ for $k \neq i, j$. A basis for the algebra S is given by the monomials $X_{i_1} \cdots X_{i_t}$; a corresponding basis for T is given by strings of indices $s = (i_1, \ldots, i_t)$. There is a natural map from $T \to S$ given by sending $s \to A_s = X_{i_1} \cdots X_{i_t}$. This map is not 1–1, but it is subjective.

We use the action of $SO(m)$ on T to describe the action on S. We compute $F^*_{abij}(s)$ by formally replacing every i index of s by an $ai + bj$

index and every j index by an $-bi + aj$ index, and by expanding the resulting expression. Let s_{ij} denote the set of all strings which can be obtained from the string s by changing $i \to j$ or $j \to i$ indices. If n denotes the number of i and j indices in s, then there are exactly 2^n elements of s_{ij}. We express

$$F^*_{abij}(s) = \sum_{s' \in s_{ij}} a^{p(s,s',i,j)} b^{q(s,s',i,j)} (-1)^{r(s,s',i,j)} s'.$$

The exponent of a denotes the number of indices which were unchanged in constructing s' from s. The exponent of b denotes the number of indices which were changed, and the exponent of (-1) denotes the number of indices which were changed from $j \to i$. With this notation, if $A = A_s$, then

$$F^*_{abij}(A_s) = \sum_{s' \in s_{ij}} a^{p(s,s',i,j)} b^{q(s,s',i,j)} (-1)^{r(s,s',i,j)} A_{s'}.$$

We can have $A_{s_1} = A_{s_2}$ in this decomposition for strings $s_1 \neq s_2$. The coefficient of the monomials reflects the multiplicity with which they can be obtained from A in the symmetric algebra.

We use a similar notation to describe the action of $SO(m)$ on $\mathscr{R}_{m,r}$. Let

$$\bar{s} = (i_1, j_1; k_1, \ldots, k_r) \qquad \text{and} \qquad g_{\bar{s}} = g_{i_1 j_1/k_1 \cdots k_r},$$

$$\bar{t} = (s_1; t_1; j_1; k_1, \ldots, k_r) \qquad \text{and} \qquad w_{\bar{t}} = w_{s_1 t_1 j_1/k_1 \cdots k_r}.$$

To compute F^*_{abij}, we formally replace every i index by an $ai + bj$ index and every j index by a $-bi + aj$ index. We compute:

$$F^*_{abij}(g_{\bar{s}}) = \sum_{\bar{s}' \in \bar{s}_{ij}} a^{p(\bar{s},\bar{s}',i,j)} b^{q(\bar{s},\bar{s}',i,j)} (-1)^{r(\bar{s},\bar{s}',i,j)} g_{\bar{s}'},$$

$$F^*_{abij}(w_{\bar{t}}) = \sum_{\bar{t}' \in \bar{t}_{ij}} a^{p(\bar{t},\bar{t}',i,j)} b^{q(\bar{t},\bar{t}',i,j)} (-1)^{r(\bar{t},\bar{t}',i,j)} w_{\bar{t}'}.$$

Let R be a polynomial. We decompose $F^*_{abij}(R) = \sum a^p b^q R_{p,q}$, where $R_{p,q}$ is composed of monomials of degree $p + q$ in the indices i and j. This decomposition is unique for the following reason: Let $F^*_{abij}R = \sum a^p b^q \bar{R}_{p,q}$ be another decomposition. Then $\sum a^p b^q (R_{p,q} - \bar{R}_{p,q}) = 0$ for all a, b. By decomposing this sum into monomials of degree n in i, j, we conclude $\sum_{p=n-q} a^p b^{n-p} (R_{p,q} - \bar{R}_{p,q}) = 0$ for all n. Let $c = a/b$; since we can choose c arbitrarily, the identity $\sum_{p=n-q} c^p(R_{p,q} - \bar{R}_{p,q}) = 0$ implies that $\bar{R}_{p,q} = R_{p,q}$. We will use this decomposition in the proof of the following lemma.

LEMMA 2.1. *Let $R \in \mathscr{R}_{m,r}$ be a polynomial.* (1) *Suppose that $g_{12/\omega}$ divides some monomial of R and that $F^*_{ab12}(R) = R$ for all admissible a and b. Then there is a multiindex ω' so that $g_{11/\omega'}$ divides some monomial of R.*

(2) *Suppose that* $\deg_1(g_{ij/\omega}) = \deg_2(g_{ij/\omega}) = 0$. Let

$$\bar{g}_{r,s} = (\partial/\partial X_1)^r (\partial/\partial X_2)^s g_{ij/\omega}.$$

Suppose that $F^*_{ab12}R = R$ and that $\bar{g}_{r,s}$ divides some monomial of R. Then $\bar{g}_{r+s,0}$ divides some monomial of R.

(3) *Suppose that $F^*_{abij}R = R$ for i and $j > k_0$. Suppose $R \notin \mathscr{Q}_{m,r}$. Then there is a variable $g_{ij/\omega}$ which divides some monomial of R such that* $\deg_k(g_{ij/\omega}) = 0$ *for* $k > k_0 + 2$.

The use of the indices 1 and 2 in statements (1) and (2) is for notational convenience. We are contracting various indices in this lemma. This contraction is the formal analog of the contraction of indices in Weyl's theorem.

Proof. We proceeed nonconstructively and assume that (1) is false. We decompose R into powers of $g_{12/\omega}$:

$$R = \sum B_j(g_{12/\omega})^j.$$

We also decompose:

$$R = F^*_{abij}R = \sum a^p b^q D_{p,q} g_{11/\omega} + \text{other terms not divisible by } g_{11/\omega}.$$

By assumption $g_{11/\omega}$ divides no monomial of R, and, hence, $D_{p,q} = 0$. Let A be a monomial of R such that $\deg_1 A + \deg_2 A = p + 1$. Since $g_{11/\omega}$ does not divide A, we express $F^*_{ab12}A = a^p b A' g_{11/\omega}$ + other terms. Suppose that $A' \neq 0$, since the exponent of b is one, we obtain the monomial $A' g_{11/\omega}$ by making all possible changes of one index $1 \to 2$ or $2 \to 1$ in the collection of strings which defines A. Since $g_{11/\omega}$ does not divide A, some variable of A must change to $g_{11/\omega}$ by changing one index. Since this variable cannot be $g_{11/\omega'}$ by hypothesis, it must be $g_{12/\omega}$. If $A = (g_{12/\omega})^k A_1$, then $F^*_{ab12}A = -ka^p b (g_{12/\omega})^{k-1} g_{11/\omega} A_1$ + other terms divisible by a higher power of b or not divisible by $g_{11/\omega}$. This implies that:

$$\sum_p D_{p,1} = -\sum j B_j (g_{12/\omega})^{j-1} = 0.$$

This identity implies that $jB_j = 0$, and, hence, $B_j = 0$ for $j > 0$. This contradicts the initial hypothesis and proves (1). The proof of this part has been detailed in order to illustrate some of the ideas which are used later. We use the exponent of b to determine the number of indices which were changed in forming that monomial. In general, we will only be interested in those terms with exponent one on b.

We prove (2) as follows: suppose that $\bar{g}_{r_1,s_1}$ divides some monomial of R. Let $n = r_1 + s_1$ and choose r maximal so that $\bar{g}_{r,n-r}$ also divides some monomial of R. Let $s = n - r$ and assume that $s > 0$. As before we decompose R into powers of $\bar{g}_{r,s}$

$$R = \sum_j B_j(\bar{g}_{r,s})^j.$$

We also decompose:

$$R = F^*_{ab12}R = \sum a^p b^q D_{p,q}\bar{g}_{r+1,s-1} + \text{other terms}.$$

Since by assumption $\bar{g}_{r+1,s-1}$ divides no monomial of R, $D_{p,q} = 0$. We proceed as before. Let A be a monomial of R such that $F^*_{ab12}A = a^p b\bar{g}_{r+1,s-1}A'$ + other terms. If $A' \neq 0$, some variable of A changes to $\bar{g}_{r+1,s-1}$ by altering one index. Since $\bar{g}_{r+2,s-2}$ does not divide A, and since $\deg_1(g_{ij/\omega}) = \deg_2(g_{ij/\omega}) = 0$, this variable must be $\bar{g}_{r,s}$. Since $F^*_{ab12}(\bar{g}_{r,s}) = -sba^{n-1}\bar{g}_{r+1,s-1}$ + other terms, we have the identity

$$0 = \sum_p D_{p,1} = -\sum_j sjB_j(\bar{g}_{r,s})^{j-1}.$$

We have assumed $s \neq 0$, and, hence, $B_j = 0$ for $j > 0$. This contradicts the assumption that $\bar{g}_{r,s}$ divides some monomial of R and completes the proof of (2).

We combine these results in the proof of (3). Since we have assumed that $R \in \mathcal{Q}_{m,r}$, we can find a variable $g_{ij/\omega}$ which divides some monomial of R. First we show that we can assume that $\deg_k(g_{ij}) = 0$ for $k > k_0 + 1$. If both indices i and $j \leqslant k_0 + 1$, then there is no need of further argument. Suppose that $i > k_0$; let F be the coordinate permutation taking $\partial/\partial X_i \to \partial/\partial X_{k_0+1}$ and $\partial/\partial X_{k_0+1} \to -\partial/\partial X_i$. Since R is invariant under F^* by assumption, $F^*(g_{ij/\omega})$ must also divide some monomial of R. We can, therefore, assume $i = k_0 + 1$. If $j \leqslant k_0 + 1$, we are done; if $j > k_0 + 1$, we apply 2.1-1 to show that $g_{k_0+1,k_0+1/\omega'}$ must divide some monomial of R.

We assume that $g_{ij/\omega}$ is chosen so that $\deg_k(g_{ij/\omega}) = 0$ for $k > k_0 + 1$. We decompose

$$g_{ij/\omega} = \left[\prod_{k>k_0+1} (\partial/\partial X_k)^{r_k}\right] g_{ij/\omega'}\,; \qquad \deg_k(g_{ij/\omega'}) = 0 \quad k > k_0 + 1.$$

We apply Lemma 2.1 (2) several times to show that if $r = \sum r_k$, then $(\partial/\partial X_{k_0+2})^r g_{ij/\omega'}$ divides some monomial of R. This proves Lemma 2.1.

The following theorem is a generalization of upper-triangular form for matrices. We use this theorem to separate variables to prove vanishing theorems in the next section.

THEOREM 2.2. *Let R be invariant under the action of $SO(m)$. Then there is a monomial A of R such that $A = BC$ for B in $\mathcal{P}_m$, C in $\mathcal{Q}_{m,r}$ and* $\deg_k(B) = 0$ *for* $k > 2L(B)$.

Proof. Let A be any monomial of R. We can factor $A = BC$ for B in $\mathcal{P}_m$, $\deg_k B = 0$ for $k > 2L(B)$, and $L(B)$ maximal. Let A be chosen from the monomials of R so that $L(B_A)$ is maximal. If $C \in \mathcal{Q}_{m,r}$, then the theorem is proved so we can assume $C \notin \mathcal{Q}_{m,r}$. We decompose $R = BR_1 +$ monomials not divisible by B. Since $\deg_k B = 0$ for $k > 2L(B)$, $F^*_{abij}B = B$ for $i, j > 2L(B)$. This implies that $F^*_{abij}R_1 = R_1$ for $i, j > 2L(B)$. We have assumed that $R_1 \notin \mathcal{Q}_{m,r}$ so we can apply Lemma 2.1 (3) to find a variable $g_{ij/\omega}$ which divides some monomial of R_1 satisfying $\deg_k(g_{ij/\omega}) = 0$ for $k > 2L(B) + 2 = L(g_{ij/\omega}B)$. This contradicts the maximality of $L(B)$ and proves Theorem 2.2.

3.

In Section 3 we will use Theorem 2.2 to separate variables and to reduce the classification of skew-invariant polynomials of order $\leqslant m$ in the tensor algebra to the corresponding classification problem for the algebras $\mathcal{Q}_{m,r}$ and $\mathcal{P}_m$. We proceed as follows.

LEMMA 3.1. *Let R be a skew invariant polynomial of order $\leqslant m$. If $R \neq 0$, then there is a monomial A of R so that:* (1) $A = BC$ *for* $B \in \mathcal{P}_m$ *and* $C \in \mathcal{Q}_{m,r}$.

(2) $\deg_k(B) = 0$ for $k > 2L(B)$, $\deg_k(C) = 0$ *for* $k \leqslant 2L(B)$, *and* $\deg_k(C) = 1$ *for* $k > 2L(B)$.

(3) *B and C consist only of second order variables.*

(4) *There are no nonzero skew-invariant polynomials of order* $<m$. *If m is odd, there are no skew-invariant polynomials of order m.*

Proof. Let $R \neq 0$. We can find a local system (X, G, D, Fr), D Riemannian, such that $R(X, G, D, Fr)(x_0) \neq 0$. We choose a new coordinate system X' in which all the first order derivatives of the metric vanish at x_0. We also choose a new orthonormal frame Fr' so that $w_{sij}(Fr', D)(x_0) = 0$. Since R is skew-invariant, $R(X', G, F', D) = \pm R(X, G, Fr, D) \neq 0$. This implies that there must be a monomial of R which contains no first order variables.

Let $R_0 \neq 0$ be the polynomial which consists of those monomials of R which contain no first order variables. Clearly R_0 is invariant under the action of $SO(m)$. We apply Theorem 2.2 to find a monomial A of R_0 such that $A = BC$ for $B \in \mathscr{P}_m$, $C \in \mathscr{Q}_{m,r}$, and $\deg_k B = 0$ for $k > 2L(B)$. We will show A will satisfy the conditions of the lemma.

Let $\bar{F}_j$ denote the diffeomorphism such that

$$\bar{F}_j(\partial/\partial X_j) = -\partial/\partial X_j \quad \text{and} \quad \bar{F}_j(\partial/\partial X_i) = \partial/\partial X_i \quad \text{for } i \neq j.$$

If A_1 is any monomial of R, $\bar{F}_j{}^*(A_1) = (-1)^{\deg_j A_1} A_1$. Since $\bar{F}_j$ reverses the orientation, $\bar{F}_j{}^*R = -R$. This implies that the index j must occur with odd degree in every monomial of R.

Since $\deg_k(B) = 0$ for $k > 2L(B)$, there must be $m - 2L(B)$ indices which occur in the monomial C. Since $C \in \mathscr{Q}_{m,r}$,

$$\operatorname{ord}(C) = \sum_{j=1}^{m} \deg_j(C) \geqslant \sum_{j>2L(B)} \deg_j(C) \geqslant m - 2L(B).$$

Since B contains no first order variables,

$$\operatorname{ord}(B) = \sum_{j=1}^{L(B)} |\, \omega_i \,| \geqslant 2L(B).$$

These two inequalities imply that

$$\operatorname{ord}(R) = \operatorname{ord}(A) = \operatorname{ord}(B) + \operatorname{ord}(C) \geqslant m - 2L(B) + 2L(B) = m.$$

Since we assumed that $\operatorname{ord}(R) \leqslant m$, all of these inequalities must in fact be equalities. We may conclude, therefore, that $\operatorname{ord}(R) = m$, $\operatorname{ord}(C) = m - 2L(B)$, and $\operatorname{ord}(B) = 2L(B)$.

Since B contains no first order variables and $\operatorname{ord}(B) = 2L(B)$, each variable of B must be order 2. Similarly, since $\deg_k(C) > 0$ for $k > 2L(B)$ and $\sum \deg_k C = \operatorname{ord}(C) = m - 2L(B)$, only the indices $2L(B) + 1, \dots, m$ can occur in C and they occur with degree 1.

We complete the proof of Lemma 3.1 by showing that C consists only of second order variables. Suppose that C is divisible by some $w_{stj/\omega}$ for $|\omega| > 1$. We can express $\partial^\omega = \partial X_{j_1} \cdots \partial/\partial X_{j_s}$ for $s > 1$. Since $\deg_{j_1} A = \deg_{j_2} A = 1$, these indices occur only in this variable of A. Let $\bar{F}$ denote the diffeomorphism such that

$$\bar{F}(\partial/\partial X_{j_1}) = \partial/\partial X_{j_2}, \qquad \bar{F}(\partial/\partial X_{j_2}) = \partial/\partial X_{j_1}$$

and

$$\bar{F}(\partial/\partial X_k) = \partial/\partial X_k \qquad \text{for} \quad k \neq j_1, j_2 .$$

$\bar{F}$ is orientation reversing, but $F^*A = A$. This contradiction implies that C consists of variables of degree $\leqslant 2$. By construction, C contains no first order variables, and, hence, C consists of only second order variables. This implies that m is even and completes the proof of Lemma 3.1.

We use this lemma to separate variables. Let k be even with $0 \leqslant k \leqslant m$. Let $\mathcal{Q}_{m,k,r}$ denote the ring generated by the variables $w_{stj/\omega}$ such that $\deg_i(w_{stj/\omega}) = 0$ for $i \leqslant k$. This is the ring of local formulas in the connection which depend only on the last $m - k$ coordinates. There is a natural isomorphism from $\mathcal{Q}_{m,k,r} \to \mathcal{Q}_{m-k,r}$ which is obtained by renumbering the indices which refer to the coordinate system.

We regard $\mathcal{P}_k$ as a subalgebra of $\mathcal{P}_m$. We will define

$$F_k \colon \mathcal{R}_{m,r} = \mathcal{P}_m \otimes \mathcal{Q}_{m,r} \to \mathcal{P}_k \otimes \mathcal{Q}_{m,k,r} .$$

These maps will enable us to separate a skew-invariant polynomial R into sums of polynomials of the form PQ where P is skew-invariant in $\mathcal{P}_k$, and Q is skew-invariant in $\mathcal{Q}_{m-k,r} = \mathcal{Q}_{m,k,r}$. This will express R in terms of products of Pontrjagin and Chern classes.

We define F_k as follows:

$$F_k(g_{j_1 j_2/\omega}) = \begin{cases} 0 & \text{if } \deg_i(g_{j_1 j_2/\omega}) > 0 \text{ for some } i > k, \\ g_{j_1 j_2/\omega} \in \mathcal{P}_k & \text{otherwise,} \end{cases}$$

$$F_k(w_{stj/\omega}) = \begin{cases} 0 & \text{if } \deg_i(w_{stj/\omega}) > 0 \text{ for some } i \leqslant k, \\ w_{stj/\omega} \in \mathcal{Q}_{m,k,r} = \mathcal{Q}_{m-k,r} & \text{otherwise.} \end{cases}$$

Let R be a skew-invariant polynomial. Let $Q_1, \ldots$ be a basis for the algebra $\mathcal{Q}_{m,k,r}$. We decompose

$$F_k(R) = \sum P_i \otimes Q_i \qquad \text{for} \quad P_i \in \mathcal{P}_k .$$

Let $\bar{F}$ be the germ of a diffeomorphism on R^k. We extend $\bar{F} = 1$ on R^{m-k}. Since none of the variables of $\mathcal{Q}_{m,k,r}$ involve the first k coordinates, $\bar{F}^* = 1$ on $\mathcal{Q}_{m,k,r}$. Since $\bar{F}$ only depends on the first k coordinates, $\bar{F}$ commutes with F_k and

$$F_k R = \pm F_k \bar{F}^* R = \pm \bar{F}^* F_k R = \pm \sum \bar{F}^* P_i \otimes Q_i .$$

Since the Q_i were a basis for $\mathcal{Q}_{m,k,r}$, this implies that $\bar{F}^* P_i = \pm P_i$ for all such diffeomorphisms $\bar{F}$. This implies by Lemma 1.2 that the P_i are skew-invariant polynomials of $\mathcal{P}_k$.

Let $P_1{}^k, \ldots$ be a basis for the skew-invariant polynomials of $\mathcal{P}_k$. We obtain a new decomposition,

$$F_k(R) = \sum P_i{}^k \otimes Q_i{}^k \qquad \text{for} \quad Q_i{}^k \in \mathcal{Q}_{m,k,r} = \mathcal{Q}_{m-k,r} .$$

We argue as before to show that $Q_i{}^k$ are skew-invariant polynomials of $\mathcal{Q}_{m-k,r}$: let $\bar{F}$ be the germ of a diffeomorphism of R^{m-k} and let $\bar{U}$ be the germ of a map from $R^{m-k} \to U(r)$. We extend these maps to R^m to be independent of the first k coordinates. As before, $\bar{F}^*$ and $\bar{U}^*$ commute with F_k since they depend trivially on the first k coordinates. Furthermore, $\bar{F}^*$ and $\bar{U}^* = 1$ on $\mathcal{P}_k$. We write

$$F_k R = \pm F_k \bar{F}^* \bar{U}^* R = \bar{F}^* \bar{U}^* F_k R = \sum P_i{}^k \otimes \bar{F}^* \bar{U}^* Q_i{}^k .$$

This implies that $\bar{F}^* \bar{U}^* Q_i{}^k = \pm Q_i{}^k$ and that the $Q_i{}^k$ are skew-invariant polynomials of $\mathcal{Q}_{m-k,r}$.

Let R be a skew-invariant polynomial of order m. Suppose that $P_i{}^k Q_i{}^k \neq 0$ for some i. Since $P_i{}^k$ is a skew-invariant polynomial of $\mathcal{P}_k$ and $Q_i{}^k$ is a skew-invariant polynomial of $\mathcal{Q}_{m-k,r}$, we apply Lemma 3.1 to show $\operatorname{ord}(Q_i{}^k) \geqslant m - k$ and $\operatorname{ord}(P_i{}^k) \geqslant k$. This implies that $\operatorname{ord}(Q_i{}^k) = m - k$ and $\operatorname{ord}(P_i{}^k) = k$.

Let f denote the direct sum of the maps F_{2k}

$$f(R) = \sum_{k=1}^{m/2} \oplus F_{2k} ; \qquad f\colon \mathcal{R}_{m,r} \to \sum_{k=1}^{m/2} \oplus (\mathcal{P}_{2k} \otimes \mathcal{Q}_{m-2k,r}).$$

Let $R \neq 0$ be a skew-invariant polynomial of order m. We apply

Lemma 3.1 to find a monomial $A = BC$ fo R such that $B \in \mathscr{P}_{2L(B)}$ and $C \in \mathscr{Q}_{m,2L(B),r}$. This implies that $F_{2L(B)}(R) \neq 0$. Therefore, f is injective when restricted to the subset of skew-invariant polynomials of order m of $\mathscr{R}_{m,r}$.

We apply these results to estimate the dimension of the subspace of skew-invariant polynomials of order m of $\mathscr{R}_{m,r}$. We define:

$c(m, r)$ = the dimension of the subspace of skew-invariant polynomials of order m of $\mathscr{R}_{m,r}$.

$c_r(m)$ = the dimension of the subspace of skew-invariant polynomials of order m of $\mathscr{Q}_{m,r}$.

$c(m)$ = the dimension of the subspace of skew-invariant polynomials of order m of $\mathscr{P}_m$.

Since f is injective, we can estimate $c(m, r)$ by the dimension of the image of f. Since f maps skew-invariant polynomials to products of skew-invariant polynomials of $\mathscr{P}_{2k}$ and $\mathscr{Q}_{m-k,r}$ we can estimate

$$c(m, r) \leqslant \sum_k c_r(m - 2k)\, c(2k).$$

In Section 4, we will aplly a slightly modified classical argument to show that $c_r(m) \leqslant \prod_r (m/2)$. In Section 5, we will show that $c(m) = 0$ unless $m \equiv O(4)$ and that $c(4k) = \prod (k)$. This will prove that

$$c(m, r) \leqslant \sum_k \prod_r \frac{(m-4k)}{2} \prod (k).$$

We have previously shown that there are $\sum_k \prod_r ((m - 4k)/2) \prod (k)$ linearly independent skew-invariant polynomials of order m which can be expressed in terms of the Pontrjagin classes of the metric and Chern classes of the connection. This proves the following classification theorem by a simple counting argument.

THEOREM 3.2. *Let R be a skew-invariant polynomial of order n of $\mathscr{R}_{m,r}$. Then $R = 0$ for $n < m$. If $n = m$, then R can be computed in terms of the Pontrjagin classes of the metric and the Chern classes of the connection.*

We can generalize this result to local formulas which take values in p-forms. Let M be an m-dimensional manifold. Let $I = (i_1, \ldots, i_p)$ for

$1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant m$. Let R_p be a collection of polynomials R_I of $\mathscr{R}_{m,r}$ for $|I| = p$. We define the p form

$$R_p(X, G, D, Fr) = \sum_{|I|=p} R_I(X, G, D, Fr)\, dX^I.$$

We let $\mathscr{R}_{m,r}^p$ denote the linear space of all such local formulas R_p. R_p is said to be an invariant map from connections and metrics to p forms if $R(X, G, D, Fr)$ is independent of X and Fr. There is a similar classification theorem for invariant elements of $\mathscr{R}_{m,r}^p$.

THEOREM 3.3. *Let R_p be an invariant map to p-forms of order n. Then $R_p = 0$ for $n < p$. If $n = p$, then R_p can be computed in terms of the Pontrjagin classes of the metric and Chern classes of the connection.*

We prove Theorem 3.3 by reducing the proof to the case $p = n$ and applying a counting argument similar to that used in the proof of Theorem 3.2.

There is a natural map $|R_p|^2 = \sum_I (R_I)^2$ mapping $\mathscr{R}_{m,r}^p \to \mathscr{R}_{m,r}$. We will use this map in the proof of the following lemma. There is a natural notion of an action by $SO(m)$ on $\mathscr{R}_{m,r}^p$.

LEMMA 3.4. *Let R_p be invariant under the action of $SO(m)$. Then there is a monomial A of some R_I such that $A = BC$ for $B \in \mathscr{P}_{2L(B)}$ and $C \in \mathscr{Q}_{m,r}$.*

Proof. We follow the proof of Theorem 2.2. Let A be a monomial of some R_I such that $A = BC$ for $B \in \mathscr{P}_{2L(B)}$ for $L(B)$ maximal. We decompose $R_I = BR_I' +$ other terms and let R_p' denote the collection R_I'. We will show that if $R_p' \notin \mathscr{Q}_{m,r}$, then there is a variable $g_{ij/\omega}$ which divides some monomial of some R_I' such that $\deg_k(g_{ij/\omega}) = 0$ for $k > 2L(B) + 2$. This will contradict the maximality of LB and prove the lemma.

Let G^* denote the extension of a Riemannian metric G to the complete exterior algebra. Then:

$$|R_p'|^2 (X, G, D, Fr) = G^*(R_p'[X, G, D, Fr], R_p'[X, G, D, Fr]).$$

Since $\deg_k(B) = 0$ for $k > 2L(B)$, F_p' is invariant under the action of F_{abij}^* for $i, j > 2L(B)$. This implies that $|R_p'|^2$ is invariant under the action of F_{abij}^* for $i, j > 2L(B)$. Since $|R_p'|^2 \notin \mathscr{Q}_{m,r}$ by assumption, we apply Lemma 2.1 (3) to find a variable $g_{ij/\omega}$ which divides some monomial of $|R_p'|^2$ such that $\deg_k(g_{ij/\omega}) = 0$ for $k > 2L(B) + 2$. This implies that the variable $g_{ij/\omega}$ must divide some monomial of R_p' and proves the lemma.

Let $R_p \in \mathscr{R}^p_{m,r}$. We say that R_p is of order n if each R_I is of order n. $R_p \neq 0$ implies that we can find a local system (X, G, D, Fr) such that $R_p(X, G, D, Fr)(x_0) \neq 0$. We choose a new coordinate system X' so that all the first order derivatives of the metric vanish. If R_p is invariant, then $R_p(X', G, D, Fr) \neq 0$. This implies that R_p must contain some monomial which contains no first order derivatives of the metric. Let $R_p{}^0$ consist of those monomials of R_p which have no first order derivatives of the metric. The map $R_p \to R_p{}^0$ is injective for such R_p.

$R_p{}^0$ is invariant under the action of $O(m)$. We apply Lemma 3.4 to find a monomial $A = BC$ of some R_I. Let R_p be of order n. Since R_I is invariant under the coordinate map $\partial/\partial X_i \to -\partial/\partial X_i$, every index of I must appear with odd degree in the monomial A. There are at most $2L(B) + \text{ord}(C)$ indices which appear in the monomial A, and, hence, $p \leqslant 2L(B) + \text{ord}(C)$. Since A contains no first order derivatives of the metric, $\text{ord}(B) \geqslant 2L(B)$, and, hence, $n = \text{ord}(B) + \text{ord}(C) \geqslant p$. This implies $R_p = 0$ for $n < p$.

If $n = p$, then all the inequalities must be equalities. Therefore, there are exactly p indices which occur in A and $\deg_k A = 0$ for k not in I. We can assume that $\bar{I} = (1,..., p)$ by a permutation of the coordinate axes. We have constructed a monomial A of $R_{\bar{I}}$ such that $\deg_k A = 0$ for $k > p$.

There is a natural injection of $\mathscr{R}_{p,r} \to \mathscr{R}_{m,r}$; we define the inverse map F_p' by

$$F_p'(\text{a monomial } A) = \begin{cases} 0 & \text{if } A \notin \mathscr{R}_{p,r}, \\ A & \text{if } A \in \mathscr{R}_{p,r}. \end{cases}$$

We use this map to define the map f_p from $\mathscr{R}^p_{m,r} \to \mathscr{R}_{p,r}$. Let R_p be invariant of order p. We define

$$f_p(R_p) = F_p'(R_{\bar{I}}).$$

For such an R, we have shown that there is a monomial A of $R_{\bar{I}}$ which belongs to $\mathscr{R}_{p,r}$. This implies that $f_p(R_p) \neq 0$, and, hence, that f_p is injective.

Let $\bar{F}$ be the germ of a diffeomorphism of R^p and let $\bar{U}$ be the germ of a map from $R^p \to U(r)$. We extend $\bar{F}$ and $\bar{U}$ to R^m in the usual fashion. Since $\bar{F}$ depends only on the first p coordinates, $\pm\bar{F}^*(dX_{\bar{I}}) = dX_{\bar{I}}$, and consequently, $\bar{F}^*\bar{U}^*(R_{\bar{I}}) = \pm R_{\bar{I}}$. Since both $\bar{F}^*$ and $\bar{U}^*$ commute with

F_p', this implies that $f_p(R_p)$ is a skew-invariant polynomial of order p in $\mathscr{R}_{p,r}$. Because f_p is injective, we can estimate that the dimension of the subspace of invariant polynomials of order p of $\mathscr{R}^p_{m,r}$ is bounded by $\sum \prod_r ((p-4k)/2) \prod (k)$. Since there are exactly this number of linearly independent invariant maps to p forms which come from combinations of Pontrjagin and Chern classes, this completes the proof of Theorem 3.3.

4.

To complete the proofs of Theorems 3.2 and 3.3, it suffices to show that $c_r(m) \leqslant \prod_r (m/2)$ and that $c(m) \leqslant \prod (m/4)$. In this section we modify a classical argument to show that $c_r(m) \leqslant \prod_r (m/2)$. We have restricted our attention to orthonormal frames in order to avoid considering terms involving the metric on the vector bundle E; we must modify the classical proof to take this into consideration.

Let w_{st} for $1 \leqslant s, t \leqslant r$ be formal variables representing the curvature tensor. For (D, Fr), we define

$$w_{st}(D, Fr)(x_0) = (DDe_s, e_t) \text{ is a 2 form on } M.$$

The w_{st} variables depend only on the connection D and the value of the frame Fr at the point x_0. We will also write

$$w_{st}(Fr, D)(x_0) = w_{st}(Fr(x_0), D).$$

Let $\mathscr{Q}^c_{m,r}$ denote the linear subspace of all complex polynomials in the w_{st} variables which are homogeneous of order $m/2$. If $Q^c \in \mathscr{Q}^c_{m,r}$, then $Q^c(Fr(x_0), D)$ is an m form on M. If $Fr(x_0)$ is a given orthonormal basis for the fiber E_{x_0}, we can always extend $Fr(x_0)$ to a frame $F(x)$ which is flat to first order with respect to the connection D at x_0;

$$w_{stj}(X, Fr, D)(x_0) = 0 \qquad \text{for all } s, t, j.$$

In such a frame, we can formally set

$$w_{st} = \sum_{j,k} w_{stj/k}\, dX_j\, dX_k .$$

Let $I = (i_1, \ldots, i_m)$ be a collection of distinct indices $1 \leqslant i_j \leqslant m$. We define sign($I$) so that $dX^I = \text{sign}(I)\, dX_1 \cdots dX_m$. If $A^c = w_{s_1t_1} \cdots w_{s_nt_n}$ for $n = m/2$, we define $\phi(A)$ by:

$$\phi(A^c) = \sum_I \text{sign}(I)\, w_{s_1t_1i_1/i_2} \cdots w_{s_nt_ni_{m-1}/i_m}.$$

ϕ is an injective map from $\mathcal{Q}^c_{m,r} \to \mathcal{Q}_{m,r}$. Let U be a unitary matrix. We define

$$U^*(w_{st}) = \sum U_{ss'}w_{s't'}(U^{-1})_{t't}.$$

Since U is independent of coordinates, from our definition of the action of U^* on $\mathcal{Q}_{m,r}$, we have the identity

$$\phi(U^*Q^c) = U^*\phi(Q^c).$$

Let Q be a skew-invariant polynomial of order m. Let Q_0 denote the polynomial which consists of those monomials of Q which contain only second order variables. The map $Q \to Q_0$ is injective by Lemma 3.1. Let $A = w_{s_1t_1i_1/i_2} \cdots w_{s_nt_ni_{m-1}/i_m}$ be a monomial of Q_0. Let $I = (i_1, \ldots, i_m)$. If $J = (j_1, \ldots, j_m)$, we form the monomial $B = w_{s_1t_1j_1/j_2} \cdots w_{s_nt_nj_{m-1}/j_m}$. Since Q is skew-invariant, the coefficients of the monomials A and B are related by the formula

$$c_Q(A)\,\text{sign}(I) = c_Q(B)\,\text{sign}(J).$$

This relation implies that Q_0 lies in the image of ϕ. Since ϕ is injective, we can find a unique polynomial Q^c of $\mathcal{Q}^c_{m,r}$ such that $\phi(Q^c) = Q_0$.

If U is a unitary matrix, then $\phi(U^*Q^c) = U^*\phi(Q^c) = U^*Q_0 = Q_0 = \phi(Q^c)$. This implies that $U^*Q^c = Q^c$. To every skew-invariant polynomial Q of order m, we have associated such a polynomial Q^c. We will show that $c_r(m) \leqslant \prod_r (m/2)$ by obtaining an estimate for the number of such polynomials Q^c satisfying $U^*Q^c = Q^c$.

We reinterpret the space $\mathcal{Q}^c_{m,r}$: let A_{ij} be a matrix. We define $w_{st}(A) = A_{st}$ and extend this to $\mathcal{Q}^c_{m,r}$. If $U^*Q^c = Q^c$, then $U^*Q^c(A) = Q^c(UAU^{-1}) = Q^c(A)$ so Q^c is invariant on the conjugacy classes of the unitary group. We use this property to obtain an estimate for the dimension of all such polynomials Q^c. We first prove the following lemma which is the analog of Lemma 1.1.

LEMMA 4.1. *Let $Q^c \neq 0$ be a polynomial of $\mathcal{Q}^c_{m\,r}$. Then there is a matrix $A = A_{ij}$ such that*

(1) $Q^c(A) \neq 0$ *and*

(2) $A_{ij} = -\bar{A}_{ji}$.

Proof. We follow the proof of Lemma 1.1. We introduce

$$x_{st} = [w_{st} - w_{ts}]/2 \qquad \text{and} \qquad y_{st} = [w_{st} + w_{ts}]/2i \quad \text{for } s \leqslant t.$$

We express w_{st} and w_{ts} in terms of the new variables:

$$w_{st} = x_{st} + iy_{st} \qquad \text{and} \qquad w_{ts} = -x_{st} + iy_{st}\,.$$

We express Q^c in terms of the new variables and write $Q^c(x_{st}, y_{xt}) \neq 0$. Let x_{st}^0 and y_{st}^0 be real constants such that $Q^c(x_{st}^0, y_{st}^0) \neq 0$. We set $A = A_{st} = x_{st}^0 + iy_{st}^0$ and $A_{ts} = -x_{st}^0 + iy_{st}^0$ for $s \leqslant t$.

Suppose that $Q^c \neq 0$ satisfies $U^*Q^c = Q^c$ for all unitary U. Let A be chosen so $Q^c(A) \neq 0$ and so that $A_{ij} = -\bar{A}_{ji}$. A is skew-Hermitian, so we can find a unitary matrix U such that UAU^{-1} is diagonal. This implies that Q^c does not vanish on at least one diagonal matrix. If we restrict the set of all such to Q^c diagonal matrices, we obtain a symmetric function of order $m/2$ in the diagonal entries. Since there are exactly $\prod_r (m/2)$ such symmetric functions, this proves that there are at most $\prod_r (m/2)$ such Q^c and proves that $c_r(m) \leqslant \prod_r (m/2)$.

5.

In this section, we will complete the proofs of Theorems 3.2 and 3.3 by showing that $c(m) \leqslant \prod (m/4)$. In Section 4, we proved a similar result for the algebra $\mathcal{Q}_m$ by using the associated polynomial Q^c of the curvature tensor. Since Q^c does not vanish on all diagonal matrices, Q^c must have at least one monomial $A_p{}^c$ of the form

$$A_p{}^c = \prod_s (w_{ss})^{p(s)} \qquad \text{for} \quad \sum p(s) = n = m/2.$$

We can assume that $p(1) \geqslant \cdots \geqslant p(r)$ by permuting the indices. For such a p, we choose A_p as any monomial of $\phi(A_p{}^c)$.

There are exactly $\prod_r (m/2)$ such monomials and they classify the skew-invariant polynomials of order m. If $c_p(Q) \equiv c_Q(A_p)$, then $Q \neq 0$ implies that $c_p(Q) \neq 0$ for some p. The functions c_p form a dual basis for the space of skew-invariant polynomials of order m of $\mathcal{Q}_{m,r}$.

We cannot apply a similar diagonalization argument to $\mathcal{P}_m$ because we cannot separate the frame for TM from the coordinate system. We can, however, prove an analogous result.

RESULT. *Let P be skew-invariant of order m. There is a monomial A of P of the form $A = g_{11/\omega_1} \cdots g_{nn/\omega_n}$ such that* $\deg_i A = 3$ *for $i \leqslant n$ and* $\deg_i A = 1$ *for $i \geqslant n$. $|\omega_i| = 2$ and there are two indices $p(i) \leqslant n$ and $x(i) > n$ so that $g_{ii/\omega_i} = g_{ii/p(i)x(i)}$.*

Such a monomial A is said to be almost diagonal. It is characterized by the functions p, x. We view p as a permutation of the indices $1 - n$ and x as a function from the indices $1 - n$ to the indices $n + 1$ through $2n = m$; we let

$$A_{p,x} = \prod_{i=1}^{n} g_{ii/p(i)x(i)} \qquad \text{for such } p, x.$$

Let q_1 be any permutation of the first n indices, and let q_2 be any permutation of the last n indices. Let $q = q_1 q_2$, and let

$$F_q(\partial/\partial X_i) = \partial/\partial X_{q(i)}\,.$$

It is clear that $F_q{}^*(A_{p,x}) = A_{q_1 p q_1^{-1}, q_2 x}$. We can, therefore, replace the permutation p by any conjugate permutation and the function x by any other function.

Let p be a permutation. We decompose p into cycles of length $(t_1, \ldots) = t(p)$ for $t_1 \geqslant \ldots$. Two permutations p and $\bar{p}$ are conjugate if and only if their cycles are the same length, i.e., $t(p) = t(\bar{p})$. Let $t = (t_1, \ldots)$ for $\sum t_i = n$. We choose A_t such that the permutation associated to A_t is of this type. The monomials A_t classify the skew-invariant polynomials in the sense that $P \neq 0$ implies that there is a monomial A_t so $c_P(A_t) \neq 0$. There are $\prod (m/2)$ such conjugacy classes of permutations, and, hence, $c(m) \leqslant \prod (m/2)$.

If E is a real vector bundle and if D is a Riemannian connection, then the relation $w_{st} = -w_{ts}$ implies that the Chern classes vanish in dimension $\equiv O(2)$. In a similar way, we will show that the permutations for such almost diagonal monomials contain only even cycles. This implies that there are no skew-invariant polynomials of order m unless $m/2$ is even, i.e., $m \equiv O(4)$. It also enables us to reduce the set of classifying monomials A_t to those with only even cycles. This proves that $c(m) \leqslant \prod (m/4)$.

We establish the existence of such almost diagonal monomials indirectly. We will show that $P \neq 0$ and P contains no almost diagonal monomials contradicts the skew-invariance of P.

Let P be a skew-invariant polynomial of order m. We first prove an analog of the Bianchi identities. Let B be a monomial of P. Let $t = t(B)$

be the type of B. We form P_t as the polynomial consisting of the monomials of P of type t. P_t is invariant under the action of $SO(m)$ so we apply 2.2 to construct a monomial A of P_t such that $\deg_k A = 0$ for $k > 2L(A)$. Since P is skew-invariant, every index must appear with odd degree in every monomial, and, hence, $2L(A) \geqslant m$. Since $L(A) = L(B)$, this proves that $2L(B) \geqslant m$.

If $P \neq 0$, we can find (X, G) such that $P(X, G)(x_0) \neq 0$. Let X' be geodetic polar coordinates centred at x_0. Since P is invariant, $P(X', G)(x_0) \neq 0$. This implies that there is a monomial A of P which does not vanish in geodetic polar coordinates. Let $t(A) = (t_1, \ldots, t_r, 0, \ldots)$. Since $2r \geqslant m$, this implies that $m = \sum_1^r t_i \geqslant \sum_1^r 2 = 2r$. This implies that all these inequalities must be equalities and that $r = m/2$ and each $t_i = 2$.

In geodetic polar coordinates, we have the identities,

$$g_{11/11}(X', G) = g_{12/11}(X', G) = g_{11/12}(X', G) = 0,$$

$$g_{11/22}(X', G) = g_{22/11}(X', G) = -g_{12/12}(X', G)/2.$$

We interpret these identities quantitatively.

LEMMA 5.1. *Let A be a monomial of P consisting of only second order variables. Then* (1) $g_{11/11}$, $g_{12/11}$, *and* $g_{11/12}$ *do not divide* A;

(2) *If* $A = g_{11/22}A'$, $B = g_{12/12}A'$ *for* $g_{11/22}$ *and* $g_{12/12}$ *not dividing* A', *then* $c_P(A) = -c_P(B)/2$.

Proof. We could prove Lemma 5.1 by expressing P in geodetic polar coordinates as a function of the curvature tensor R_{ijkl}. We then reexpress the resulting expression in terms of the $g_{ij/kl}$ variables and use the Bianchi identities to prove (1) and (2). Instead, however, we will prove these results directly by using the invariance of P under a suitable group of nonlinear coordinate transformations.

Let $\bar{X}$ be the canonical coordinate system of R^m and let $F(\bar{X})$ be the germ of a diffeomorphism

$$F_1 = \bar{X}_1 + a\bar{X}_1^3 \qquad F_j = \bar{X}_j \quad \text{for } j > 1.$$

Then we compute

$$F_*(\partial/\partial\bar{X}_1) = (1 + 3a\bar{X}_1^2)\, \partial/\partial X_1, \qquad F_*(\partial/\partial\bar{X}_j) = \partial/\partial X_j \quad \text{for } j > 1.$$

Since $g_{ij} = O(|X|)$, and $g_{11} = 1 + O(|X|)$, we compute

$$F^*(g_{11}) = g_{11} + 6a\bar{X}_1{}^2 + O(|X|^3), \qquad F^*(g_{ij}) = g_{ij} + O(|X|^3) \quad \text{otherwise.}$$

For $|\omega| < 3, F_*(\partial/\partial\bar{X}^\omega) = \partial/\partial X^\omega$, and, therefore,

$$F^*(g_{11/11}) = g_{11/11} + 12a$$

and

$$F^*(g_{ij/\omega}) = g_{ij/\omega} \quad \text{for } |\omega| < 3 \quad \text{otherwise.}$$

Suppose that some monomial of P is divisible by $g_{11}/_{11}$. Let $A = g^j_{11/11}A'$ be a monomial of P of type (2,...). Then $F^*A = 12ajg^{j-1}_{11/11}A' +$ other terms. Let $B = g^{j-1}_{11/11}A'$. We decompose

$$F^*P = \sum a^i P_i\,.$$

Since a is arbitrary, $P_i = 0$ for $i > 0$. Since F^*A makes a nonzero contribution to P_1, there must be some other monomial A' of P such that $F^*(A_1) = c(A_1)aB$ with $c(A_1) \neq 0$. Suppose that A_1 contains a first order variable. Then $F^*(g_{ij/k}) = g_{ij/k}$ implies that every monomial of F^*A_1 contains a first order variable. Therefore, A_1 consists of only second order variables. Since $F^*(g_{ij/\omega}) = g_{ij/\omega}$ for $|\omega| = 2$ unless $g_{ij/\omega} = g_{11/11}$, we decompose $A_1 = g^k_{11/11}A_1{}'$. Then $F^*A_1 = 12akg^{k-1}_{11/11}A_1{}' +$ other terms. Since F^*A_1 makes a nonzero contribution to $ag^{j-1}_{11/11}A'$, we conclude that $A_1 = A$. This contradiction shows that $g_{11/11}$ divides no such monomial of P.

In a similar fashion we prove that $g_{12/11}$ divides no such monomial of P. Let F be the diffeomorphism

$$F_2 = \bar{X}_2 + a\bar{X}_1{}^3 \qquad \text{and} \qquad F_j = \bar{X}_j \quad \text{for } j \neq 2.$$

As before

$$F_*(\partial/\partial\bar{X}_1) = \partial/\partial\bar{X}_1 + 3aX_1{}^2\partial/\partial X_2\,, \qquad F_*(\partial/\partial\bar{X}_j) = \partial/\partial X_j \quad \text{for } j > 1.$$

We compute

$$F^*(g_{12/11}) = g_{12/11} + 6a,$$

$$F^*(g_{ij/\omega}) = g_{ij/\omega} \quad \text{otherwise } |\omega| < 3.$$

We argue as before to show that $F^*P = P$ implies $g_{12/11}$ divides no such monomial of P.

We complete the proof of (1) as follows: Let F be the diffeomorphism with

$$F_1 = \bar{X}_1 + a\bar{X}_1^2\bar{X}_2 \qquad F_j = \bar{X}_j \quad \text{for } j > 1.$$

Then

$$F_*(\partial/\partial\bar{X}_1) = \partial/\partial X_1 + 2a\bar{X}_1\bar{X}_2\partial/\partial X_1 ,$$

$$F_*(\partial/\partial\bar{X}_2) = \partial/\partial X_2 + a\bar{X}_1^2\partial/\partial X_1 , \qquad F_*(\partial/\partial\bar{X}_j) = \partial/\partial X_j \quad \text{for } j > 2.$$

We compute

$$F_*(g_{11/12}) = g_{11/12} + 4a, \qquad F^*(g_{12/11}) = g_{12/11} + 2a,$$

$$F^*(g_{ij/\omega}) = g_{ij/\omega} \quad \text{otherwise for } |\, \omega \,| < 3.$$

In this case, there are two variables of order 2 which we must consider. However, we have already shown that $g_{12/11}$ divides no monomial of type (2,...). We can argue as before to show that $g_{11/12}$ divides no such monomial.

We complete the proof of Lemma 5.1 by using invariance under the action of $SO(m)$. Let

$$A = g_{11/22}A', \qquad B = g_{12/12}A', \qquad C = g_{11/11}A'.$$

We set $n = \deg_1(A) + \deg_2(A)$. Let F_{ab12} be the linear coordinate rotation. We decompose $F^*P = a^{n-1}bc(P)\, g_{11/12}A' +$ other terms. Since $F^*P = P$; we conclude that since $g_{11/12}$ divides no monomial of P, $c(P) = 0$. Let A_1 be a monomial which makes a nonzero contribution to $c(P)$. Since $g_{11/12}$ does not divide A_1, there is a variable $g_{ij/\omega}$ of A_1 which changes to $g_{11/12}$ by altering one index. Therefore, $A_1 = A$, B, or C. We have already proved that C is not a monomial of P. Since $g_{11/22}$ and $g_{12/12}$ do not divide A' by hypothesis,

$$F^*(A) = -2a^{n-1}bg_{11/12}A' + \cdots \qquad \text{and} \qquad F^*(B) = -a^{n-1}bg_{11/12}A' + \cdots .$$

This implies $0 = -c(P) = 2c_P(A) + c_P(B)$ and completes the proof of Lemma 5.1.

For the remainder of this section, we will restrict our attention to monomials which consist solely of second order terms. The following lemma shows that there is a monomial of P in which $m/2$ indices have been contracted. We will use this lemma to prove that there exist almost diagonal monomials of P.

We define the notion of touching for use in the following lemma. We say that the index i touches the index j in the monomial A if A is divisible by either the variable $g_{ij/kk'}$ or by the variable $g_{kk'/ij}$ for some indices k and k'. We say that the index i is contracted in A if i touches itself in A.

LEMMA 5.2. *Let P be as previously. Then there is a monomial A such that* (1) $\deg_k A \leqslant 3$ *for all* k;

(2) *every index of degree* 3 *touches itself and exactly one index of degree* 1;

(3) *there are exactly* $m/2$ *indices of degree* 1 *and* $m/2$ *indices of degree* 3.

Such a monomial is said to be fully contracted.

Proof. Consider the set of all monomials A of P such that $\sum(\deg_k A)^2$ is minimal. Among all these monomials, let A be chosen such that the number of indices which touch themselves in A is maximal. We show A satisfies (1)–(3).

Let i be an index such that $\deg_i A = 1$. The index i touches an index j in A. Suppose that $\deg_j A = 1$. Let F_{ij} be the rotation such that

$$F_{ij}(\partial/\partial X_i) = \partial/\partial X_j\,,\ F(\partial/\partial X_j) = \partial/\partial X_i \text{ and}$$
$$F_{ij}(\partial/\partial X_k) = \partial/\partial X_k \text{ otherwise.}$$

Since $\deg_i A = \deg_j A = 1$, and i touches j in A, $F_{ij}^*(A) = A$. F_{ij}^* maps monomials to monomials and $F_{ij}^* P = -P$. This implies A is not a monomial of P. This contradiction shows $\deg_j A > 1$.

Let $\deg_j(A) + 1 = n$ be even, and let k_0 denote the number of indices distinct from i and j which touch themselves in A. Let P_0 be the polynomial consisting of those monomials B of P such that

(1) $\deg_k B = \deg_k A$ for $k \neq i, j$;

(2) the number of indices $k \neq i, j$ which touch themselves in B is k_0.

P_0 is invariant under the action of the linear rotations F_{abij}^* since our defining conditions did not depend on the indices i and j. For such a monomial B, $\deg_i B + \deg_j B = n$, and, hence, $\deg_i^2 B + \deg_j^2 B \leqslant 1 + (n-1)^2$. This implies $\sum \deg_k^2 B \leqslant \sum \deg_k^2 A$. Since $\sum \deg_k^2 A$ is minimal, these inequalities must be equalities. Therefore, $\sum \deg_k^2 B$ is minimal. Since $\deg_i^2 B + \deg_j^2 B = 1 + (n-1)^2$, we conclude that either $\deg_i B = 1$ or $\deg_j B = 1$.

Since the index i touches j in A, A is divisible by either $g_{ij/\omega}$ or by $g_{kk'/ij}$. We assume the former for notational convenience and write

$$A = g_{ij/\omega}A'.$$

We decompose $F^*_{abij}P_0 = P_0 = a^{n-1}bc(P_0)\, g_{ii/\omega}A' +$ other terms. Since P is skew-invariant and $\deg_i(g_{ii/\omega})$ is even, $c(P_0) = 0$. Since $F^*_{abij}(A)$ makes a nonzero contribution to $c(P_0)$, there must be at least one other monomial B of P_0 which also makes a nonzero contribution. Let $B \neq A$ be that monomial such that $F^*_{abij}(B) = a^{n-1}bc(B)\, g_{ii/\omega}A' +$ other terms.

Since the exponent of b is one, one variable of B changes to give $g_{ii/\omega}A'$ by altering one index $i \to j$ or $j \to i$. In the later case, $B = A$. Therefore, B changes to $g_{ii/\omega}A'$ by changing an $i \to j$. This implies that $\deg_i B = 3$ and that B is divisible by $g_{ii/\omega'}$. Since either $\deg_i B = 1$ or $\deg_j B = 1$, this implies $\deg_j B = 1$ and $n = 4$. The index i touches itself in B, and, consequently, there are $k_0 + 1$ indices which touch themselves in B. This implies that $k_0 + 1$ indices touch themselves in A and that j must touch itself in A.

We summarize these results as follows: Let $\deg_i A = 1$. The index i touches an index j such that $\deg_j A = 3$. The index j touches itself in A and touches no other index in A. Let S denote the set of indices i such that $\deg_i A = 1$ and let T denote the complementary set. The relation of touching gives an injective map of $S \to T$. This shows that $|S| \leqslant |T| = m - |S|$ and that $|S| \leqslant m/2$. We estimate $2m = \sum_S \deg_k A + \sum_T \deg_k A \geqslant |S| + 3|T| = 3m - 2|S|$. This implies that $m/2 \leqslant |S|$ and that $|S| = |T| = m/2$. The map given by touching must be bijective. This proves every index is of degree 1 or 3 in A and that the indices of degree 3 touch themselves and exactly one other index of degree 1. This proves the lemma.

We consider only fully contracted monomials for the remainder of this section. We define the function $k(A)$ to measure the extent to which such monomials are almost diagonal:

$$\begin{aligned} k(A) &= -1 \text{ if } A \text{ is not fully contracted or if } c_P(A) = 0,\\ &= p \text{ if } A = g_{11/\omega_1} \cdots g_{pp/\omega_p}A',\\ &= 0 \quad \text{otherwise.} \end{aligned}$$

We will prove that there are monomials for which $k(A) = n = m/2$.

We will complete the proof that $c(m) \leqslant \prod (m/4)$ by showing that the permutation defined by such an A contains only even cycles.

LEMMA 5.3. *Let A be a monomial with $k(A) > -1$. Then* (1) *Let i and j be indices such that* $\deg_i A = 3$, $\deg_j A = 1$, *and i and j touch in A. There is a unique monomial $\bar{A}$ which is formed by interchanging an i and j index. $k(\bar{A}) > -1$ and $c_P(A) + c_P(\bar{A}) = 0$.*

(2) *Let i, j, r, s be indices which are not necessarily distinct. Suppose $A = g_{ij/rs}A'$, set $A_1 = g_{rs/ij}A'$. Then $k(A_1) > -1$ and $c_P(A) = c_P(A_1)$.*

(3) *Suppose that $k(A)$ is maximal among all the monomials of P. Let $i > k(A)$. Then $g_{ii/\omega}$ and $g_{rs/ii}$ do not divide A for $r \neq s$. If* $\deg_j A = 1$, *then $g_{ij/\omega}$ and $g_{rs/ij}$ also do not divide A for $r \neq s$.*

Proof. Let A_0 be the monomial which is formed by changing the single j to an i index. For example, if $A = g_{ii/\omega_1}g_{ij/\omega_2}A'$, then $A_0 = g_{ii/\omega_1}g_{ii/\omega_2}A'$. Since A_0 is not a monomial of P, we express $F^*_{abij}P = a^3bc(P)\,A_0 +$ other terms and conclude $c(P) = 0$. The only monomials which make a contribution to $c(P)$ are A and $\bar{A}$, and, hence, $0 = c(P) = c_P(A) + c_P(\bar{A})$. It is clear that since A is maximally contracted, so is $\bar{A}$, and, hence, $k(\bar{A}) > -1$.

We prove (2) by considering different cases. It is clear that A_1 is maximally contracted and we need only prove the coefficient relation. We assume first $i = j$ and $r = s$; from Lemma 5.1 (1) $r \neq i$. Since $k(A) > -1$, $\deg_i A' = 1$ and $g_{ii/rr}$, $g_{ir/ir}$, and $g_{rr/ii}$ cannot divide A'. We apply Lemma 5.1 (2) to show $c_P(g_{ii/rr}A') = -c_P(g_{ir/ir}A')/2 = c_P(g_{rr/ii}A')$.

Next we assume that $i = j$, but that $r \neq s$, since A is maximally contracted we assume for notational convenience that $\deg_r A = 3$ and $\deg_s A = 1$. Again by Lemma 5.1 (1), $i \neq r$ and $i \neq s$. We apply the procedure of (1) to construct $\bar{A} = g_{ii/rr}\bar{A}'$ and $\bar{A}_1 = g_{rr/ii}\bar{A}'$ such that $c_P(A) + c_P(\bar{A}) = c_P(A_1) + c_P(\bar{A}_1) = 0$. We apply the previous paragraph to show that $c_P(\bar{A}) = c_P(\bar{A}_1)$, and, hence, $c_P(A) = c_P(A_1)$. The case in which $i \neq j$ and $r = s$ follows from this case.

We finally assume that $i \neq j$, $r \neq s$; since A is fully contracted all these indices are distinct. For notational convenience we assume $\deg_r A = 3$ and $\deg_s A = 1$. We construct $\bar{A} = g_{ij/rr}\bar{A}'$ and $\bar{A}_1 = g_{rr/ij}\bar{A}'$; by Lemma 5.2 (1) we conclude $c_P(A_1) + c_P(\bar{A}_1) = c_P(A_1) + c_P(\bar{A}_1)$. From the previous paragraph, we have $c_P(\bar{A}) = c_P(\bar{A}_1)$, and, hence, $c_P(A) = c_P(A_1)$.

We prove (3) as follows: Decompose $A = BC$ for $B = g_{11/\omega_1} \cdots g_{pp/\omega_p}$

and $p = k(A)$. Suppose first that $g_{ii/\omega}$ divides A. Since $i > p$, $g_{ii/\omega}$ divides C. By a permutation of the coordinate axes, we may assume $i = p + 1$ which contradicts the maximality of $k(A)$. We assume that $g_{rs/ii}$ divides A for $r \neq s$. Since $r \neq s$, $g_{rs/ii}$ divides C. We write $C = g_{rs/ii}C'$. We apply (2) to conclude that $g_{ii/rs}C'B$ is also a fully contracted monomial of P. This contradicts what we have just proved.

In a similar fashion, we use (1) to prove that if either $g_{ij/\omega}$ or $g_{rs/ij}$ divides such a monomial A, then $g_{ii/\omega}$ or $g_{rs/ii}$ divides a monomial $\bar{A}$ which also satisfies $k(\bar{A}) = p$. This contradicts the preceding paragraph and the lemma is proved.

We can now show that there exist almost diagonal monomials of P. Let A be a monomial with $k(A)$ maximal. We assume that $k(A) < n = m/2$. There must be some index $i > k(A)$ such that i touches itself in A. As a consequence of Lemma 5.3 (3), this variable must be of the form $g_{rr/ii}$. Also as a consequence of Lemma 5.3 (3), $r \leqslant k(A)$. We can assume for notational simplicity that $r = 1$ by a permutation of coordinates.

The monomial A is maximally contracted, and, therefore, $\deg_1 A = 3$. This implies that the index 1 appears in one other variable. Suppose that $A = g_{11/ii}g_{1x(1)/\omega'}A'$. We apply Lemma 5.3 (1) to construct $\bar{A} = g_{1x(1)/ii}g_{11/\omega'}A'$ with $k(\bar{A}) = k(A)$. This contradicts Lemma 5.3 (3), and, therefore, $g_{11/ii}g_{rs/1x(1)}A' = A$. If $r \neq s$, we apply Lemma 5.3 (2) to construct $A_1 = g_{11/ii}g_{1x(1)/rs}A'$ such that $k(A_1) = k(A)$. This contradicts what we have just proved, and, therefore, A is of the form

$$A = g_{11/ii}g_{22/1x(1)}A'$$

(by a permutation of the coordinate axes we can assume $r = 2$).

We take q maximal so that $k(A)$ is maximal and A is of the form

$$A = g_{11/ii}g_{22/1,x(1)} \cdots g_{qq/q-1,x(q-1)}A'.$$

Since $\deg_q A' = 1$, q must appear in some variable of A'. We suppose first that $A' = g_{qx(q)/\omega}A_0$. We apply Lemma 5.3 (1) a total of q times to the pair $(j, x(j))$ to construct the fully contracted monomial of P

$$\bar{A} = g_{1x(1)/ii}g_{2x(2)/11} \cdots g_{qx(q)/\omega}A_0 .$$

We also apply Lemma 5.3 (2) to construct the fully contracted monomial of P:

$$\bar{A}_1 = g_{ii/1x(1)}g_{11/2x(2)} \cdots g_{qq/\omega}A_0 .$$

Since $k(\bar{A}_1) = k(A)$, this contradicts Lemma 5.3 (3) and shows that A must be divisible by $g_{rs/qx(q)}$. By Lemma 5.3 (3), this implies $r = s \leqslant k(A)$. We can assume $r = q + 1$ by a permutation of the coordinate axes. This contradicts the maximality of q and completes the proof that there exist almost diagonal monomials

$$A = g_{11/\omega_1} \cdots g_{nn/\omega_n} \qquad n = m/2.$$

At the beginning of this section, we described a permutation of the indices 1–n which was associated to such a monomial A. Suppose that this permutation has a cycle of odd length. We can assume that

$$A = g_{11/2x(2)} \cdots g_{s-1,s-1/sx(s)} g_{ss/1x(1)} A'.$$

We assume s is odd: $s > 1$ by Lemma 5.1 (1). We will show that this implies that $c_P(A) = 0$ and contradicts the fact $k(A) = m/2$. We apply Lemma 5.3 (1) a total of s-times to construct

$$A_1 = g_{2x(2)/11} \cdots g_{sx(s)/s-1,s-1} g_{1x(1)/ss} A',$$

such that $c_P(A_1) = c_P(A)$. We apply Lemma 5.3 (2) a total of s times to construct

$$\bar{A}_1 = g_{22/1x(1)} \cdots g_{ss/s-1,x(s-1)} g_{11/sx(s)} A'.$$

Since s is *odd*, $c_P(A) = (-1)^s c_P(\bar{A}_1) = -c_P(A_1)$. Let q be the permutation defined by $q(j) = s + 1 - j$ and $q(x(j)) = x(s + 1 - j)$ for $j \leqslant s$; $q(k) = k$ otherwise. F_q is orientation preserving and $F_q{}^*(\bar{A}_1) = g_{s-1,s-1/sx(s)} \cdots g_{11/2x(2)} g_{ss/1x(1)} A' = A$. This implies $c_P(A) = -c_P(\bar{A}_1) = -c_P(F_q{}^*\bar{A}_1) = -c_P(A)$, and, hence, $c_P(A) = 0$. This completes the proof that $c(m) = 0$ unless $m \equiv O(4)$ and $c(4k) = \prod (k)$.

6.

If M is a spin manifold, we can represent the DeRham complex by using the spin complex with coefficients in the contragredient representation. We show that $P_n{}^m = 0$ for $n < m$ and that $P_m{}^m = cE_m$ since we can imbed any germ of a metric on R^m in a spin manifold. We compute $d = 1$ by integrating over the classifying manifold $M = S^2 \times \cdots \times S^2$. This proves the classical result that $\int_M E_m d \operatorname{vol} = \mathscr{X}(M)$ for any manifold M.

The DeRham complex is functorially defined independent of an orientation. The polynomials $P_n{}^m$ are invariant in $\mathscr{P}_m$; they are skew-invariant as maps from metrics to m forms. We cannot conclude that $\operatorname{ord}(P) < m$ implies $P = 0$ based only on the assumption that P is invariant. We will need an additional axiom which we motivate as follows: Let M be a spin manifold and let P be the principal SO frame bundle for TM. Let Q be the double covering SPIN bundle giving the spin structure. Let ρ: SPIN $\to SO$ be the usual representation of SPIN on R^m given by Clifford multiplication, $\rho(x)\,y = xyx^{-1}$. For further details on Clifford modules, the reader is referred to [1]. Then $Q_\rho \times R^m = TM$. We extend ρ to a representation of SPIN on the module Cliff(R^m). There is a functorial identification of Cliff(R^n) with the exterior algebra on R^n which shows

$$Q \underset{\rho}{\otimes} \operatorname{Cliff}(R^m) = \Lambda(TM).$$

Since M has a Riemannian structure, we identify TM and T^*M. There are four other natural representations of SPIN on the Clifford algebra. Let SPIN act on Cliff(R^m) $\otimes$ C by multiplication from the left. This representation decomposes into $2^{m/2}$ equivalent representations $\Delta^\pm$. We can also let SPIN act by multiplication by the inverse on the right. Similarly, this representation decomposes into $2^{m/2}$ equivalent representations $\Delta'^\pm$. We construct corresponding vector bundles $\Delta^\pm$ and $\Delta'^\pm$ from the principal bundle Q. We can use these four bundles to reconstruct the signature and DeRham complexes.

It is clear from the definition that:

$$\rho = (\Delta^+ \oplus \Delta^-) \otimes (\Delta'^+ \oplus \Delta'^-),$$

and, hence,

$$\Lambda(T^*M) = (\Delta^+ \oplus \Delta^-) \otimes (\Delta'^- \oplus \Delta'^+).$$

Let $e_1, \ldots, e_m$ be an orthonormal basis for R^m. Let $e = e_1 \cdots e_m$ in Cliff(R^m). Since e commutes with SPIN, we define a corresponding element e of Cliff(TM). The choice of e or $-e$ amounts to an orientation on M. By definition, multiplication by e on $\Delta^\pm$ from the left is just ± 1, while multiplication by e^{-1} on Δ from the right is also ± 1. Since $e^2 = 1$, we can decompose $\Lambda(TM)$ into eigenspaces $\Lambda^\pm_e$. We can also decompose Λ into spaces Λ^0 which commute and anticommute with e. We compute

$$\Lambda^+ - \Lambda^- = (\Delta^+ - \Delta^-) \otimes (\Delta'^+ \oplus \Delta'^-)$$

and

$$\Lambda^e - \Lambda^0 = (\Delta^+ - \Delta^-) \otimes (\Delta'^+ - \Delta'^-).$$

The leading symbol of the differential operator for each complex is Clifford multiplication. By functoriality, we can express the lower order symbol of the operator in terms of the first derivatives of the metric. This term will vanish for geodetic polar coordinates; hence, the differential operators will agree. We compute P_n^m by considering the spin complex $(\Delta^+ - \Delta^-)$ with coefficients in the virtual bundle $(\Delta'^+ - \Delta'^-)$. We apply the results from the Introduction to show that $P_n^m(G)(x)$ vanishes identically for $n < m$ and G a spin metric. This implies $P_n^m = 0$. We conclude that P_m^m can be computed in terms of the Pontrjagin classes of M and Chern classes of $(\Delta'^+ - \Delta'^-)$.

If we can show the the Chern classes of this virtual bundle can be computed as Pontrjagin and Euler classes of TM, then we could express $P_m^m = cE_m + \bar{P}_m^m$ where $\bar{P}_m^m$ is skew-invariant. Since both P_m^m and E_m do not depend on the orientation of M, $\bar{P}_m^m = 0$ and $P_m^m = cE_m$. This completes the proof of the result given in the Introduction for the DeRham complex.

We must describe those elements of $\mathscr{P}_m$ which can be defined by the Chern classes for some functorially defined vector bundle. Let P_p be a collection of polynomials P_I which takes values in p forms. We assume that P is invariant under orientation preserving diffeomorphisms. (This is certainly true for the Chern classes of $\Delta'^+ - \Delta'^-$.)

Let F: $\partial/\partial X_1 \to -\partial/\partial X_1$. We decompose $P_I = P_I^1 + P_I^2$ for

$$P_I^1 = (P_I + F^*P_I)/2 \qquad \text{and} \qquad P_I^2 = (P_I - F^*P_I)/2.$$

Let P_1 denote the collection P_I^1 and P_2 the collection P_I^2. P_1 is an invariant map from metrics to $2k$ forms; we can apply our previous classification theorem to P_1. However, P_I^2 is a skew-invariant map from metrics to $2k$ forms and we will need an additional property to express the fact that P_p comes from a Chern class.

Let C be a characteristic class; our model will be $P_p = C(\Delta'^-)$. Let $\bar{G}_1$ be the germ of a Riemannian metric R^{m-1}. We give the flat metric to the last coordinate and let $\bar{G} \times 1$ be the product metric on R^m. Since the connection defined by the metric is flat in the last coordinate, we can define a vector bundle on R^{m-1} which pulls back to Δ'^-. By functoriality, $C(\Delta'^-)$ is the pull-back of a p form on R^{m-1}. This implies that if $\deg_m(I) > 0$, then $P_I(\bar{G}_1 \times 1) = 0$. Suppose that A is a monomial of such a P_I. If $\deg_m(A) = 0$, then we can find a flat metric such that $P_I(\bar{G}_1 \times 1) \neq 0$. This implies that $\deg_m(A) > 0$ for all such A. We define

$$P_p \text{ is complete if } \deg_m(I) > 0, \qquad c_{P_I}(A) \neq 0 \text{ implies } \deg_m(A) > 0.$$

Of course, if P_p is invariant under the action of $SO(m)$, we can replace the index m by any index. If $p = m$, then P is complete if and only if $F'_{m-1}(P) = 0; F'_{m-1} : \mathscr{P}_m \to \mathscr{P}_{m-1}$.

THEOREM 6.1. *Let P_p be complete of order n and invariant under orientation preserving diffeomorphisms. If $n < p$, then $P = 0$. If $n = p$ and $p < m$, then P is invariant under all diffeomorphisms and can be computed in terms of the Pontrjagin classes. If $n = p = m$, then we decompose $P = cE_m + \bar{P}$ where E_m is the Euler class and $\bar{P}$ is a characteristic class.*

We combine this result with the results of Section 3 to show that the characteristic classes of $\Delta'^{\pm}$ can be computed in terms of the Pontrjagin classes and the Euler class of TM.

We can prove that the polynomials $P_n{}^m$ are complete without using the spin representation by using the multiplicative property of the DeRham complex. Let E be an elliptic complex over the manifold M and F a complex over N. There is a natural elliptic complex $E \otimes F$ over $M \times N$. Let $f(t, x, d_E)$ denote the asymptotic expansion for E and $f(t, x', d_{E'})$ the corresponding expansion for F. Then $f(t, x, d_E) f(t, x', d_F)$ is the corresponding expansion for $E \otimes F$. This implies that

$$B_n(x, x', d_{E \otimes F}) = \sum_p B_p(x, d_E)\, B_{n-p}(x, d_F).$$

Since we have given S^1 a homogeneous metric, $\int B_n(x, d) = 0$ implies that $B_n(x, d)$ vanishs for the DeRham complex on S^1. We apply the product rule to show that $P_m{}^n(\bar{G} \times 1) = 0$ for all such $\bar{G}$ and hence the polynomials $P_m{}^n$ are complete.

Proof of Theorem 6.1. First we decompose $P_p = P_p{}^1 + P_p{}^2$ such that $P_p{}^1$ is invariant and $P_p{}^2$ is skew-invariant. P_p is complete implies that both $P_p{}^1$ and $P_p{}^2$ are complete. We, therefore, assume $P_p = P_p{}^2$. If $P \neq 0$, then there must be a monomial A of some P_I which consists of only second and higher order variables. By Lemma 3.4, A consists of only second order variables and $n = p$. If $\deg_m(I) > 0$, then $\deg_m(A) > 0$ by hypothesis. By permuting the coordinate axes, this implies that $\deg_i(I) > 0$ implies $\deg_i(A) > 0$ for any index i. On the other hand, if $\deg_i(I) = 0$, then $\deg_i A$ must be odd since P_p is a skew-invariant map to forms. This implies $\deg_i(A) > 0$ for all i. Since $\deg_k(A) = 0$ for $k > 2L(A) = n$, this implies $m \leqslant n$, and, hence, $m = p = n$.

Let A be a monomial of P containing no first order terms. Since every index must appear in A, $2m = \sum \deg_k A \geqslant 2m$ and every index appears with degree 2. Furthermore, A consists only of second order variables. By Lemma 2.1 (1), we may assume that A is divisible by $g_{11/jk}A'$. Since $\deg_1 A = 2$, $\deg_1(j, k) = 0$, and we can apply Lemma 2.1 (2) to conclude that A can be chosen divisible by $g_{11/22}A'$. Since $\deg_1 A + \deg_2 A = 4$, $\deg_1 A' + \deg_2 A' = 0$. We proceed inductively to show that $A = g_{11/22} \cdots g_{m-1,m-1/mm}$ must be a monomial of P.

This monomial classifies the subspace of all complete invariant polynomials of order m. Since E_m is complete and invariant, we set

$$c = c_P(g_{11/22} \cdots g_{m-1,m-1/mm})/c_{E_m}(g_{11/22} \cdots g_{m-1,m-1/mm}).$$

Since $P - cE_m$ is complete and invariant and since the monomial $g_{11/22} \cdots g_{m-1,m-1/mm}$ does not appear in this polynomial, $P - cE_m = 0$.

7.

In this section, we present some combinatorial results concerning the asymptotic behavior of the eigenvalues for the Laplace operator. These results were obtained by combinatorial methods which are not of special interest. For part of the computations, a computer program was used. Further details are available from the author.

Let M be a smooth compact m dimensional Riemannian manifold without boundary. Let $D_p{}^m$ be the Laplace operator $dd^* + d^*d$ acting on p forms. The operator $D_p{}^m$ is positive, elliptic, and self-adjoint. If μ_i are the eigenvalues and θ_i the eigenforms of $D_p{}^m$, we let

$$f(t, x, D_p{}^m) = \sum \exp(-t\mu_i)(\theta_i, \theta_i)(x).$$

It is well known [3, 6] that $f(t, x, D_p{}^m)$ is well defined for $\mathrm{Re}(t) > 0$ and has an asymptotic expansion as $t \to 0^+$ of the form,

$$f(t, x, D_p{}^m) = \sum_{n=0}^{\infty} B_n(x, D_p{}^m)\, t^{(m-n)/2} \qquad (B_n = 0 \text{ for } n\text{-odd}).$$

The functions $B_n(x, D_p{}^m)$ depend only on the Riemannian metric of the manifold. We use the notation $P^m_{n,p}$ to denote these invariants of the metric. In the notation of the Introduction

$$P_n{}^m = \sum_p (-1)^n P^m_{n,p}.$$

It is very difficult to compute these invariants explicitly. We have, however, been able to obtain some results along this line. Let K denote the scalar curvature on M and let K_N denote $(-d^*d)^{(n-2)/2}K$. Let d_n be the constant $d_n = 2^{n/2}(1 \cdot 3 \cdot \cdots \cdot (n+1))$. Then

$$P^m_{n,p} = \left(n\binom{m-2}{p} + n\binom{m-2}{p-2} - (2n+4)\binom{m-2}{p-1}\right) K_n \Big/ (d_n(4\pi)^{m/2})$$
$$+ \text{ lower order terms.}$$

This result also permits us to prove that $P_n{}^m \neq 0$ for $n > m$ and m, n even, i.e., that there exist Riemannian metric so that the invariants $P_n{}^m(x, \text{metric}) \neq 0$.

For a two dimensional manifold, we have computed the invariants $P^2_{n,0}$ for $n = 0, 2, 4, 6, 8$ by using a computer program. Let K denote the scalar curvature and let $D = -d^*d$ denote the ordinary Laplacian. We will let $DKDK$ denote $D(K \cdot DK)$ and $(DK)^2$ denote $DK \cdot DK$. With this notation, our computations were as follows:

$$P^2_{0,0} = 1/4\pi d_0\,,$$

$$P^2_{2,0} = 2K/4\pi d_2\,,$$

$$P^2_{4,0} = (4DK + 4K^2)/4\pi d_4\,,$$

$$P^2_{6,0} = (6DDK + 8DKK + 12KDK + 32/3K^3)/4\pi d_6\,,$$

$$P^2_{8,0} = (8DDDK + 12DDKK + 24DKDK + 8(DK)^2 + 16KDDK$$
$$+ 40DKKK + 72KKDK + 48K^4)/4\pi d_8\,.$$

We are currently extending our computer program to cover the cases $n > 8$ and $m > 2$.

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